A-Math (Revision) <Kaiwen :) Sec 4 Revision>

↳ Algebra

A1: Quadratic Functions

↳ Completing the sq.

$y = ax^2 + bx + c \Rightarrow y = a(x-h)^2 + k$.

TP: (h, k).

Eg: $y = x^2 - 6x + 12$

$y = (x-3)^2 - 9 + 12 = (x-3)^2 + 3$.

$\frac{\text{coeff of } x}{2}$

Expand

$x^2 - 6x + 9$ ← Extra

* coeff must be 1.

$y = 2x^2 + 6x - 4$

$y = 2(x^2 + 3x - 2)$

* coeff must be +ve

$y = -x^2 - 6x - 9$

$y = -(x^2 + 6x + 9)$

factorising.

↳ $y = ax^2 + bx + c$ always +ve / -ve.

↳ Conditions:

1. $b^2 - 4ac < 0$

No intersection

↳ No sol^{n}.

2. +ve : a +ve

-ve : a -ve

A2: Equations and Inequalities

↳ $b^2 - 4ac$ (discriminant). (D)

Quadratic

1. 2 real and distinct $D > 0$
2. 2 real and equal $D = 0$
3. No real roots $D < 0$

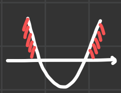
* 4. real. $D \geq 0$

Line / curve

1. 2 intersection pt $D > 0$
2. 1 intersection pt / tangential $D = 0$
3. No intersection $D < 0$
4. Intersection $D \geq 0$

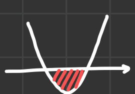
↳ Inequalities

$> / \geq$



Trick
Stand here!

$< / \leq$



Trick
Stand here!

* Special Qn

SHOW

↳ prove,

find the value of D.

* complete the sq.

A2.1.8.

↳ coeff of x^2 must be +ve.

$-x^2 + 2x + 3 < 0$

$-(x^2 - 2x - 3) < 0$

$x^2 - 2x - 3 > 0$.

⋮

A3: Surds.

↳ Rationalisation. Pg 41.

* Conjugation.

$\frac{3}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2}$

flip the sign.

* Trick: surds denominator, always use calculator.

* Qn type: $\sqrt{80}$.

$\sqrt{80} = \sqrt{2^4 \times 5}$

$= \sqrt{2^4} \times \sqrt{5}$

$= 4\sqrt{5}$

prime factors.

Pg 40.

$\sqrt{2} \times \sqrt{3}$.

$\sqrt{2} \times \sqrt{6}$

$= 2\sqrt{3}$.

Ex

$2\sqrt{5} = \sqrt{2^2 \times 5}$

$= \sqrt{20}$

A4: Polynomials

deg g(x) > deg r(x).

↳ Long Division.

$f(x) = g(x)\, q(x) + r(x)$

divisor, quotient, remainder

$$\begin{array}{r|l} & q(x) \\ \hline g(x) & f(x) \\ & \\ \hline & r(x) \\ \hline \end{array}$$

↳ Remainde Theorem

$f(x) = \underbrace{g(x)\, q(x)}_{\text{go to } 0} + r(x)$

" objective"

* Ex. $f(x) = \underbrace{(x-2)\, q(x)}_{\text{go to } 0} + r(x)$

$\therefore x = 2$.

A4: Polynomials (continued)

* Factor Theorem.

$f(x) = g(x)\,q(x)$

Find $q(x)$? → long div.

Factor: can divide without remainder.

* Ex: $f(x) = \underbrace{(2x-1)}_{\text{go to } 0.}q(x)$

$\therefore x = \frac{1}{2}$

↳ Cubic eqn.

* type calculator.

→ 1. 1 beautiful, 2 gross

2. 3 beautiful.

↳ Integer

↳ rational no.

* can tie to $b^2 = 4ac$.

decimals. ↓ quad formula.

* Smart "trial and error"

Pg 63.

⇒ case 1: 1 use in LD. to solve for quadratic to get the 2 gross.

↳ cubic identities

↳ $a^3+b^3 = (a+b)(a^2-ab+b^2)$ opp

↳ $a^3-b^3 = (a-b)(a^2+ab+b^2)$. no "2"

A5: Simultaneous Eqn.

↳ 1 linear, 1 non-linear.

* Substitution.

* Common Mistakes

1. Avoid fractions.

$x+2y=3$

- $x = 3-2y$ ✓

- $y = \frac{3-x}{2}$ × avoid.

2. $\left(\frac{x+2}{3}\right)^2 + 3y = 7.$ * identity

$\frac{[x^2+4x+4]}{[9]} + 3y = 7$

* rmb to "sq"

$x^2+4x+4 + 9(3y) = 9(7).$

* multiply throughout.

A6: Partial Fractions.

PF

- proper
- Improper ↓ LD

* $\frac{P(x)}{Q(x)} \therefore \frac{x^2+4}{x^2+9}.$

If the deg. same, improper!

$$g(x)\overline{\big)\,f(x)}\quad q(x)$$

$r(x)$

1. Linear factor
2. Repeated Linear
3. Non-factorisable

$$\frac{f(x)}{g(x)} = \frac{r(x)}{g(x)} + q(x).$$

1. $\frac{}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$

2. $\frac{}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$

3. $\frac{}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+D}{(x^2+c^2)}$

A7: Binomial Theorem.

↳ General (r+1) term.

* $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

* Independent term of x

↳ x^0

* $^nC_r = \frac{n!}{r!(n-r)!}$

↳ $\binom{n}{1} = n$

$\binom{n}{2} = \frac{n(n-1)}{2}$

$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$

* powers are n

$(1+3x)^n$.

A7.1.4 Pg 103.

pairing

* $(x^2+x+1)(a+bx+cx^2+dx^3+\ldots)$

Find coeff of x^3?

A7.1.3.(a)(ii) Pg 99.

* $(x^2+x+1)\left(a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}+\ldots\right)$

x-term?

A8: Exponential & Logarithm.

↳ Exponential $\quad \ln x = \log_e x$

e^x / a^x

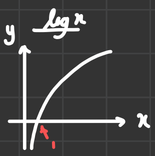
↳ Logarithms $\quad \lg x = \log_{10} x$

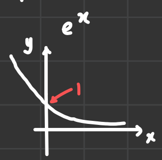
$\log_a b$.

* changing btw exp and log $\quad$ Pg 108 for arrows.

$$\log_a b = x \iff b = a^x$$

↳ sketching of graphs





* cannot cross to -ve y $\qquad$ * cannot cross to -ve x

↳ Laws of logarithm.

* $\log_a U + \log_a V = \log_a UV \neq \log_a(U+V)$

* $\log_a U - \log_a V = \log_a \frac{U}{V}. \neq \log_a(U-V)$

* change of base:

$$\log_U V = \frac{\log_a V}{\log_a U}$$

* Qn variant

↳ Powers!

$$\left.\begin{array}{l} e^{2x} + e^{x} = 1 \\ U^2 + U = 1 \end{array}\right\} \text{substitution!}$$

↳ G1. Trigonometry

Geometry

Ratios

TOA CAH SOH

$\tan\theta = \frac{\text{opposite}}{\text{Adjacent}}$

$\cos\theta = \frac{\text{Adjacent}}{\text{Hypothenuse}}$

$\sin\theta = \frac{\text{opposite}}{\text{Hypothenuse}}$

cotθ   cosecθ   secθ

$\cot\theta = \frac{1}{\tan\theta}$ $\text{cosec}\,\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$

✱ cscθ

ASTC

S	A
T	C

A: all ratios are positive

S: Sine is +ve.

T: Tangent is +ve

C: cosine is +ve

S	A
- sin > 0 - cos < 0 - tan < 0	- sin > 0 - cos > 0 - tan > 0
- sin < 0 - cos < 0 - tan > 0	- sin < 0 - cos > 0 - tan < 0
T	C

S	A
$\theta = 180° - \alpha$ (π)	$\theta = \alpha$
$\theta = 180° + \alpha$ (π)	$\theta = 360° - \alpha$ (2π)
T	C

Identity: $\cos\theta = \frac{\sqrt{3}}{2}$

↓

>0 or <0 ?

↓ ↓

A/C   S/T

Ex.

$\tan A = \frac{5}{12}$ $\cos B = -\frac{3}{5}$

A, B same quadrant

Quadrant: 3rd.

Quadrant

S	A
2nd	1st
3rd	4th
T	C

Acute: $\theta > 90$ 1st

obtuse: $0 \le \theta \le 180°$ 2nd

reflex: $180° \le \theta \le 360°$ 3rd/4th.

✱ Special angles

		sinθ	cosθ	tanθ
$\sqrt{\frac{0}{4}}$	0	0	1	0
$\sqrt{\frac{1}{4}}$	30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\sqrt{\frac{2}{4}}$	45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\sqrt{\frac{3}{4}}$	60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\sqrt{\frac{4}{4}}$	90	1	0	∞

Ex

$\sin 105°$ exact form

$= \sin(60° + 45°)$

Addition formulae.

Identities

Quotient: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Pythagorean:

$\sin^2 A + \cos^2 A = 1$ ✱

$\sec^2 A = 1 + \tan^2 A$ ✱

$\text{cosec}^2 A = 1 + \cot^2 A$ ✱

Addition:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

+ : −

− : +

Double Angle:

$\sin 2A = 2\sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

Half angle

Equations

Eqn for $0 \le \theta \le 360°$ or $0 \le \theta \le 2\pi$. range!

Step-by-step

$\rightarrow \cos\theta = \frac{\sqrt{3}}{2}$.

If the value is -ve, must take the +ve.

α (basic angle) $= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= 30°$

cannot solve straight for θ.

S	A
T	C

$\therefore \theta = \alpha$ or $\theta = 360° - \alpha$

$= 30°$ $= 330°$ ✗

check if in the range

R-formula

$a\sin A + b\cos A = R\cos(A \pm \alpha)$

✱ $a\sin A + b\cos A = R\sin(A \pm \alpha)$

$R = \sqrt{a^2 + b^2}$ $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

Proof

$a\sin A + b\cos A = R\sin(A + \alpha) = R\sin A\cos\alpha + R\cos A\sin\alpha$

Compare coefficient,

$R\cos\alpha = a$ —①

$R\sin\alpha = b$ —②

R: ①² + ②²,

$R^2\sin^2\alpha + R^2\cos^2\alpha = a^2 + b^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = a^2 + b^2$

$R = \sqrt{a^2 + b^2}$

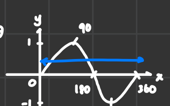
α: $\frac{②}{①}$, $\frac{\sin\alpha}{\cos\alpha} = \frac{b}{a}$

$\tan\alpha = \frac{b}{a}$

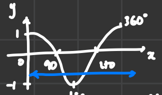
$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

Graphs

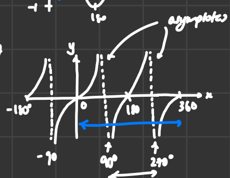
y = sinθ



y = cosθ



y = tanθ



- Amplitude
- Period
- Vertical translation

$y = a\sin bx + c$

amplitude   period   vertical translation

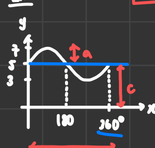
How long complete 1 cycle

>0 ↑

<0 ↓

$\boxed{b = \frac{360°}{\text{period}}}$

Ex



$y = a\sin bx + c$

a = 2.

b = 1

c = 5

G2: Coordinate Geometry

↳ Eqn: 1 grad. 1 pt.

↳ gradient: $\frac{y_2-y_1}{x_2-x_1}$

↳ Midpt $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ *

↳ Distance: $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

↳ Area $= \frac{1}{2}\left|\begin{matrix} x_1 & x_2 & \dots & x_1 \\ y_1 & y_2 & \dots & y_1 \end{matrix}\right|$ *

↘ + ↗ − repeat!

| |: modulus

⇒ change −ve → +ve.

* ⊥ lines *

product of gradient = −1.

* Alternate form for line *

$y-y_1=m(x-x_1)$.

↳ ⊥ bisector:

- 1 grad: ⊥ of the line. (−1).
- 1 pt: Midpt of the line.

↳ ⊥ bisector cuts line in ½

G3: Further Coordinate Geometry

* Standard

Eqn: $(x-a)^2+(y-b)^2=r^2$

Centre: (a,b)

radius: r

* General

Eqn: $x^2+y^2+2gx+2fy+c=0$

centre: $(-g,-f)$

radius: $\sqrt{g^2+f^2-c}$

* reflection.

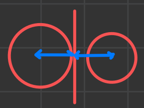
↳ radius do not change

↳ x-axis: x-coordinate do not change

y-coordinate −y.

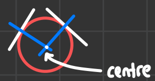
y-axis: y-coordinate do not change

x-coordinate −x.

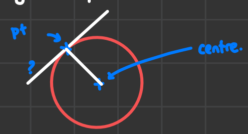


↳ Normals (calculus)

↳ normal will cut the centre.



* Tangent of a pt. on the circle



- Find grad of normal
  ↳ pt and centre.
- ⊥ (−1).
- Eqn: grad ✓, pt. (x).

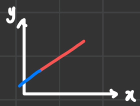
G4: Linear Law

* $y=ab^x$

$\lg y = x\lg b + \lg a$

$Y = mX + C$ ⇐ looking for.

* Tip: Extrapolate all the way to the axes.



Calculus

C1. Differentiation

↳ Standard:

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

$$\frac{d}{dx}(a) = 0.$$

$$\frac{d}{dx}(x) = 1$$

product: $\frac{d}{dx}(uv) = \frac{du}{dx}(v) + \frac{dv}{dx}(u)$

quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$

* Both u and v must have x

↳ Identities

Trigo:

$$\frac{d}{dx}(\sin(ax+b)) = \underset{\text{diff. x}}{\underline{a}}\cos(ax+b).$$

$$\frac{d}{dx}(\cos(ax+b)) = -\underline{a}\sin(ax+b)$$

$$\frac{d}{dx}(\tan(ax+b)) = \underline{a}\sec^2(ax+b).$$

Exp: $\frac{d}{dx}(e^{ax+b}) = \underset{\text{diff.}}{\underline{a}}e^{ax+b}$

ln:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}.$$

↳ Increasing / Decreasing $f^{\underline{n}}$.

$\frac{dy}{dx} > 0$: ↑

$\frac{dy}{dx} < 0$. ↓.

↳ Stationary pt:

$\frac{dy}{dx} = 0.$

* Nature: 2 test:

1st Derivative

x		α	
dy/dx		0	
shape		–	

↑ $x-0.01$ ↑ $x+0.01$

put in cal.

/‾\ : max

_/ : min

/‾/ , \\‾\\ . Inflexion.

* 2nd Derivative test.

↳ $\frac{d^2y}{dx^2}$ (differentiate 1 more time).

$\frac{d^2y}{dx^2} > 0$: minimum

$\frac{d^2y}{dx^2} < 0$: maximum

$\frac{d^2y}{dx^2} = 0$: Inconclusive.

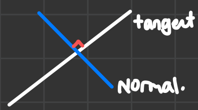
↳ Gradient, Tangent, Normals.

↳ $M_{Tan} \times M_{normal} = -1.$

$y = mx + c$

↑ grad.

$\frac{dy}{dx}$: grad of tangent.



↳ Maxima/Minima.

↳ $\frac{dy}{dx} = 0.$

* Tip. must show why min/max.

↳ use 1st test / 2nd test.

↳ Rate of change

↳ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}.$

$$\frac{dy}{dx} = \frac{dy}{\square} \times \frac{\square}{dx}$$

$\square$: Identical.

Ex: Find rate of change of x.

⇒ $\frac{dx}{dt}.$

* Decreasing : $\underline{-ve}$.

C2: Integration

↳ Standard.

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$$

✱ Must write

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

diff. the (ax+b).

✱ Trick: $\int a f(x) \, dx = a \int f(x) \, dx$.

↳ Identities

diff. the ✱.

trigo.

$$\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c$$

$$\int \sec^2(ax+b) \, dx = \frac{1}{a}\tan(ax+b) + c.$$

Exp: $\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b}$

diff. of power

reciprocal:

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln(ax+b) + c$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln(f(x)) + c$$

↳ Definite Integrals.

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

$$\int_a^a f(x) \, dx = 0.$$

$$-\int_a^b f(x) \, dx = \int_b^a f(x) \, dx$$

↳ Differentiation + Integration. Pg 160/161.

Ex: $\frac{d}{dx}\left[(x+1)\sqrt{2x-1}\right] = \frac{3x}{\sqrt{2x-1}}$ ←

Find $\int \frac{2x}{3\sqrt{2x-1}} \, dx$

From part (a)

Method: $\int \frac{2x}{3\sqrt{2x-1}} \, dx = \frac{2}{9} \int \frac{3x}{\sqrt{2x-1}} \, dx$

Num: "Add" 2, "remove" 3

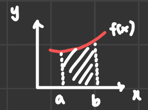
Num Denom.

Denom: "add" 3.

denom

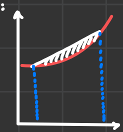
↳ Area under the curve.

$$A = \int_a^b f(x) \, dx$$



✱ Shape – Area or vice versa.

Ex:



∴ ◻ – Area under curve.

C3: Kinematics

Displacement

$\int y$ Area gradient $\frac{dy}{dx}$

Velocity

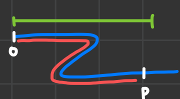
$\int y$ Area gradient $\frac{dy}{dx}$

Acceleration

✱ Instantaneous rest: $v = 0$.

3rd second: btw 2nd and 3rd.

✱ Distance vs Displacement.



– : Distance travelled

– : Displacement.

✱ +ve: assume to the right

-ve: assume to the left.