1 A matrix $\mathbf{M}$ has eigenvalues 1, 2 and 4, with corresponding eigen vectors $\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}3\\2\\2\end{pmatrix}$ and $\begin{pmatrix}1\\4\\3\end{pmatrix}$.

Determine $\mathbf{M}^n$, giving your answer as a single matrix. [5]

1	$\mathbf{M} = \mathbf{PDP}^{-1}$ $= \begin{pmatrix}2&3&1\\1&2&4\\1&2&3\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&4\end{pmatrix}\begin{pmatrix}2&3&1\\1&2&4\\1&2&3\end{pmatrix}^{-1}$ $\mathbf{M}^n = \begin{pmatrix}2&3&1\\1&2&4\\1&2&3\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&4\end{pmatrix}^n\begin{pmatrix}2&3&1\\1&2&4\\1&2&3\end{pmatrix}^{-1}$ $= \begin{pmatrix}2&3&1\\1&2&4\\1&2&3\end{pmatrix}\begin{pmatrix}1&0&0\\0&2^n&0\\0&0&4^n\end{pmatrix}\begin{pmatrix}2&3&1\\1&2&4\\1&2&3\end{pmatrix}^{-1}$ $= \begin{pmatrix}2&3(2^n)&4^n\\1&2(2^n)&4(4^n)\\1&2(2^n)&3(4^n)\end{pmatrix}\begin{pmatrix}2&7&-10\\-1&-5&7\\0&1&-1\end{pmatrix}$ $= \begin{pmatrix}4-3(2^n) & 14-15(2^n)+4^n & -20+21(2^n)-4^n\\ 2-2(2^n) & 7-10(2^n)+4(4^n) & -10+14(2^n)-4(4^n)\\ 2-2(2^n) & 7-10(2^n)+3(4^n) & -10+14(2^n)-3(4^n)\end{pmatrix}$

2 The sequence $\{u_n\}$ is given by the recurrence relation $u_{n+2} = \dfrac{u_{n+1}}{u_n} + \left(\dfrac{u_n}{u_{n+1}}\right)^2$ for $n \geq 1$, together with positive initial values $u_1 = x$ and $u_2 = y$.

(a) In the case when $x = 1$ and $y = 1$, the sequence $\{u_n\}$ exhibits *alternating convergence.*

That is, as $n \to \infty$, $u_n \to \begin{cases} \alpha & \text{when } n \text{ is even,} \\ \beta & \text{when } n \text{ is odd,} \end{cases}$

for some constants α and β.

(i) Write down, correct to 4 decimal places, the value of α and the value of β. [2]

(ii) Explain why, in the case when $x = 1$ and $y = 2$, the long-term behaviour of $\{u_n\}$ is

$$u_n \to \begin{cases} \beta & \text{when } n \text{ is even,} \\ \alpha & \text{when } n \text{ is odd.} \end{cases}$$ [1]

(b) You are given that, for all positive values of x and y, the sequence $\{u_n\}$ exhibits alternating convergence.

(i) Suppose that, for a chosen pair of positive initial values, the alternate limits are (in either order) p and q. Show that p and q must satisfy the relationship $p^3 + q^3 = p^2q^2$. [2]

(ii) Deduce a value for x and y such that $\{u_n\}$ is a constant sequence. [2]

(a)(i)	Using GC, $\alpha = 2.2287$, $\beta = 1.8997$ (to 4 dp)
(a)(ii)	Old sequence: $u_1 = 1,\ u_2 = 1,\ u_3 = 2,\ u_4 = 2.25,\ u_5 = 1.9151,\ \ldots$ New sequence: $u_1 = 1,\ u_2 = 2,\ u_3 = 2.25,\ u_4 = 1.9151,\ \ldots$ This is because the terms in the sequence in (ii) starts from the second term of the sequence in (i) and in the same order. Hence, $u_n \to \begin{cases} \beta & \text{when } n \text{ is even,} \\ \alpha & \text{when } n \text{ is odd.} \end{cases}$
(b)(i)	Suppose when $n \to \infty$, $u_n \to p$, $u_{n+1} \to q$, $u_{n+2} \to p$ $p = \dfrac{q}{p} + \left(\dfrac{p}{q}\right)^2$ $p = \dfrac{q^3 + p^3}{pq^2}$ $p^2q^2 = q^3 + p^3$ (shown)
(b)(ii)	For $\{u_n\}$ to be constant sequence, when $n \to \infty$, $u_n \to p,$ where $p = q$ when n is even or odd. $p^2p^2 = p^3 + p^3$ $p^4 - 2p^3 = 0$ $p^3(p-2) = 0$ $p = 0$ (rejected since $p > 0$) or $p = 2$ Hence $x = y = 2$.

3 **(a)** Let $f(x) = px^5 + qx^4$, where p and q are real constants and $p \neq 0$. The value of the integral $I = \int_{-a}^{a} f(x)\, dx$ is to be estimated using Simpson's Rule with four strips.

Find an expression for the error obtained when using this method. [7]

(b) Let $g(x) = px^5 + qx^4 + rx^3 + sx^2 + tx + u$, where p, q, r, s, t and u are real constants and $p \neq 0$.

The value of the integral $J = \int_{-a}^{a} g(x)\, dx$ is to be estimated using Simpson's Rule with four strips.

Write down, with justification, the expression for the error obtained when using this method. [1]

(a)

$$I = \int_{-a}^{a} (px^5 + qx^4)\, dx$$

$$= p\int_{-a}^{a} x^5\, dx + q\int_{-a}^{a} x^4\, dx$$

$$= \quad 0 \quad + 2q\left[\frac{x^5}{5}\right]_0^a$$

$$= \frac{2}{5}qa^5$$

x	$-a$	$-\frac{a}{2}$	0	$\frac{a}{2}$	a
$f(x)$	$pa^5 + qa^4$	$-\frac{p}{32}a^5 + \frac{q}{16}a^4$	0	$\frac{p}{32}a^5 + \frac{q}{16}a^4$	$pa^5 + qa^4$

By Simpson's rule,

$$I \approx \frac{a}{6}\left[(-pa^5 + qa^4) + (pa^5 + qa^4) + 4(-\frac{p}{32}a^5 + \frac{q}{16}a^4 + \frac{p}{32}a^5 + \frac{q}{16}a^4)\right]$$

$$= \frac{a}{6}\left[\frac{5}{2}qa^4\right]$$

$$= \frac{5}{12}qa^5$$

$$\text{Error} = \left(\frac{5}{12} - \frac{2}{5}\right)qa^5 = \frac{1}{60}qa^5$$

(b) The error is $\frac{1}{60}qa^5$. This is because Simpson's Rule is exact (i.e. no error) for cubic functions.

4 **(a)** Exponential growth and decay can be modelled by the differential equation $\frac{\mathrm{d}N}{\mathrm{d}t} = \lambda N$, where N is the measure of some physical quantity at time $t \geq 0$ and λ is a parameter. Given an initial positive value, N_0, of N,

(i) solve this differential equation, [2]

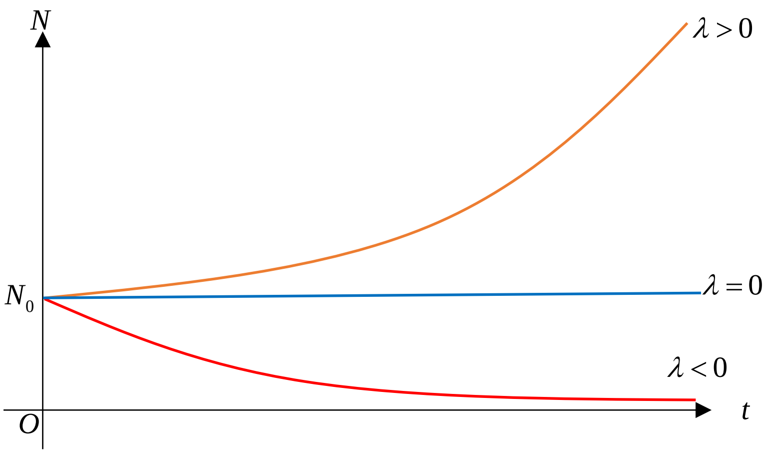
(ii) sketch, on one diagram, the behaviours of N according to the possible values of λ. [3]

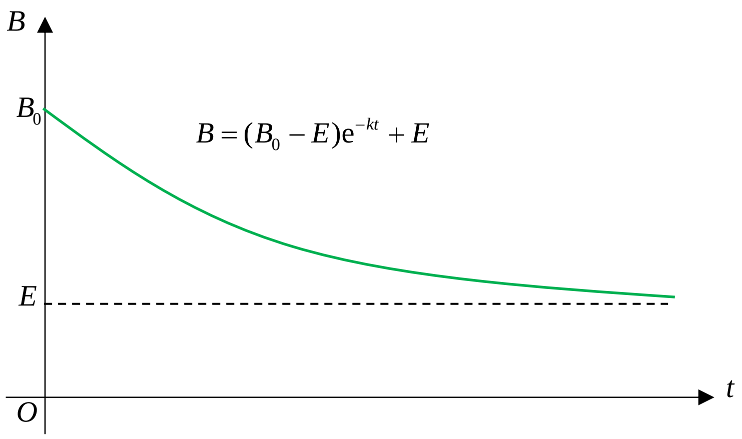
(b) *Newton's Law of Cooling* states that the rate at which the temperature of an object is changing at any instant is proportional to the difference between the temperature of the object and the temperature of its surroundings at that instant.

A container of liquid is placed in a room which has a constant temperature E. Initially, the temperature of the liquid is B_0, where $B_0 > E$. At time t later, the temperature of the liquid is B.

(i) Write down and solve the differential equation which describes this situation. [3]

(ii) Sketch the solution curve for B against t. [1]

(a) (i)	$\frac{\mathrm{d}N}{\mathrm{d}t} = \lambda N$ $\int \frac{1}{N}\,\mathrm{d}N = \lambda \int 1\,\mathrm{d}t$ $\ln N = \lambda t + C \; (\because N > 0)$ $N = \mathrm{e}^{\lambda t + C}$ $N = A\mathrm{e}^{\lambda t}$ $t = 0, N = N_0 \Rightarrow A = N_0$. $\therefore N = N_0 \mathrm{e}^{\lambda t}$
	

(b)	$\dfrac{dB}{dt}=-k(B-E),$ where k is a positive constant $\displaystyle\int \frac{1}{B-E}\,dB=-\int k\,dt$ $\ln(B-E)=-kt+C\ (\because B>E)$ $B-E=Ae^{-kt}$ $B=Ae^{-kt}+E$ $t=0,\ B=B_0 \Rightarrow A=B_0-E$ $\therefore B=(B_0-E)e^{-kt}+E$
	

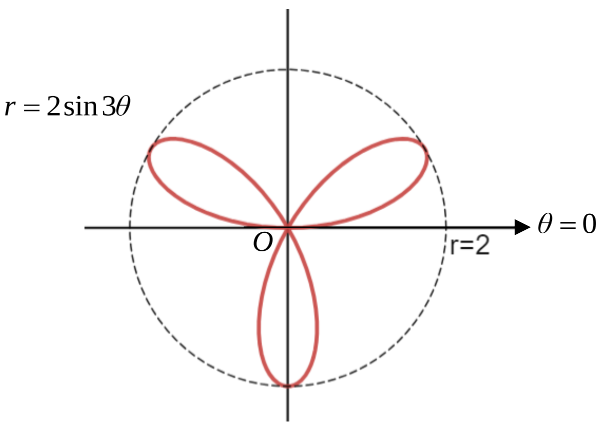
5 **(a)** Use de Moivre's theorem to find an expression for $\sin 3\theta$ in terms of $\sin\theta$. [3]

(b) A curve C has cartesian equation $\left(x^2+y^2\right)\left(x^2+y^2-6y\right)+8y^3=0$.

(i) Show that the polar equation of the curve is $r=2\sin 3\theta$. [2]

(ii) Sketch C. You may assume that the curve is defined only for those values of θ for which $r \geq 0$. [2]

(iii) Determine an integral that gives the exact value of the total length of one loop of C. Use your calculator to evaluate this integral, giving your answer correct to 3 decimal places. [3]

(a)	By de Moivre's theorem, $(\cos\theta+\mathrm{i}\sin\theta)^3=\cos 3\theta+\mathrm{i}\sin 3\theta$ $(\cos\theta+\mathrm{i}\sin\theta)^3=(\cos\theta)^3+3(\cos\theta)^2(\mathrm{i}\sin\theta)+3(\cos\theta)(\mathrm{i}\sin\theta)^2+(\mathrm{i}\sin\theta)^3$ $=\cos^3\theta+3\mathrm{i}\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-\mathrm{i}\sin^3\theta$ $=\left(\cos^3\theta-3\cos\theta\sin^2\theta\right)+\mathrm{i}\left(3\cos^2\theta\sin\theta-\sin^3\theta\right)$ Comparing imaginary part: $\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$ $=3\left(1-\sin^2\theta\right)\sin\theta-\sin^3\theta$ $=3\sin\theta-4\sin^3\theta$
(b)(i)	$\left(x^2+y^2\right)\left(x^2+y^2-6y\right)+8y^3=0$ Since $x^2+y^2=r^2$, $x=r\cos\theta$, $y=r\sin\theta$ $r^2\left(r^2-6r\sin\theta\right)+8(r\sin\theta)^3=0$ $r^4-6r^3\sin\theta+8r^3\sin^3\theta=0$ $r^3\left(r-6\sin\theta+8\sin^3\theta\right)=0$ $r=0$ or $r-6\sin\theta+8\sin^3\theta=0$ (rej $\because r\geq 0$) $r-6\sin\theta+8\sin^3\theta=0$ $r=6\sin\theta-8\sin^3\theta$ $=2\left(3\sin\theta-4\sin^3\theta\right)$ $=2\sin 3\theta$ (shown)
(b)(ii)	

(b)(iii)	$r = 2\sin 3\theta$ $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 6\cos 3\theta$ Total length of l loop of $C = \int_0^{\frac{\pi}{3}} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{3}} \sqrt{4\sin^2 3\theta + \left(6\cos 3\theta\right)^2}\ \mathrm{d}\theta$ $= 4.455$ (3dp)

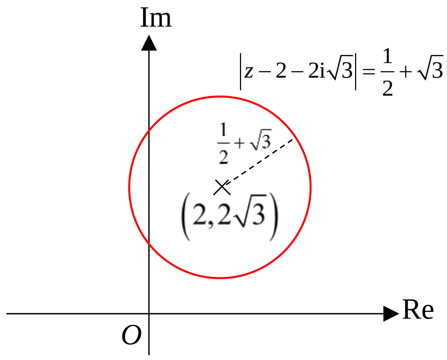
6 Complex numbers $z = x + \mathrm{i}y$ are represented by points $P(x, y)$ in the complex plane.

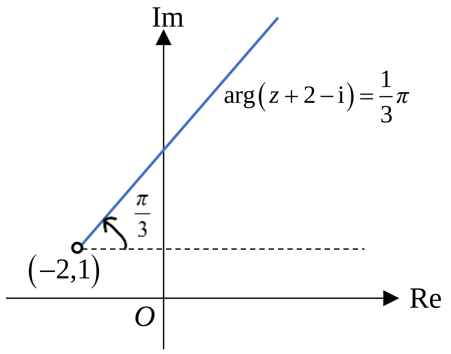
(a) On separate Argand diagrams, sketch the loci L_1 and L_2 given by

$$\left|z - 2 - 2\mathrm{i}\sqrt{3}\right| = \frac{1}{2} + \sqrt{3} \quad \text{and} \quad \arg(z + 2 - \mathrm{i}) = \frac{1}{3}\pi .$$ [4]

(b) By finding cartesian equations for these loci, prove that L_1 and L_2 meet tangentially. [7]

(a)





(b)

For $\left|z - 2 - 2\mathrm{i}\sqrt{3}\right| = \frac{1}{2} + \sqrt{3}$, cartesian equation is

$$(x-2)^2 + \left(y - 2\sqrt{3}\right)^2 = \left(\frac{1}{2} + \sqrt{3}\right)^2 \quad \text{---(1)}$$

For $\arg(z + 2 - \mathrm{i}) = \frac{1}{3}\pi$, cartesian equation is

$$y - 1 = \tan\left(\frac{1}{3}\pi\right)(x+2)$$

$$y - 1 = \sqrt{3}(x+2)$$

$$y = 1 + \sqrt{3}(x+2) \quad \text{---(2)}$$

Substitute (2) into (1), we have

$$(x-2)^2 + \left(1 + \sqrt{3}(x+2) - 2\sqrt{3}\right)^2 = \left(\frac{1}{2} + \sqrt{3}\right)^2$$

$$(x-2)^2 + \left(1 + \sqrt{3}x\right)^2 = \left(\frac{1}{2} + \sqrt{3}\right)^2$$

$$x^2 - 4x + 4 + 1 + 2\sqrt{3}x + 3x^2 = \frac{1}{4} + \sqrt{3} + 3$$

$$4x^2 + \left(2\sqrt{3} - 4\right)x + \frac{7}{4} - \sqrt{3} = 0$$

$$\text{Discriminant} = \left(2\sqrt{3} - 4\right)^2 - 4(4)\left(\frac{7}{4} - \sqrt{3}\right)$$

$$= 12 - 16\sqrt{3} + 16 - 16\left(\frac{7}{4} - \sqrt{3}\right)$$

$$= 28 - 16\sqrt{3} - 28 + 16\sqrt{3}$$

$$= 0$$

There is only one real root. Since the half line L_2 intersects the circle L_1 at only one point, they must be tangential to each other.

7 For different values of the parameter t, the equations of three planes are $x+2y+3z=1$, $3x-y+tz=-4$ and $2x+(t+4)y+(t+1)z=-3$.

Determine the common points, if they exist, of these planes in each of the cases

- $t=7$,
- $t=2$,
- $t=5$.

Full working must be shown. [10]

7

Consider $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & -1 & t & -4 \\ 2 & t+4 & t+1 & -3 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -7 & t-9 & -7 \\ 0 & t & t-5 & -5 \end{array}\right)$

When $t=7$, $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -7 & -2 & -7 \\ 0 & 7 & 2 & -5 \end{array}\right) \xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -7 & -2 & -7 \\ 0 & 0 & 0 & -12 \end{array}\right)$

The last row implies $0x+0y+0z=-12$ which is inconsistent.
The system does not have a solution which means that the 3 planes do not intersect at a common point.

When $t=2$, $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -7 & -7 & -7 \\ 0 & 2 & -3 & -5 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{7}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & -5 \end{array}\right)$

$\xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & -7 \end{array}\right)$

$-5z=-7 \Rightarrow z=\frac{7}{5}$

$y+z=1 \Rightarrow y=-\frac{2}{5}$

$x+2y+3z=1 \Rightarrow x=1+\frac{4}{5}-\frac{21}{5}=-\frac{12}{5}$

$\therefore$ The 3 planes intersect at the common point $\left(-\frac{12}{5}, -\frac{2}{5}, \frac{7}{5}\right)$.

When $t=-5$, $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -7 & -14 & -7 \\ 0 & -5 & -10 & -5 \end{array}\right) \xrightarrow[R_3 \to -\frac{1}{5}R_2]{R_2 \to -\frac{1}{7}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right)$

$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$y+2z=1 \Rightarrow y=1-2z$

$x+2y+3z=1 \Rightarrow x=1-2(1-2z)-3z=-1+z$

The 3 planes intersect at the common line $r=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$. .

Full workings must be shown without using GC. But you should use your GC to double check your answer.

8 When modelling an alternating current in an electrical circuit, the variables are sometimes written as complex numbers.

A combination of *Ohm's Law* and *Krichoff's Law* gives the following relationship between voltage V, current I, resistance R, capacitance C and inductance L present in the circuit at time t:

$$V = IR + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}\int I \, \mathrm{d}t, \qquad (*)$$

where all variables are in standard units.

(a) It is given that V is of the form $V_0(\cos\omega t + \mathrm{i}\sin\omega t)$, where ω is a measure of the frequency of the oscillating input voltage. By writing $Q = \int I \, \mathrm{d}t$, express (*) as a second-order differential equation in Q. [1]

(b) In a given circuit, $R = 10$, $L = 5$ and $C = 0.04$. By writing the particular integral of Q in the form $q(\cos\omega t + \mathrm{i}\sin\omega t)$, where q is a complex number, and considering real and imaginary parts of Q in the differential equation in part **(a)** separately, express Q as a function of time t where $\omega = 5$ and $V_0 = 200$. [10]

(c) Write the current I as a function of t. [2]

(a)	$Q = \int I \, \mathrm{d}t,$ $\frac{\mathrm{d}Q}{\mathrm{d}t} = I$ $\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = \frac{\mathrm{d}I}{\mathrm{d}t}$ $V = \frac{\mathrm{d}Q}{\mathrm{d}t}R + L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{1}{C}Q$ $V = L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q$
(b)	When $R = 10$, $L = 5$ and $C = 0.04$. $V = 5\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + 10\frac{\mathrm{d}Q}{\mathrm{d}t} + 25Q$ Auxiliary equation: $5m^2 + 10m + 25 = 0$ $m^2 + 2m + 5 = 0$ $m = \frac{-2 \pm \sqrt{4 - 4(5)}}{2}$ $= -1 \pm 2\mathrm{i}$ General solution: $Q = \mathrm{e}^{-t}(A\cos 2t + B\sin 2t)$ Particular integral: $Q = (a + b\mathrm{i})(\cos\omega t + \mathrm{i}\sin\omega t)$ $\frac{\mathrm{d}Q}{\mathrm{d}t} = (a + b\mathrm{i})\omega(-\sin\omega t + \mathrm{i}\cos\omega t)$ $\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = (a + b\mathrm{i})\omega^2(-\cos\omega t - \mathrm{i}\sin\omega t)$ When $\omega = 5$ and $V_0 = 200$, $V = 200(\cos 5t + \mathrm{i}\sin 5t)$. $V = 125(a + b\mathrm{i})(-\cos 5t - \mathrm{i}\sin 5t) + 50(a + b\mathrm{i})(-\sin 5t + \mathrm{i}\cos 5t) + 25(a + b\mathrm{i})(\cos 5t + \mathrm{i}\sin 5t)$ $= [(-100a - 50b)\cos 5t + (100b - 50a)\sin 5t] + \mathrm{i}[(50a - 100b)\cos 5t + (-100a - 50b)\sin 5t]$

	Comparing real and imaginary parts, $200=-100a-50b$ $0=100b-50a$ From GC, $a=-\frac{8}{5}, b=-\frac{4}{5}$ Particular integral is $Q=(-\frac{8}{5}-\frac{4}{5}\mathrm{i})(\cos 5t+\mathrm{i}\sin 5t)$ General solution is $Q=\mathrm{e}^{-t}(A\cos 2t+B\sin 2t)+(-\frac{8}{5}-\frac{4}{5}\mathrm{i})(\cos 5t+\mathrm{i}\sin 5t)$ $I=\frac{\mathrm{d}Q}{\mathrm{d}t}$ $=-\mathrm{e}^{-t}\left[A\cos 2t+B\sin 2t\right]+\mathrm{e}^{-t}\left[-2A\sin 2t+2b\cos 2t\right]+(-8-4\mathrm{i})(-\sin 5t+\mathrm{i}\cos 5t)$

9 This question concerns conic sections with equations of the form

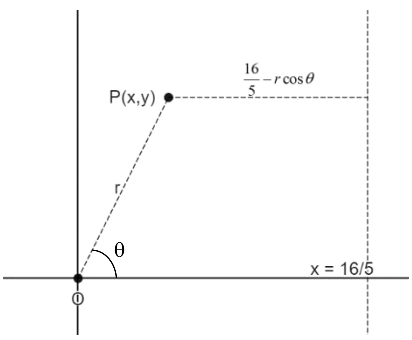
$$Ax^2 + By^2 + Cx + Dy + E = 0.$$

(a) The hyperbola H is given by $A = b^2$, $B = -a^2$, $C = D = 0$ and $E = -a^2b^2$. The *conjugate* hyperbola to H, denoted by J, is given by $A = -b^2$, $B = a^2$, $C = D = 0$ and $E = -a^2b^2$.
Determine the relationship between e (the eccentricity of H) and f (the eccentricity of J). [3]

(b) The conic section K is given by $A = 16$, $B = -9$, $C = -160$, $D = 0$ and $E = 256$.

(i) Identify the nature of K and determine its eccentricity. Give the coordinates of its foci and the equations of its directrices.

(ii) Explain why it is appropriate to express the equation of K in the form $r = \dfrac{p}{1 + q\cos\theta}$ and determine the values of the constants p and q in this case. [3]

(a)	$H: \; b^2x^2 - a^2y^2 - a^2b^2 = 0 \Rightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $J: \; -b^2x^2 + a^2y^2 - a^2b^2 = 0 \Rightarrow \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$ $\left.\begin{array}{l} e^2 = 1 + \dfrac{b^2}{a^2} \\ f^2 = 1 + \dfrac{a^2}{b^2} \end{array}\right\} \dfrac{e^2}{f^2} = \dfrac{b^2}{a^2} \Rightarrow ae = bf$
(b)(i)	$H: \; 16x^2 - 9y^2 - 160x + 256 = 0$ $16(x^2 - 10x) - 9y^2 = -256$ $16(x-5)^2 - 9y^2 = -256 + 16(5)^2 = 144$ $\dfrac{(x-5)^2}{9} - \dfrac{y^2}{16} = 1$ Eccentricity $= \sqrt{\dfrac{9+16}{9}} = \dfrac{5}{3}$ Foci are $\left(5 \pm \dfrac{5}{3}(3), 0\right) = (10, 0)$ and $(0, 0)$ Directrices are $x = 5 \pm \dfrac{3}{5/3}$, ie. $x = \dfrac{34}{5}$ and $x = \dfrac{16}{5}$
(ii)	Since the origin is a focus, it is convenient to express K in polar form with origin as the pole. 

	By the directrix-eccentricity-focus definition, $r = e\left(\frac{16}{5} - r\cos\theta\right)$. $r = \frac{5}{3}\left(\frac{16}{5} - r\cos\theta\right)$ $r\left(1 + \frac{5}{3}\cos\theta\right) = \frac{16}{3}$ $r = \dfrac{\frac{16}{3}}{1 + \frac{5}{3}\cos\theta}$ $\therefore p = \frac{16}{3}, \quad q = \frac{5}{3}.$

10 In a nature reserve, a project was commenced to keep records of the population of a certain type of ground-nesting bird, on a monthly basis. The total population, *n* months after records began, is given by P_n where

$$P_n = A_n + B_n + C_n.$$

In this equation,

A_n is the number of ageing adults (that is, those too old to breed)

B_n is the number of adults of breeding age, and

C_n is the number of "children" in the population (that is, those too young to breed).

For positive constants α, β, γ and δ, each between 0 and 1, the numbers in each group are modelled by the following system of recurrence relations.

$$A_{n+1} = A_n + \alpha B_n - \beta A_n$$

$$B_{n+1} = B_n + \gamma C_n - \alpha B_n$$

$$C_{n+1} = C_n + \delta B_n - \gamma C_n$$

(a) (i) Give an interpretation of each of α, β, γ and δ. [4]

(ii) Hence give one criticism of this population model. [1]

At the start of the project, it is known that $A_0 = 200$, $B_0 = 1050$ and $C_0 = 250$. Estimates for the population parameters are $\alpha = 0.04$, $\beta = 0.1$, $\gamma = 0.05$ and $\delta = 0.01$.

(b) (i) Use these figures to calculate the values of A_1, B_1, C_1 and A_2, B_2, C_2. [3]

(ii) Hence give a second criticism of this population model. [1]

(iii) Suggest an improvement to the model that would overcome this issue. [1]

When $n = 100$, the original model gave $A_{100} = 50$, $B_{100} = 98$ and $C_{100} = 35$ (all to the nearest integer).

(c) Describe what this suggests in the long term for this population of ground-nesting birds. [1]

One of the researchers on the project says that a much simpler model could be used which would give the same output when $n = 100$. The simpler model is

$$A_{n+1} = R_A \times A_n$$

$$B_{n+1} = R_B \times B_n$$

$$C_{n+1} = R_C \times C_n$$

for suitable monthly scale factors R_A, R_B and R_C.

(d) Calculate each of these monthly scale factors and suggest a common scale factor for all three population groups. [4]

(a)(i)	α represents the **proportion** of adults of breeding age at time *n* becoming ageing adults at time $n+1$. β represents the **proportion** of ageing adults at time *n* not surviving to time $n+1$. γ represents the **proportion** of children at time *n* becoming adults of breeding age at time $n+1$. δ represents the **proportion** of adults of breeding age at time *n* giving birth to children at time $n+1$.
(a)(ii)	This model does not take into consideration external factors such as disease/starvation that might affect the bird's population at any growth stage. E.g., the model assumes that all children and adults of breeding age would not die before growing to become adults of breeding age and ageing adults eventually.
(b)(i)	$A_1 = 200 + 0.04(1050) - 0.1(200) = 222$ $B_1 = 1050 + 0.05(250) - 0.04(1050) = 1020.5$ $C_1 = 250 + 0.01(1050) - 0.05(250) = 248$ $A_2 = 222 + 0.04(1020.5) - 0.1(222) = 240.62$ $B_2 = 1020.5 + 0.05(248) - 0.04(1020.5) = 992.08$ $C_2 = 248 + 0.01(1020.5) - 0.05(248) = 245.805$
(b)(ii)	The model results in population sizes which are not whole numbers.
(b)(iii)	An improvement would be to include a ceiling or floor function to each of the recurrence relations so that only whole numbers are evaluated and produced.
(c)	The figures provided indicated that the population sizes of each type of ground-nesting birds decreases significantly from the start of the project, suggesting that the population of each type of ground-nesting bird will eventually decreases to zero in the long term. Hence the population of the ground-nesting bird is predicted to become extinct. Note: By just stating that the population "tends to zero in the long-term." is insufficient. You need to correctly contextualise that comment. A better response would be to mention that "the population of birds is predicted to become extinct in the long run".
(d)	$A_{n+1} = R_A \times A_n$ $= (R_A)^2 \times A_{n-1}$ $= (R_A)^3 \times A_{n-2}$ $= \ldots$ $= (R_A)^{n+1} \times A_0$ Similarly, $B_{n+1} = (R_B)^{n+1} \times B_0$ and $C_{n+1} = (R_C)^{n+1} \times C_0$ $50 = (R_A)^{100}\, 200 \Rightarrow R_A = \left(\frac{1}{4}\right)^{\frac{1}{100}} = 0.98623 = 0.986$ (3sf) $98 = (R_B)^{100}\, 1050 \Rightarrow R_B = \left(\frac{7}{75}\right)^{\frac{1}{100}} = 0.97656 = 0.977$ (3sf) $35 = (R_C)^{100}\, 250 \Rightarrow R_C = \left(\frac{7}{50}\right)^{\frac{1}{100}} = 0.98053 = 0.981$ (3sf) A common scale factor would be the $\frac{1}{3}\left[\left(\frac{1}{4}\right)^{100} + \left(\frac{7}{75}\right)^{100} + \left(\frac{7}{50}\right)^{100}\right] = 0.981$ (3sf)