ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Promo Paper (100 marks) **27 Sept 2023**

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
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8	
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10	
11	
12	
Total	

1 The sum of the first n terms, S_n, of a sequence $\{u_n\}$ is given by $S_n = -2n^2 + n$.
Prove that the sequence $\{u_n\}$ is an arithmetic progression. [3]

2 The curve C has equation $y = \frac{x^2+2}{x-1}$, where $x \neq 1$.

(i) Draw a sketch of the curve C, clearly indicate the equation(s) of its asymptote(s) and the coordinates of any intersection with the axes. [3]

(ii) By drawing an additional graph on the diagram drawn in (i), state the number of real root(s) of the equation $x^2+2=(x-1)(x^2+1)$. [2]

3 A curve C has equation $4yx^2 + xy^2 = 5$.

(i) Find the equation of the normal at point A with coordinates $(1,1)$. [3]

(ii) The normal at point A meets the x and y axes at the points Q and R respectively. Find the exact area of OQR. [2]

4 (a) Find $\int \cot^2 2x \, dx$. [2]

(b) Write down the constants A and B such that, for all values of x, $-2x-1 = A(-2x-2)+B$.

Hence, find $\int \frac{-2x-1}{\sqrt{4-2x-x^2}} \, dx$. [4]

5 (a) Find $\int \sin 3x\left(\sec^2 3x - \sin x\right) dx$. [3]

(b) If $k > 2$, find $\int_0^k \left|e^x - 2\right| dx$ in terms of k. Leave your answer in exact form. [3]

6 Referred to the origin O, the points P and Q are such that $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are not parallel and non-zero vectors. The point R lies on OQ such that $\overrightarrow{OR} = k\overrightarrow{OQ}$, where k is a constant. S lies on line segment PR such that $PS : SR = 2 : 3$ and T is on PQ such that $PT : TQ = 1 : 3$.

(i) Find $\overrightarrow{OS}$ in terms of $\mathbf{p}$, $\mathbf{q}$ and k. [1]

(ii) Given that O, S and T are collinear, find k. [3]

(iii) If $OQ = 5$, show that the shortest distance from T to OQ can be expressed $\beta|\mathbf{p}\times\mathbf{q}|$, where β is a constant to be found. [3]

7 The functions f and g are defined by

$$f : x \mapsto \begin{cases} -x-2, & -5 \le x < 0, \\ (x-1)^2, & 0 \le x \le 5, \end{cases}$$

$$g : x \mapsto |x| - 1, \quad -5 \le x \le 5.$$

(i) Find the value of $f(0)$. [1]

(ii) Sketch the graph of f, indicating all axial intercepts and end-points of the graph. Hence explain why f^{-1} does not exist. [4]

(iii) Explain why fg exists. Hence find the exact range of fg. [2]

(iv) If the domain of f is restricted to $[-5,0)$, find the set of values of x that satisfy the equation $f^{-1}(x) = f(x)$. [1]

8 (a) (i) Find $\frac{d}{dx}\left(\frac{x^2}{1-x^2}\right)$. [1]

(ii) Hence find $\int \frac{2x \ln x}{(1-x^2)^2} dx$. [3]

(b) Using the substitution $x = \sin\theta$, where $0 < \theta < \frac{\pi}{2}$, show that

$$8\int x^2\sqrt{1-x^2}\, dx = \sin^{-1} x - p(x)\sqrt{1-x^2} + C,$$

where $p(x)$ is a cubic polynomial to be determined. [5]

9 (i) State a sequence of three transformations which transform the graph with equation $x^2 + y^2 = 1$ to the graph with equation $\frac{(x+1)^2}{4} + \frac{y^2}{9} = 1$. [3]

(ii) Sketch, on the same diagram, the graphs of $y = |2x-1|$ and $9(x+1)^2 + 4y^2 = 36$ for $y \ge 0$. State clearly the axial intercepts of both the graphs and the coordinates of the intersections between the two graphs. [5]

(iii) Hence solve the inequality $\sqrt{36 - 9(x+1)^2} \le 2|2x-1|$. [2]

(iv) Using your answer in (iii), find the exact range of values of x for which

$$\sqrt{36 - 9\left(\frac{1}{x}+1\right)^2} \le 2\left|\frac{2}{x} - 1\right|.$$ [2]

10 Customers who open a savings account with SRF bank will earn an interest rate of 0.2% per month based on the amount in the account on the last day of each month. The interest is added to the account at the end of each month.

(a) Peter has a savings account with the SRF bank. He puts \$2000 into his savings account at the beginning of each month starting from 1st January 2023. Find the amount he has in his savings account on 31st December 2027 after the interest has been added if no withdrawal is made. Show your workings clearly. [4]

(b) Sandy, another customer of SRF bank, has retired from work starting 1st January 2023. Since then, she will no longer do any deposit of money into her own savings account with the bank. Instead, she will be using the money saved in that account for her monthly spending. She has \$750 000 in her SRF bank savings account on 1st January 2023.

(i) As she still has some money with her for her daily spending in January 2023, she will only commence her monthly withdrawal of \$$k$ from her savings account in the middle of each month, starting from February 2023 onwards. Show that the amount of money in her savings account, at the end of n complete months from 1st January 2023, after the interest has been added, is given by $1.002^n(\$750000)-\$501k\left(1.002^{n-1}\right)+\$501k$. [4]

(ii) Taking k to be 3000, find the amount of money left in her savings account immediately after her last complete withdrawal of \$3000. [4]

11 With respect to the origin O, the position vectors of the points A and B are $\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\mathbf{i}-4\mathbf{k}$ respectively. The equation of the planes p_1 and p_2 are given by

$$p_1:\ \mathbf{r}\cdot\begin{pmatrix}1\\3\\0\end{pmatrix}=1 \text{ and } p_2:\ \mathbf{r}\cdot\begin{pmatrix}0\\7\\\alpha\end{pmatrix}=-4\alpha, \text{ where } \alpha \text{ is a real constant with } \alpha>0.$$

(i) Find the position vector of the foot of perpendicular from A to p_1. [3]

(ii) Verify that B lies on both p_1 and p_2. Show that the vector equation of the line of intersection, L, of p_1 and p_2 is of the form

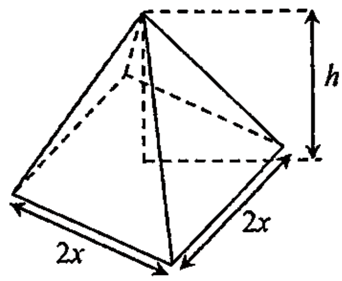
$$\mathbf{r}=\begin{pmatrix}1\\0\\-4\end{pmatrix}+s\begin{pmatrix}3\alpha\\-\alpha\\7\end{pmatrix},\ s\in\ .$$ [2]

(iii)(a) The point $C(-1,1,-4)$ is equidistant from both p_1 and p_2. Calculate the value of α for this case. Hence find in scalar product form the equation of the plane p_3 in which both the line L and the point C lie. [5]

(b) The plane p_4 has equation $y-z=2$.

Given that the angle between L and p_4 is $\frac{\pi}{6}$, find the value of α. [3]

12 In an amusement park, the construction of a model Egyptian pyramid is done in 2 stages. In the first stage, a mould that consists of only 4 identical isosceles triangles as its slanted sides, is created using wooden boards with negligible thickness which are joined together as shown in the diagram below.



- The pyramid has an open square base with sides $2x$ m.
- The height of the pyramid is h m.

It is given that the volume of the model has a fixed value of 1 m^3.

(i) If A is the external surface area of the 4 slanted sides, show that

$$A=\frac{\sqrt{9+16x^6}}{x}.$$ [3]

[The volume of a square-based pyramid is $\frac{1}{3}\times$ base area $\times$ height]

(ii) Using differentiation, find the exact value of x that gives the minimum A. [4]

It is now decided that $x = 1$. In the second stage, the mould is placed on the ground, and at time $t = 0$ seconds, cement is poured into the mould through a hole at the tip of the pyramid at a constant rate of $0.005\ m^3/s$.

(iii) If the volume and the height of cement in the mould at time t seconds are given by $V\ m^3$ and w m respectively, show that the volume of the cement in the mould is given by $V=1-\frac{64}{27}\left(\frac{3}{4}-w\right)^3$. [2]

(iv) Find the rate of change of the height of cement 1 minute after the cement has been poured. [5]

1 **Solution**

$u_n = S_n - S_{n-1}$

$= -2n^2 + n - \left[-2(n-1)^2 + (n-1)\right]$

$= 2(-2n+1)+1$

$= -4n+3$

$u_n - u_{n-1}$

$= -4n+3-\left[-4(n-1)+3\right]$

$= -4$ (constant)

Therefore, $\{u_n\}$ is an arithmetic progression.

2 **Solution**

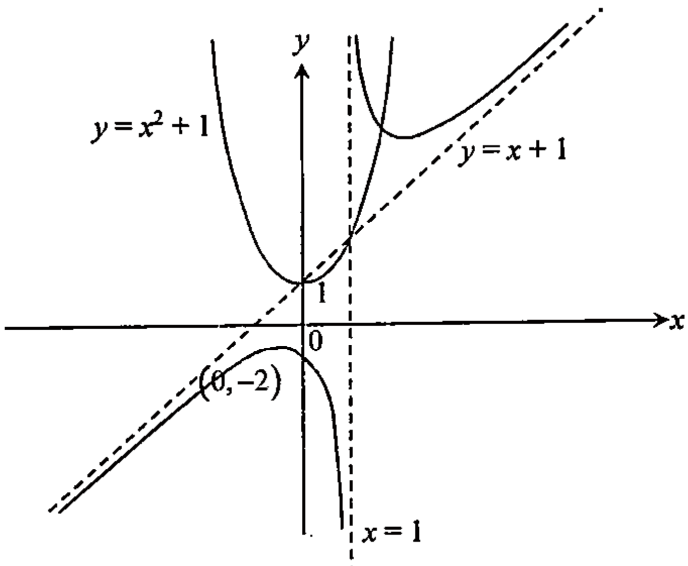
(i) When $x = 0, y = -2$

$y = \dfrac{x^2+2}{x-1} = x+1+\dfrac{3}{x-1}$

$x = 1$ and

$y = x + 1$, are equations of the asymptotes

(ii)



$x^2 + 2 = (x-1)(x^2+1)$

$\Rightarrow \dfrac{x^2+2}{x-1} = x^2+1$

By adding an additional graph in (i), i.e. $y = x^2 + 1$, no. of real root is 1.

3 **Solution**

(i)

$4yx^2 + xy^2 = 5$

$4x^2 \frac{dy}{dx} + 8xy + y^2 + 2xy\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(4x^2 + 2xy\right) = -\left(y^2 + 8xy\right)$

$\frac{dy}{dx} = \frac{-\left(y^2 + 8xy\right)}{4x^2 + 2xy}$

Gradient of tangent to the curve at $(1,1) = -\frac{3}{2}$

So gradient of normal to the curve at $(1,1)$ $= \frac{2}{3}$

Equation of the normal at $(1,1)$ is $y - 1 = \frac{2}{3}(x-1)$

$\Rightarrow y = \frac{2}{3}x + \frac{1}{3}$

(ii) When $x = 0$, $\Rightarrow y = \frac{1}{3}$. So $R\left(0, \frac{1}{3}\right)$

When $y = 0$, $\Rightarrow x = -\frac{1}{2}$. So $Q\left(-\frac{1}{2}, 0\right)$

Area of $OQR = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{12}$ units^2

4 **Solution**

(a)

$\int \cot^2 2x \, dx = \int \operatorname{cosec}^2 2x - 1 \, dx$

$= -\frac{1}{2}\cot 2x - x + C$

(b) $-2x - 1 = 1(-2x - 2) + 1$

$\int \frac{-2x-1}{\sqrt{4-2x-x^2}} dx = \int \frac{-2x-2}{\sqrt{4-2x-x^2}} dx + \int \frac{1}{\sqrt{4-2x-x^2}} dx$

$= 2\sqrt{4-2x-x^2} + \int \frac{1}{\sqrt{5-(x+1)^2}} dx$

$= 2\sqrt{4-2x-x^2} + \sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$

5 **Solution**

(a) $\int \sin 3x\left(\sec^2 3x-\sin x\right)\,dx$

$=\int\left[\sec(3x)\tan(3x)-\sin(3x)\sin x\right]\,dx$

Or

$=\int\left\{\sin(3x)\left[\cos(3x)\right]^{-2}-\sin(3x)\sin x\right\}\,dx$

$=\frac{1}{3}\sec 3x+\frac{1}{2}\int \cos 4x-\cos 2x\ dx$

$=\frac{1}{3}\sec 3x+\frac{1}{2}\left(\frac{\sin 4x}{4}-\frac{\sin 2x}{2}\right)+C$

(b) $\int_0^k\left|e^x-2\right|\,dx=\int_0^{\ln 2}2-e^x\ dx+\int_{\ln 2}^k e^x-2\ dx$

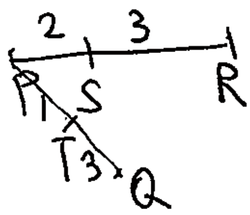
$=\left[2x-e^x\right]_0^{\ln 2}+\left[e^x-2x\right]_{\ln 2}^k$

$=2\ln 2-2+1+e^k-2k-\left(2-2\ln 2\right)$

$=4\ln 2-3+e^k-2k$

6 **Solution**

(i) By ratio theorem,



$\overrightarrow{OS}=\frac{3\mathbf{p}+2k\mathbf{q}}{5}$

(ii) $\overrightarrow{OT}=\frac{3\mathbf{p}+\mathbf{q}}{4}$

$\overrightarrow{OS}=\alpha\overrightarrow{OT}$, where α is real.

$\frac{3\mathbf{p}+2k\mathbf{q}}{5}=\alpha\left(\frac{3\mathbf{p}+\mathbf{q}}{4}\right)$

Comparing coefficients of $\mathbf{p}$ and $\mathbf{q}$:

$\frac{3}{5}=\frac{3}{4}\alpha\Rightarrow\alpha=\frac{4}{5}$

$\frac{2}{5}k=\frac{1}{4}\alpha\Rightarrow k=\frac{1}{2}$

(iii) Shortest distance from the point T to OQ

$=\left|\overrightarrow{OT}\times\frac{\overrightarrow{OQ}}{\left|\overrightarrow{OQ}\right|}\right|$

$=\left|\left(\frac{3\mathbf{p}+\mathbf{q}}{4}\right)\times\frac{\mathbf{q}}{5}\right|$

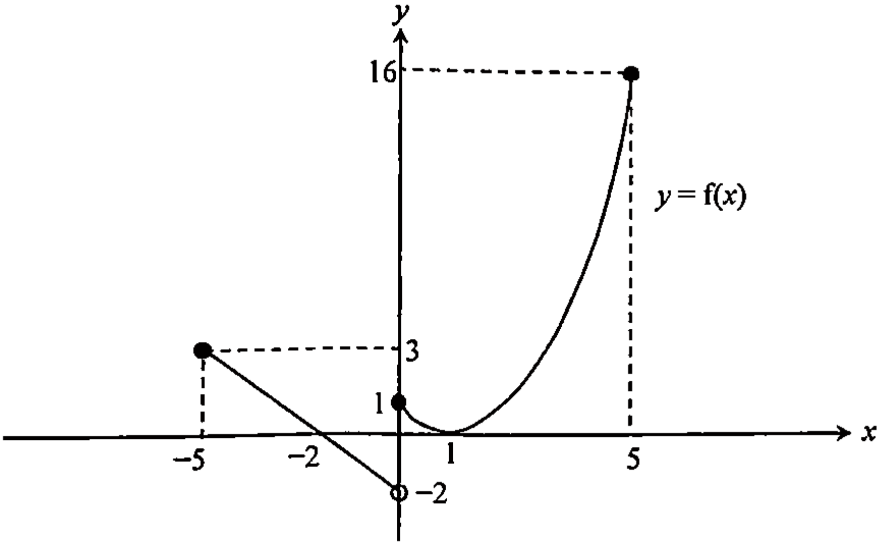
$=\frac{1}{20}\left|(3\mathbf{p}\times\mathbf{q})+(\mathbf{q}\times\mathbf{q})\right|=\frac{3}{20}\left|\mathbf{p}\times\mathbf{q}\right|\quad(\because\mathbf{q}\times\mathbf{q}=0)$

$\beta=\frac{3}{20}$

7 Solution

(i) $f(0)=(0-1)^2=1$

(ii)



Since the line $y = 0.5$ cuts the curve more than once, f is not a one-one function and so f^{-1} does not exist.

(iii)

$R_g=[-1,4]$

$D_f=[-5,5]$

Since $R_g \subseteq D_f$ so fg(x) exist.

Using graph of f and $R_g=[-1,4]$,

$[-5,5]\xrightarrow[g]{}[-1,4]\xrightarrow[f]{}(-2,-1]\cup[0,9]$

$R_{fg}=(-2,-1]\cup[0,9]$

(iv) $y=f^{-1}(x)$ is a reflection of $y=f(x)$ about the line $y=x$ and solving $f^{-1}(x)=f(x)$ is to find the number of intersections between the two curves. From the sketch above, for the domain $[-5,0)$, the two curves will intersect in the set $\{x\in\mathbb{R},-2<x<0\}$.

8 Solution

$$\frac{d}{dx}\left(\frac{x^2}{1-x^2}\right)=\frac{(1-x^2)(2x)-x^2(-2x)}{(1-x^2)^2}$$

$$=\frac{2x}{(1-x^2)^2}$$

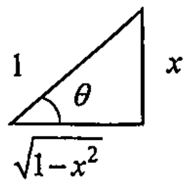
(ii)

$$\int\frac{2x\ln x}{(1-x^2)^2}\,dx=\frac{x^2\ln x}{1-x^2}-\int\frac{x^2}{1-x^2}\cdot\frac{1}{x}\,dx$$

$=\dfrac{x^2\ln x}{1-x^2}-\displaystyle\int\frac{x}{1-x^2}\,dx$

$=\dfrac{x^2\ln x}{1-x^2}+\dfrac{1}{2}\ln\left|1-x^2\right|+C$

(b) $x=\sin\theta \Rightarrow \dfrac{dx}{d\theta}=\cos\theta$



$8\displaystyle\int x^2\sqrt{1-x^2}\,dx=8\int\sin^2\theta\sqrt{1-\sin^2\theta}\,(\cos\theta)\,d\theta$

$=2\displaystyle\int(2\sin\theta\cos\theta)^2\,d\theta$

$=2\displaystyle\int(\sin 2\theta)^2\,d\theta$

$=\displaystyle\int(1-\cos 4\theta)\,d\theta$

$=\theta-\dfrac{1}{4}\sin 4\theta+C$

$=\theta-\dfrac{1}{4}\cdot 2\sin 2\theta\cos 2\theta+C$

$=\theta-\sin\theta\cos\theta\left(1-2\sin^2\theta\right)+C$

$=\sin^{-1}x-x\sqrt{1-x^2}\left(1-2x^2\right)+C$

Where $p(x)=x\left(1-2x^2\right)$

9 **Solution**

(i) Method 1:

$x^2+y^2=1$

$\xrightarrow{(1)} x^2+\left(\dfrac{y}{3}\right)^2=1$

$\xrightarrow{(2)} \left(\dfrac{x}{2}\right)^2+\left(\dfrac{y}{3}\right)^2=1$

$\xrightarrow{(3)} \left(\dfrac{x+1}{2}\right)^2+\left(\dfrac{y}{3}\right)^2=1$

(1) Scaling parallel to the y-axis by a factor of 3 –
(2) Scaling parallel to the x-axis by a factor of 2 –
(3) Translation of 1 unit in negative x–direction –

Method 2:

$x^2 + y^2 = 1$

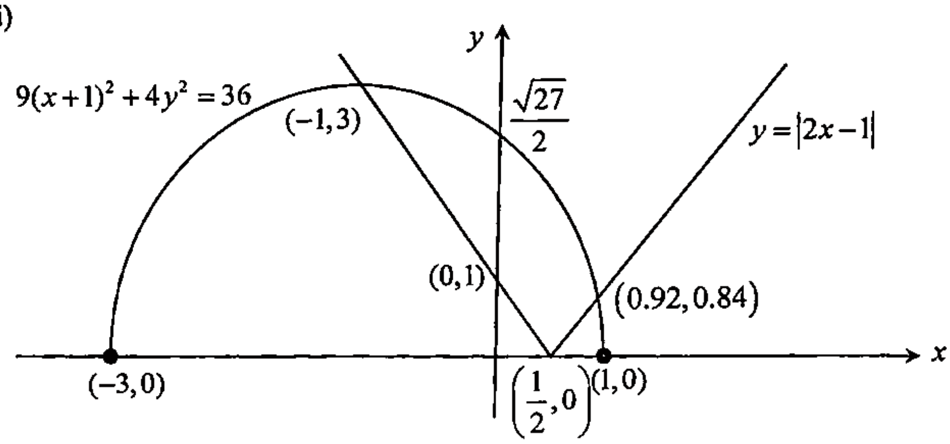
$\xrightarrow{(1)} x^2 + \left(\frac{y}{3}\right)^2 = 1$

$\xrightarrow{(2)} \left(x + \frac{1}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\xrightarrow{(3)} \left(\frac{x+1}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

(1) Scaling parallel to the y-axis by a factor of 3 –
(2) Translation of 0.5 unit in the negative x–direction –
(3) Scaling parallel to the x-axis by a factor of 2 –

(ii)



(iii) From the sketch,

$\therefore -3 \le x \le -1 \quad \text{or} \quad 0.92 \le x \le 1$

(iv) Replace x with $\frac{1}{x}$

$\therefore -3 \le \frac{1}{x} \le -1 \quad \text{or} \quad 0.92 \le \frac{1}{x} \le 1$

$1 \le -\frac{1}{x} \le 3 \quad \text{or} \quad 1 \le x \le \frac{25}{23}$

$\frac{1}{3} \le -x \le 1 \quad \text{or} \quad 1 \le x \le \frac{25}{23}$

$-1 \le x \le -\frac{1}{3} \quad \text{or} \quad 1 \le x \le \frac{25}{23}$

10 Solution

(i)

Mth	End
1	$1.002(\$2000)$
2	$1.002^2(\$2000)+1.002(\$2000)$
3	$1.002^3(\$2000)+1.002^2(\$2000)+1.002(\$2000)$
n	$1.002^n(\$2000)+1.002^{n-1}(\$2000)+...+1.002^2(\$2000)+1.002(\$2000)$

Amount in the saving account on 31st Dec 2027 =

$=1.002^{60}(\$2000)+1.002^{59}(\$2000)+...+1.002^2(\$2000)+1.002(\$2000)$

$=\dfrac{1.002(2000)\left(1.002^{60}-1\right)}{1.002-1}$

$=\$127616.46$

(b)(i)

Mth	Beginning	End
1	$750000	$(\$750000)1.002$
2	$(\$750000)1.002$	$(\$750000)1.002^2-1.002(\$k)$
3	$(\$750000)1.002^2-1.002(\$k)$	$1.002^3(\$750000)-1.002^2(\$k)$ $-1.002(\$k)$
4	$1.002^3(\$750000)-1.002^2(\$k)$ $-1.002(\$k)$	$1.002^4(\$750000)-1.002^3(\$k)$ $-1.002^2(\$k)-1.002(\$k)$
	...	
n		$1.002^n(\$750000)-1.002^{n-1}(\$k)$ $-...-1.002^2(\$k)-1.002(\$k)$

Amount at the end of n months

$=1.002^n(\$750000)-1.002^{n-1}(\$k)-...-1.002^2(\$k)-1.002(\$k)$

$=1.002^n(\$750000)-\dfrac{1.002(\$k)\left(1.002^{n-1}-1\right)}{1.002-1}$

$=1.002^n(\$750000)-\$501k\left(1.002^{n-1}\right)+\$501k$

(ii) $1.002^n(\$750000)-\$501(3000)\left(1.002^{n-1}\right)+\$501(3000)\geq 0$

n	$Y_1=1.002^n(\$750000)-\$1503000\left(1.002^{n-1}\right)+\$1503000$
346	5754.856
347	2760.366
348	−240.113

$n\leq 347.92$ (from graph)

Max number of months for complete withdrawal of $3000 is 347 months (28 years 11 months)

Amount remaining in the saving account immediately after the last withdrawal of $\$3000 = \$5754.86 - \$3000 = \$2754.86$

11 Solution

(i) Let F be the foot of perpendicular from A to the plane p_1.

$$\ell_{AF}: \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$$

Since F lies on line AF,

$$\overrightarrow{OF} = \begin{pmatrix} 1+\lambda \\ 2+3\lambda \\ 1 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}$$

As F is also on the plane p_1,

$$\begin{pmatrix} 1+\lambda \\ 2+3\lambda \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 1$$

$$\lambda + 1 + 6 + 9\lambda = 1$$

$$\lambda = -\frac{3}{5}$$

$$\overrightarrow{OF} = \begin{pmatrix} 1 - \frac{3}{5} \\ 2 + 3\left(-\frac{3}{5}\right) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \\ 1 \end{pmatrix}$$

(ii) $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 1(1) + 0(3) - 4(0) = 1$

And $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 7 \\ \alpha \end{pmatrix} = 1(0) + 0(7) - 4(\alpha) = -4\alpha$

Hence $B\,(1, 0, -4)$ lies on both p_1 and p_2.

$$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 7 \\ \alpha \end{pmatrix} = \begin{pmatrix} 3\alpha \\ -\alpha \\ 7 \end{pmatrix}$$

$\therefore$ Eqn of L is $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + s \begin{pmatrix} 3\alpha \\ -\alpha \\ 7 \end{pmatrix}, s \in \mathbb{R}$ (Shown)

(iii) (a)

B lies on both planes $\Rightarrow$ can use $\overrightarrow{BC}$ (or any $\overrightarrow{XC}$, where X is a random point on plane) to project onto each normal to find perpendicular distance from C to each plane.

$$\overrightarrow{BC}=\begin{pmatrix}-1\\1\\-4\end{pmatrix}-\begin{pmatrix}1\\0\\-4\end{pmatrix}=\begin{pmatrix}-2\\1\\0\end{pmatrix}$$

$$\frac{\left|\overrightarrow{BC}\cdot \underset{\sim}{n}_1\right|}{\left|\underset{\sim}{n}_1\right|}=\frac{\left|\overrightarrow{BC}\cdot \underset{\sim}{n}_2\right|}{\left|\underset{\sim}{n}_2\right|}$$

$$\frac{\left|\begin{pmatrix}-2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\3\\0\end{pmatrix}\right|}{\sqrt{10}}=\frac{\left|\begin{pmatrix}-2\\1\\0\end{pmatrix}\square\begin{pmatrix}0\\7\\\alpha\end{pmatrix}\right|}{\sqrt{49+\alpha^2}}$$

$$\frac{|-2+3+0|}{\sqrt{10}}=\frac{|0+7+0|}{\sqrt{49+\alpha^2}}$$

$$\sqrt{49+\alpha^2}\times 1=\sqrt{10}\times 7$$

$$49+\alpha^2=490$$

$$\alpha^2=441\Rightarrow\ \alpha=\pm 21=21\ (\text{Since }\alpha>0)$$

Since L and C lie on p_3, $\overrightarrow{BC}$ and direction of L are parallel to p_3.

$$\begin{pmatrix}-2\\1\\0\end{pmatrix}\times\begin{pmatrix}3\alpha\\-\alpha\\7\end{pmatrix}=\begin{pmatrix}-2\\1\\0\end{pmatrix}\times\begin{pmatrix}3(21)\\-21\\7\end{pmatrix}=\begin{pmatrix}7\\14\\-21\end{pmatrix}=7\begin{pmatrix}1\\2\\-3\end{pmatrix}$$

Equation of p_3:

$$\mathbf{r}\cdot\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}-1\\1\\-4\end{pmatrix}\square\begin{pmatrix}1\\2\\-3\end{pmatrix}$$

$$\mathbf{r}\cdot\begin{pmatrix}1\\2\\-3\end{pmatrix}=13$$

(iii)(b) $p_4:\ \mathbf{r}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=2$

$$\sin\frac{\pi}{6}=\frac{|\mathbf{n}.\mathbf{d}_L|}{|\mathbf{n}||\mathbf{d}_L|}=\frac{\left|\begin{pmatrix}0\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}3\alpha\\-\alpha\\7\end{pmatrix}\right|}{\sqrt{2}\sqrt{9\alpha^2+\alpha^2+49}}$$

$$\frac{1}{2}=\frac{|-\alpha-7|}{\sqrt{2}\sqrt{9\alpha^2+\alpha^2+49}}$$

$\sqrt{10\alpha^2+49}=\sqrt{2}\left|-\alpha-7\right|=\sqrt{2}(\alpha+7)\ (\because \alpha>0)$

$10\alpha^2+49=2\alpha^2+28\alpha+98$

$8\alpha^2-28\alpha-49=0$

$\Rightarrow \alpha=\dfrac{28\pm\sqrt{784-4(8)(-49)}}{16}$

$\alpha=\dfrac{7}{4}+\sqrt{\dfrac{147}{16}}$ or 4.78 (since $\alpha>0$)

12 Solution

(i) $1=\dfrac{1}{3}(4x^2)(h)$

$h=\dfrac{3}{4x^2}$

Let A be the surface area of the mould.

$A=4\left(\dfrac{1}{2}\right)(2x)\sqrt{h^2+x^2}$

– For finding height of triangle $\sqrt{h^2+x^2}$

$=4x\sqrt{h^2+x^2}$

$=4x\sqrt{\dfrac{9}{16x^4}+x^2}$

$=\dfrac{1}{x}\sqrt{9+16x^6}$

(ii) $\dfrac{dA}{dx}=\dfrac{1}{x}\left(\dfrac{1}{2}\right)(16x^6+9)^{-\frac{1}{2}}(96x^5)-\dfrac{1}{x^2}\sqrt{16x^6+9}$

$\dfrac{dA}{dx}=\dfrac{48x^4}{\sqrt{9+16x^6}}-\dfrac{\sqrt{9+16x^6}}{x^2}$

$\dfrac{dA}{dx}=\dfrac{32x^6-9}{x^2\sqrt{9+16x^6}}$

When $\dfrac{dA}{dx}=0$

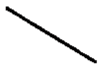
$32x^6-9=0$

Using G.C,

$x=\left(\dfrac{9}{32}\right)^{\frac{1}{6}}$

x	0.80	$\left(\dfrac{9}{32}\right)^{\frac{1}{6}}$	0.81
$\dfrac{dA}{dx}$	–0.263	0	0.0156
Sketch	\	—	/

OR

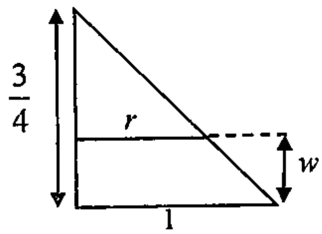
x	$\left(\frac{9}{32}\right)^{\frac{1}{6}}-0.01$	$\left(\frac{9}{32}\right)^{\frac{1}{6}}$	$\left(\frac{9}{32}\right)^{\frac{1}{6}}+0.01$
$\frac{dA}{dx}$	-0.279	0	0.275
Sketch			

OR $\left.\frac{d^2A}{dx^2}\right|_{x=\left(\frac{9}{32}\right)^{\frac{1}{6}}} = 27.7$

(iii) Height of pyramid $= \frac{3}{4}$ m

When $x = 1$, $h = \frac{3}{4}$

Let V and w be the volume and height of concrete in the pyramid respectively.



$$\frac{r}{1} = \frac{\frac{3}{4} - w}{\frac{3}{4}}$$

$$\therefore r = \frac{4}{3}\left(\frac{3}{4} - w\right)$$

$$V = 1 - \frac{1}{3}(2r)^2\left(\frac{3}{4} - w\right)$$

$$V = 1 - \frac{4}{3}\left(\frac{4}{3}\right)^2\left(\frac{3}{4} - w\right)^3$$

$$V = 1 - \frac{64}{27}\left(\frac{3}{4} - w\right)^3$$

OR

$$\frac{V_{air}}{1} = \left(\frac{\frac{3}{4} - w}{\frac{3}{4}}\right)^3$$

$$V_{air} = \frac{64}{27}\left(\frac{3}{4} - w\right)^3$$

$$V = V_{pyramid} - V_{air}$$

$$\therefore V = 1 - \frac{64}{27}\left(\frac{3}{4} - w\right)^3$$

(iv) $\frac{dV}{dw} = \frac{64}{9}\left(\frac{3}{4} - w\right)^2$

$$\therefore \frac{dw}{dt} = \frac{dw}{dV} \times \frac{dV}{dt} = \frac{9(0.005)}{64\left(\frac{3}{4} - w\right)^2}$$

1 min later, $V = 0.3$

$$\therefore 0.3 = 1 - \frac{64}{27}\left(\frac{3}{4} - w\right)^3$$

$w = 0.0840719987$

$w = 0.084072$

Hence $\dfrac{dw}{dt} = \dfrac{9(0.005)}{64\left(\frac{3}{4} - 0.084072\right)^2}$

$= 0.00159$ m/s (3 s.f)