Name: Solutions	Register No.:	Class:

CRESCENT GIRLS' SCHOOL
SECONDARY FOUR 2023
PRELIMINARY EXAMINATION

ADDITIONAL MATHEMATICS **4049/02**

Paper 2 **25 August 2023**

Candidates answer on the Question Paper. **2 hours 15 mins**

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your name and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
If working is needed for any question it must be shown with the answer.
Omission of essential working will result in loss of marks.
The use of an approved scientific calculator is expected, where appropriate.
If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures. Give answers in degrees to one decimal place.
For π, use either your calculator value or 3.142, unless the question requires the answer in terms of π.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total of the marks for this paper is **90**.

For Examiner's Use

Question	1	2	3	4	5	6	7	8	9	10	11
Marks											

Table of Penalties		Qn. No.		
Presentation	–1			90
Significant Figures/ Units	–1		Parent's/Guardian's Signature	

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-r)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\Delta = \frac{1}{2}ab\sin C$$

1 (i) The equation of a curve is $f(x)=\dfrac{\sqrt{x^2+1}}{3x+4}$, $x \neq p$.

Find $f'(x)$, simplifying your answer as a single fraction. Hence determine the gradient of the tangent at the point on the curve where $x = 0$. [5]

$$f'(x)=\frac{(3x+4).\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)-(x^2+1)^{\frac{1}{2}}(3)}{(3x+4)^2}$$

$$=\frac{\frac{x(3x+4)}{\sqrt{x^2+1}}-3\sqrt{x^2+1}}{(3x+4)^2}$$

$$=\frac{x(3x+4)-3(x^2+1)}{\sqrt{x^2+1}(3x+4)^2}$$

$$=\frac{4x-3}{\sqrt{x^2+1}(3x+4)^2}$$

When $x = 0$, gradient of tangent is $\dfrac{4(0)-3}{\sqrt{0^2+1}(3(0)+4)^2}$

$$=-\frac{3}{16}$$

(ii) State the value of p. [1]

$$p=-1\frac{1}{3}$$

2 By using a suitable substitution, show that $3e^{\sqrt{4x}} - 4 = e^{\sqrt{x}}$ has only one solution and find its value correct to 2 significant figures. [5]

$3\left(e^{\sqrt{x}}\right)^2 - 4 = e^{\sqrt{x}}$

Let $y = e^{\sqrt{x}}$

$3y^2 - y - 4 = 0$

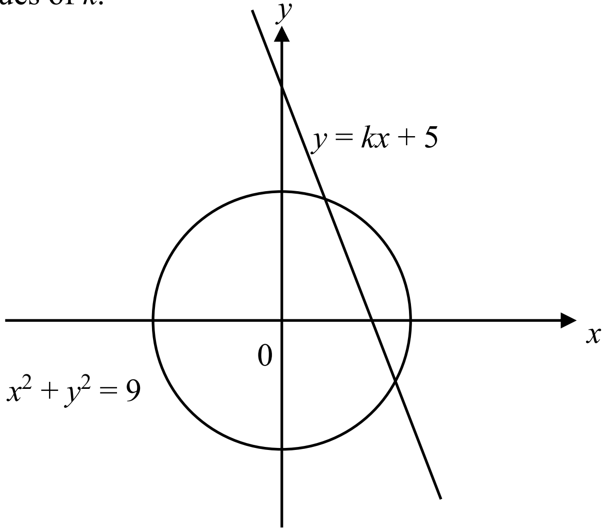
$(3y-4)(y+1) = 0$

$y = \frac{4}{3}$ or $y = -1$

$e^{\sqrt{x}} = \frac{4}{3}$ or $e^{\sqrt{x}} = -1$ (NA since $e^{\sqrt{x}} > 0$)

$\sqrt{x} = \ln\frac{4}{3} \Rightarrow x = 0.083$ (to 2 s.f.)

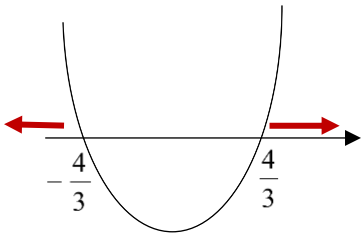
3 The diagram below shows a circle $x^2 + y^2 = 9$ and a straight line $y = kx + 5$.
Find the range of values of *k*. [5]



Sub $y = kx + 5$ into $x^2 + y^2 = 9$

$x^2 + (kx+5)^2 = 9$
$x^2 + k^2x^2 + 10kx + 25 - 9 = 0$
$x^2(1+k^2) + 10kx + 16 = 0$
Since line cuts the curve at 2 distinct points, $b^2 - 4ac > 0$.
$(10k)^2 - 4(1+k^2)(16) > 0$
$100k^2 - 64 - 64k^2 > 0$
$4(9k^2 - 16) > 0$
$(3k+4)(3k-4) > 0$

$k < -\frac{4}{3}$ or $k > \frac{4}{3}$



Since $k < 0$, $k < -\frac{4}{3}$.

4 (a) Show that $\dfrac{d}{dx}\ln(\sin x+\cos x)=\tan\left(\dfrac{\pi}{4}-x\right)$. [4]

$$\frac{d}{dx}\ln(\sin x+\cos x)=\frac{1}{\sin x+\cos x}\,\frac{d}{dx}(\sin x+\cos x)$$

$$=\frac{\cos x-\sin x}{\sin x+\cos x}$$

$$=\frac{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x}+\frac{\cos x}{\cos x}}$$

$$=\frac{1-\tan x}{1+\tan x}$$

$$=\frac{\tan\frac{\pi}{4}-\tan x}{\tan\frac{\pi}{4}+\tan x}$$

$$=\tan\left(\frac{\pi}{4}-x\right) \quad \text{(shown)}$$

(b) Solve $\tan\left(\dfrac{\pi}{4}-x\right)=-3$ for $0\le x\le\pi$. [2]

$$\tan\left(\frac{\pi}{4}-x\right)=-3$$

$$\tan\left[-\left(x-\frac{\pi}{4}\right)\right]=-3$$

$$\tan\left(x-\frac{\pi}{4}\right)=3$$

Basic angle = 1.2490

$$x-\frac{\pi}{4}=1.2490$$

$$x=2.03 \quad \text{(to 3 s.f.)}$$

5 (i) Differentiate $3x\cos\frac{1}{2}x$ with respect to x. [3]

$$\frac{\mathrm{d}}{\mathrm{d}x}3x\cos\frac{1}{2}x = 3x\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\frac{1}{2}x\right)+\left(\cos\frac{1}{2}x\right)\frac{\mathrm{d}}{\mathrm{d}x}(3x)$$

$$= 3x\left(-\frac{1}{2}\sin\frac{1}{2}x\right)+\left(\cos\frac{1}{2}x\right)3$$

$$= -\frac{3}{2}x\sin\frac{1}{2}x+3\cos\frac{1}{2}x$$

(ii) Hence, find the exact value of $\int_0^{\frac{\pi}{3}} x\sin\frac{1}{2}x\ \mathrm{d}x$. [4]

$$\frac{\mathrm{d}}{\mathrm{d}x}3x\cos\frac{1}{2}x = -\frac{3}{2}x\sin\frac{1}{2}x+3\cos\frac{1}{2}x$$

$$\frac{3}{2}x\sin\frac{1}{2}x = 3\cos\frac{1}{2}x-\frac{\mathrm{d}}{\mathrm{d}x}3x\cos\frac{1}{2}x$$

$$x\sin\frac{1}{2}x = 2\cos\frac{1}{2}x-\frac{\mathrm{d}}{\mathrm{d}x}2x\cos\frac{1}{2}x$$

$$\int_0^{\frac{\pi}{3}} x\sin\frac{1}{2}x\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} 2\cos\frac{1}{2}x\,\mathrm{d}x-\left[2x\cos\frac{1}{2}x\right]_0^{\frac{\pi}{3}}$$

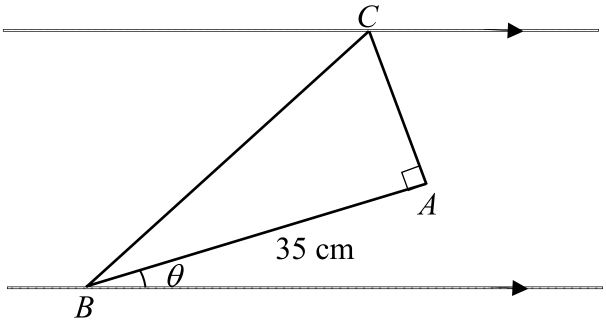
$$= \left[4\sin\frac{1}{2}x-2x\cos\frac{1}{2}x\right]_0^{\frac{\pi}{3}}$$

$$= 4\sin\frac{1}{2}\left(\frac{\pi}{3}\right)-2\left(\frac{\pi}{3}\right)\cos\frac{\pi}{6}$$

$$= 4\left(\frac{1}{2}\right)-\frac{2\pi}{3}\left(\frac{\sqrt{3}}{2}\right)$$

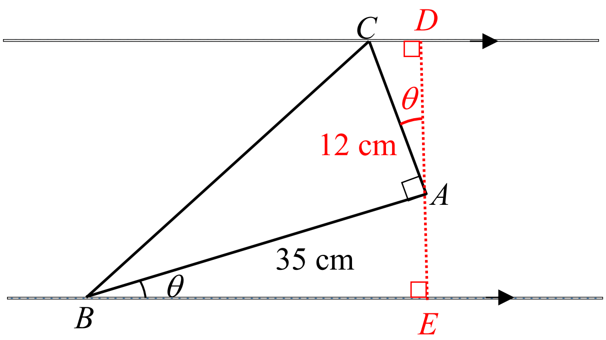
$$= 2-\frac{\sqrt{3}\pi}{3}$$

6 The figure (not drawn to scale) shows a right-angled triangle ABC constructed between two parallel lines.



The area of triangle ABC is 210 cm^2. AB = 35 cm and makes an acute angle θ with one of the lines.

(i) Show that the distance between the parallel lines, $d = (12\cos\theta + 35\sin\theta)$ cm. [2]



$$\frac{1}{2}(350(AC) = 210$$

$$AC = 12\,\text{cm}$$

$$\cos\theta = \frac{AD}{12} \Rightarrow AD = 12\cos\theta$$

$$\sin\theta = \frac{AE}{35} \Rightarrow AE = 35\sin\theta$$

$$d = AD + AE$$
$$= (12\cos\theta + 35\sin\theta)\text{ cm}$$

(ii) Express d in the form $R\cos(\theta-\alpha)$, where R is a constant and α is an angle in radians. [3]

$12\cos\theta + 35\sin\theta = R\cos(\theta-\alpha)$

$R = \sqrt{12^2 + 35^2}$
$= 37$

$\alpha = \tan^{-1}\left(\frac{35}{12}\right)$
$= 1.2405$

$d = 37\cos(\theta - 1.24)$

(iii) Find the value of θ when d = 28 cm. [2]

$37\cos(\theta - 1.2405) = 28$

$\cos(\theta - 1.2405) = \frac{28}{37}$
Basic angle = 0.71246

$\theta - 1.2405 = 0.71246$ (rejected)
$\theta - 1.2405 = 2\pi - 0.71246 - 2\pi$
$\theta - 1.2405 = -0.71246$ $\Rightarrow \theta = 0.528$ (to 3 s.f.)

7 The coefficient of $\frac{1}{x^3}$ is 512 in the expansion of $\left(\frac{2}{x}+px^2\right)^9$, where $p < 0$.

(i) By first working out the general term of $\left(\frac{2}{x}+px^2\right)^9$, find the value of p. [3]

$\left(\frac{2}{x}+px^2\right)^9$

$T_{r+1}=\binom{9}{r}\left(\frac{2}{x}\right)^{9-r}\left(px^2\right)^r$

$=\binom{9}{r}(2)^{9-r}(p^r)x^{3r-9}$

For the term in $\frac{1}{x^3}$, $3r-9=-3$

$r=2$

$\binom{9}{2}(2)^7(p^2)=512$

$p^2=\frac{1}{9}$

$p=\frac{1}{3}$ (rejected since $p < 0$) or $p=-\frac{1}{3}$

(ii) Using the results in (i),

(a) show that the coefficient of the first term in the expansion of $\left(\frac{2}{x}+px^2\right)^9$ is also 512. [1]

$T_{r+1}=\binom{9}{r}(2)^{9-r}\left(-\frac{1}{3}\right)^r x^{3r-9}$

First term, $r=0$

Coeff. of $T_1=\binom{9}{0}(2)^9\left(-\frac{1}{3}\right)^0$ = 512 (shown)

(b) find the $\frac{1}{x^6}$ term in the expansion of $\left(\frac{2}{x}+px^2\right)^9$. [2]

For $\frac{1}{x^6}$ term, $3r-9=-6$

$$r=1$$

$$T_2=\binom{9}{1}(2)^{9-1}\left(-\frac{1}{3}\right)^1 x^{-6}$$

$$= -\frac{768}{x^6}$$

(c) explain why the term in $\frac{1}{x^4}$ does not exist in the expansion of

$\left(\frac{2}{x}+px^2\right)^9\left(\frac{1}{8x}+\frac{1}{12}x^2\right)$. [2]

$$\left(\frac{2}{x}-\frac{x^2}{3}\right)^9\left(\frac{1}{8x}+\frac{x^2}{12}\right)$$

$$=\left(....+\frac{512}{x^3}-\frac{768}{x^6}+...\right)\left(\frac{1}{8x}+\frac{x^2}{12}\right)$$

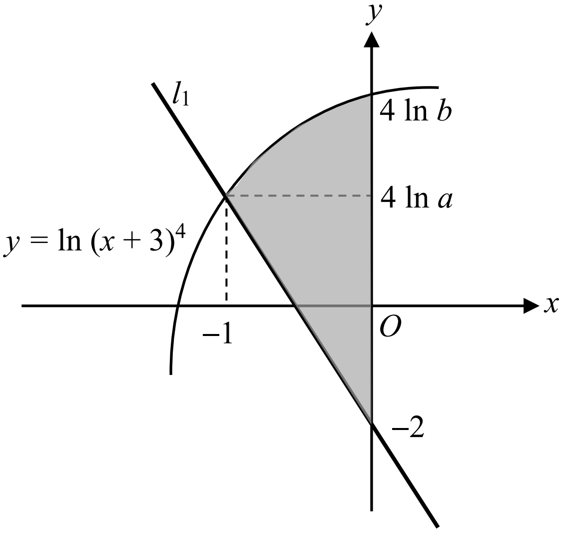
Coefficient of $\frac{1}{x^4}= 512\left(\frac{1}{8}\right)-768\left(\frac{1}{12}\right)$

$$=0$$

Hence the the term in $\frac{1}{x^4}$ does not exist in the expansion of

$\left(\frac{2}{x}+px^2\right)^9\left(\frac{1}{8x}+\frac{1}{12}x^2\right)$.

8 The diagram below shows the curve $y = \ln(x+3)^4$ which cuts the y-axis at $(0,\ 4\ln b)$.
The line l_1, cuts the y-axis at (0, −2) and meets the curve at $(-1,\ 4\ln a)$.



(i) Find the value of a and of b. [2]

$y = \ln(x+3)^4$ cuts the y-axis,

$4\ln(0+3) = 4\ln b$

$b = 3$

When $x = -1$,

$4\ln(-1+3) = 4\ln a$

$a = 2$

(ii) Find the equation of l_1 giving your answer in the form $y=(p\ln q+r)x+s$ where p, q r and s are integers. [2]

Gradient of l_1 is $\dfrac{4\ln 2-(-2)}{-1-0}=-4\ln 2-2$

Equation of l_1 is $y=\left(-4\ln 2-2\right)x-2$

(iii) Calculate the area of the shaded region, giving your answer to 3 decimal places. [6]

$y = 4\ln(x+3)$

$e^{\frac{y}{4}}=x+3$

$x=e^{\frac{y}{4}}-3$

Area of shaded region

$= \dfrac{1}{2}\left(4\ln 2+2\right)(1)+\left[-\int_{4\ln 2}^{4\ln 3}\left(e^{\frac{y}{4}}-3\right)\mathrm{d}y\right]$

$= 2\ln 2+1-\left[4e^{\frac{y}{4}}-3y\right]_{4\ln 2}^{4\ln 3}$

$= 2\ln 2+1-\left[4e^{\frac{4\ln 3}{4}}-3(4\ln 3)-\left(4e^{\frac{4\ln 2}{4}}-3(4\ln 2)\right)\right]$

$= 2\ln 2+1-4(3)+12\ln 3+4(2)-12\ln 2$

$= 3.252$ units^2 (to 3 d.p.)

9 Two particles, P and Q, each moving in a straight line passes a point, Z, at the same instant. The displacement of P, s_P m is given by $s_P = \frac{t^3}{3} - \frac{3t^2}{2} + 5t$ where t is the time in seconds after passing Z.

The particle Q passes Z with a velocity of 11 m/s and its acceleration, a_Q m/s^2 is given by $a_Q = 2t - 6$ where t is the time in seconds after passing Z.

(a) Find the value of t for which the velocities of P and Q are equal. [4]

$$s_P = \frac{t^3}{3} - \frac{3t^2}{2} + 5t$$
$$v_P = \frac{ds_P}{dt} = t^2 - 3t + 5$$
$$a_Q = 2t - 6$$
$$v_Q = \int 2t - 6 \, dt$$
$$= t^2 - 6t + c \text{, where } c \text{ is an arbitrary constant}$$
When $t = 0$, $v_Q = 11$, $c = 11$
$$v_Q = t^2 - 6t + 11$$
$$v_P = v_Q$$
$$t^2 - 3t + 5 = t^2 - 6t + 11$$
$$3t = 6 \Rightarrow t = 2$$

(b) Explain why particle Q will always move in the same direction after passing Z. [2]

$$v_Q = t^2 - 6t + 11$$
$$= t^2 - 6t + 3^2 - 3^2 + 11$$
$$= (t-3)^2 + 2$$

For all $t > 0$, $(t-3)^2 > 0$, $(t-3)^2 + 2 \geq 2$.
Since $(t-3)^2 + 2 > 0$, $v > 0$.
Q will always move in the same direction after passing Z.

Alternative method

$$v_Q = t^2 - 6t + 11$$
$$b^2 - 4ac = (-6)^2 - 4(1)(11)$$
$$= -8 < 0$$

OR
Since $b^2 - 4ac < 0$ and $a > 0$, v_Q is always

$t^2 - 6t + 11 = 0$ has no real values of t implying that Q will not come to instantaneous rest.
Hence Q will always move in the same direction after passing Z.

(c) Determine, with explanation, if there is any instance when particle Q is ahead of particle P. [4]

$s_Q = \int t^2 - 6t + 11 \, dt$

$= \frac{t^3}{3} - 3t^2 + 11t + d$, where d is an arbitrary constant

When $t = 0$, $s_Q = 0$, $d = 0$

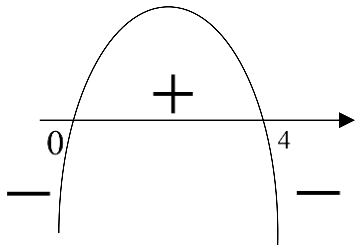
$s_Q = \frac{t^3}{3} - 3t^2 + 11t$

$s_Q - s_P = \frac{t^3}{3} - 3t^2 + 11t - \left(\frac{t^3}{3} - \frac{3t^2}{2} + 5t\right)$

$= -\frac{3t^2}{2} + 6t$

$= \frac{3}{2}t(4-t)$

When Q overtakes P, $s_Q - s_P > 0$



$\frac{3}{2}t(4-t) > 0$

$\frac{3}{2}t(4-t) > 0$

$0 < t < 4$

Particle Q is ahead of particle P when $0 < t < 4$.

(d) Find the average velocity of particle Q in the first 5 seconds. [2]

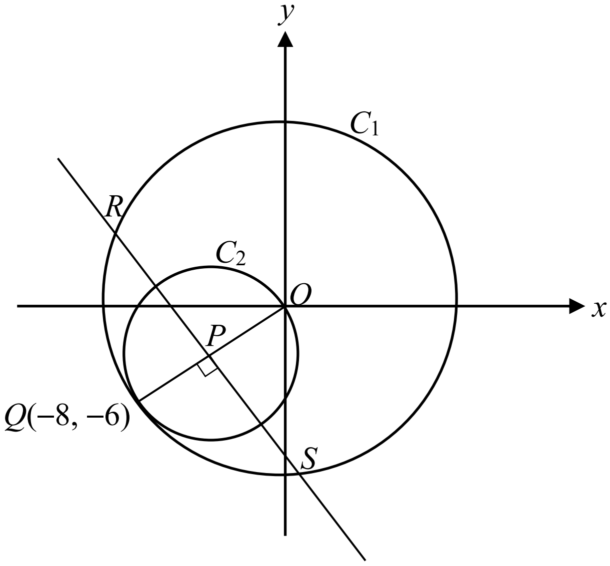
When $t = 5$, $s_Q = \frac{5^3}{3} - 3(5)^2 + 11(5)$

$= 21\frac{2}{3}$ m

Average velocity of particle Q in the first 5 seconds = $21\frac{2}{3} \div 5$

$= 4\frac{1}{3}$ m/s

10 The diagram shows two circles C_1 and C_2.



C_1 has its centre at the origin O while C_2 passes through O and has its centre at P. The point $Q(-8, -6)$ lies on both circles and OQ is the diameter of C_2.

(i) Find the equations of C_1 and C_2. [5]

Radius of C_1: $OQ = \sqrt{(-8)^2 + (-6)^2}$

$= 10$ units

Equation of C_1 is $x^2 + y^2 = 100$.

Coordinates of $P = \left(\dfrac{0+(-8)}{2}, \dfrac{0+(-6)}{2} \right) = (-4, -3)$

Radius of C_2: $OP = \dfrac{1}{2}(10)$

$= 5$ units

Equation of C_2 is $(x+4)^2 + (y+3)^2 = 25$.

The line through P perpendicular to OQ meets the circle C_1 at the points R and S.

(ii) Show that the x-coordinates of R and S are $a - b\sqrt{3}$ and $a + b\sqrt{3}$ respectively, where a and b are integers to be determined. [7]

Gradient of $OQ = \frac{-6}{-8} = \frac{3}{4}$

$\therefore$ gradient of $RS = -\frac{4}{3}$

Equation of RS is $y + 3 = -\frac{4}{3}(x + 4)$

$$y = -\frac{4}{3}x - \frac{25}{3}$$

$y = -\frac{4}{3}x - \frac{25}{3}$ ------ (1)

$x^2 + y^2 = 10$ ------ (2)

Sub (1) into (2):

$$x^2 + \left(-\frac{4}{3}x - \frac{25}{3}\right)^2 = 100$$

$$x^2 + \frac{16}{9}x^2 + \frac{200}{9}x + \frac{625}{9} - 100 = 0$$

$$\frac{25}{9}x^2 + \frac{200}{9}x - \frac{275}{9} = 0$$

$$x^2 + 8x - 11 = 0$$

$$x = \frac{-8 \pm \sqrt{64 - 4(-11)}}{2}$$

$$= \frac{-8 \pm 6\sqrt{3}}{2}$$

$$= -4 \pm 3\sqrt{3}$$

x-coordinates of R and S are $-4 - 3\sqrt{3}$ and $-4 + 3\sqrt{3}$ respectively where $a = -4$ and $b = 3$.

11 (a) The population P, in millions of a city, recorded in the month of January for various years is modelled by the equation $P = 10 + at^n$, where t is the time measured in years from January 2002 and a and n are constants.

The values are tabulated below.

Year	2005	2012	2017	2022
P	20.4	73.2	126.2	188.9

(i) On the grid opposite, plot $\lg(P-10)$ against $\lg t$ for the given data and draw a straight-line graph to estimate the values of a and n, giving your answers to one decimal place. [6]

$\lg(P-10) = \lg at^n$

$\lg(P-10) = n\lg t + \lg a$

Year	2005	2012	2017	2022
t	3	10	15	20
$\lg t$	0.47	1.00	1.18	1.30
$\lg(P-10)$	1.02	1.80	2.07	2.25

Correct table of values
All points plotted correctly
Best-fit straight line

Gradient, $n = \dfrac{2.10-1.20}{1.20-0.60}$

= 1.5 (to 1 d.p.)

$\lg a = 0.3$

$a = 10^{0.3}$

= 1.9953

= 2.0 (to 1 d.p.)

(ii) Use your graph to determine the year in which the population reached 100 millions. [2]

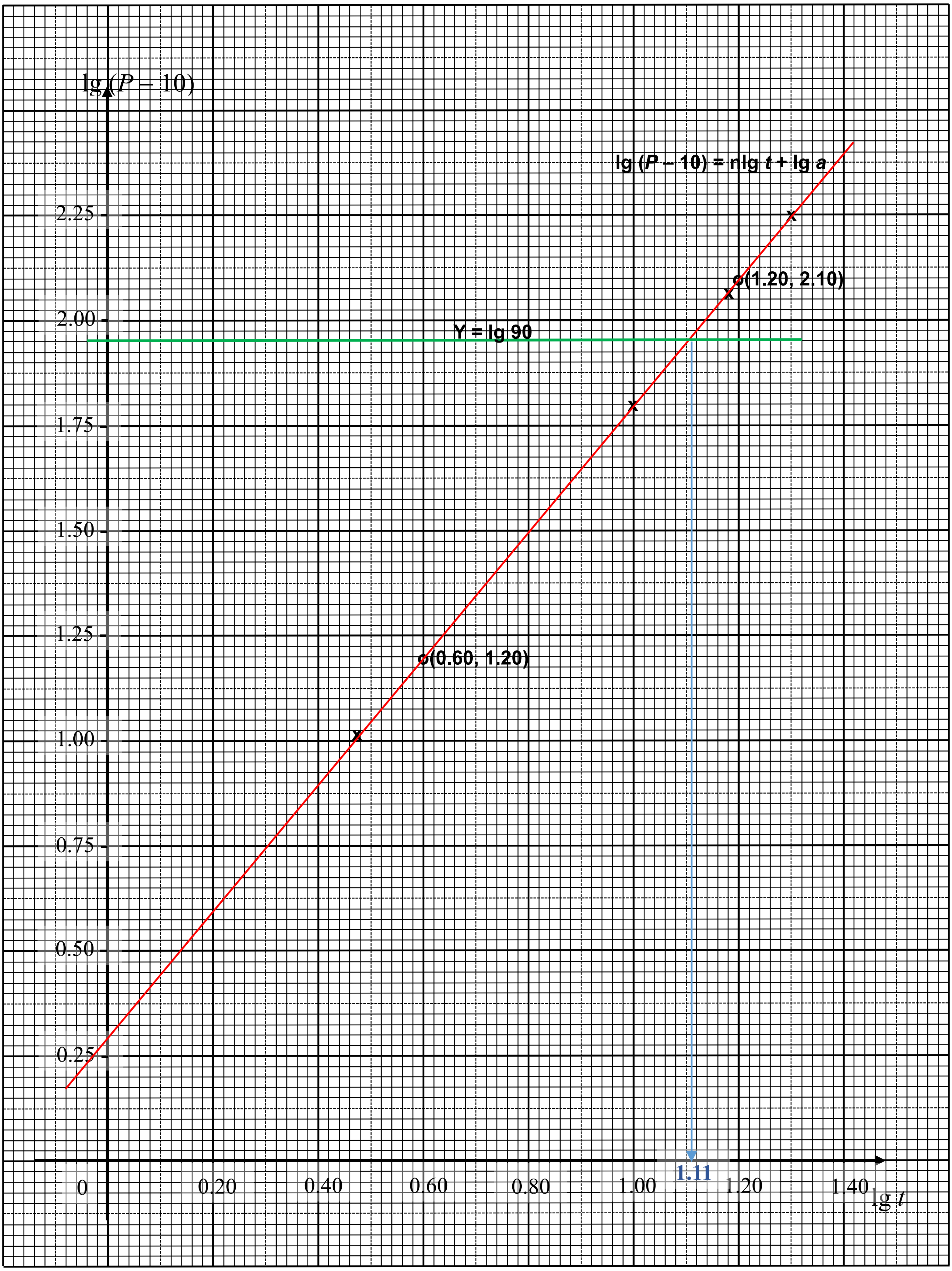
$P = 100$

Draw $Y = \lg 90$ (i.e. Y = 1.95)

$\lg t = 1.11$

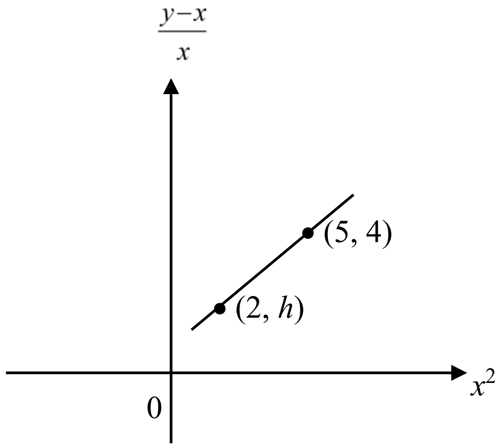
$t = 10^{1.11} = 12.9$

The year which the population reaches 100 millions is 2014.



lg (P – 10)
lg (P – 10) = nlg t + lg a
(1.20, 2.10)
Y = lg 90
(0.60, 1.20)
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0
0.20
0.40
0.60
0.80
1.00
1.11
1.20
1.40
lg t

(b) The diagram shows part of a straight-line graph, passing through the points $(2, h)$ and (5, 4), and representing the equations $2x^3 + kx = 3y$, where k and h are constants.
Find the value of h and of k. [4]



Gradient = $\dfrac{4-h}{5-2} = \dfrac{4-h}{3}$

Equation of the line is $Y - 4 = \dfrac{4-h}{3}(X - 5)$

$$\frac{y-x}{x} - 4 = \frac{4-h}{3}(x^2 - 5)$$

$$3(y-x) - 12x = (4-h)(x^3 - 5x)$$

$$3y - 3x - 12x = 4x^3 - 20x - hx^3 + 5hx$$

$$3y = (4-h)x^3 + (5h-5)x$$

Compare with $3y = 2x^3 + kx$

$4 - h = 2 \Rightarrow h = 2$

$5h - 5 = k \Rightarrow k = 5(2) - 5 = 5$

End of Paper