# Anglo-Chinese School

(Independent)

THE BEST IS YET TO BE

## PRELIMINARY EXAMINATION 2022

## YEAR 6 IB DIPLOMA PROGRAMME

**MATHEMATICS: ANALYSIS AND APPROACHES**

**HIGHER LEVEL**

**PAPER 1**

Tuesday 13 September 2022

2 hours

**Candidate Session Number**

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**INSTRUCTIONS TO CANDIDATES**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the boxes provided.
- Section B: answer all questions on the writing paper provided. Fill in your session number on each answer sheet, and attach them to this examination paper using the string provided.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.
- Questions with an asterisk(*) are common to both HL & SL papers.

| Section A (55 Marks) | | Section B (55 Marks) | |
|---|---|---|---|
| **Question** | **Marks** | **Question** | **Marks** |
| **1** | | **10** | |
| **2** | | | |
| **3** | | | |
| **4** | | **11** | |
| **5** | | | |
| **6** | | | |
| **7** | | **12** | |
| **8** | | | |
| **9** | | | |
| **Subtotal** | | **Subtotal** | |
| **TOTAL** | **/ 110** | | |

**This question paper consists of 12 printed pages including this cover page.**

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines if necessary.

Do **not** write solutions on this page.

**1.** *[Maximum Mark: 7]

(a) Given that $\sin A = \dfrac{1}{\sqrt{3}}$ and that $A$ is acute.

(i) Find the exact value of $\tan A$.

(ii) Hence, find the value of $\tan\left(\dfrac{A}{2}\right)$ in the form $\sqrt{m} - \sqrt{n}$, where $m$ and $n$ are integers. [4]

(b) Solve the equation $2\sin x = \tan x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. [3]

**2.** *[Maximum Mark: 5]

Consider the expression $\left(x^2+\frac{k}{x}\right)^n$ where $k$ and $n$ are positive integers.

(a) Given that there are 11 terms in the expansion, state the value of $n$. [1]

(b) For this value of $n$, find the value of $k$ given that the coefficient of $x^{14}$ is $180$. [4]

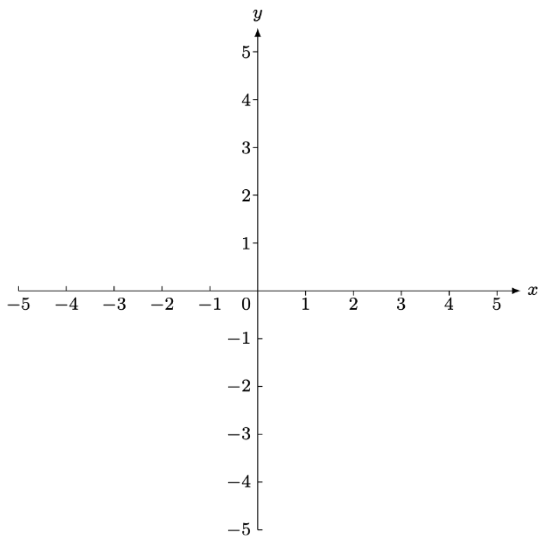
**3.** *[Maximum Mark: 7]

Consider the function $f(x) = \dfrac{x}{cx+6}$, where $c \neq 0$.

(a) The line $x = -2$ is a vertical asymptote to the graph of $y = f(x)$. Find the value of $c$. [1]

(b) State the equation of the horizontal asymptote to the graph of $y = f(x)$. [1]

(c) On the set of axes below, sketch the graph of $y = f(x)$, indicating all asymptotes and any axial intercept(s). [3]




(d) Hence, solve the inequality $0 < f(x) < 2$. [2]

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**4.** [Maximum Mark: 6]

An arithmetic sequence $u_1, u_2, u_3, \ldots$ has $u_1 = 1$ and common difference $d \neq 0$. Given that $u_2, u_3$ and $u_6$ are the first three terms of a geometric sequence.

(a) Find the value of $d$. [3]

(b) Given that in the arithmetic sequence $u_n = -17$, determine the value of $\sum_{r=1}^{n} u_r$. [3]

**5.** [Maximum Mark: 5]

Prove, by Mathematical Induction, that $6^n - 1$ is always divisible by $5$, for integers $n \geq 2$.

**6.** [Maximum Mark: 4]

Prove by contradiction that for any positive integer $a$ and $b$, where $a \neq b$, $a + b \geq 2\sqrt{ab}$.

**7.** [Maximum Mark: 4]

The graph of $y = f'(x)$, $0 \le x < 5$ is shown in the following diagram. The curve intersects the $x$-axis at $(1,0)$ and (4,0) and has a minimum at $x = 3$.
Given that $f(0) = 1$ and that the shaded area enclosed by the curve $y = f'(x)$, the $x$-axis and the $y$-axis is $1$. The area enclosed by the curve $y = f'(x)$ and the $x$-axis between $x = 1$ and $x = 4$ is $5$.

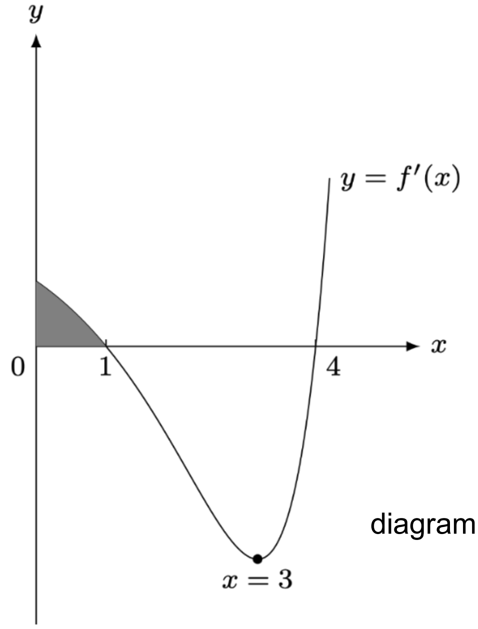


diagram not to scale

(a) Write down the $x$-coordinate of the point of inflexion on the graph of $y = f(x)$. [1]

(b) Hence or otherwise, find the value of $f(1)$ and $f(4)$. [3]

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**8.** [Maximum Mark: 8]

A continuous random variable $X$ has probability density function $f(x)$ given by

$$f(x)=\begin{cases} k & 2<x\le 3, \\ k(x-2) & 3<x\le 4, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the value of $k$ is $\frac{2}{5}$. [3]

(b) Determine the value of the median, $m$. [5]

**9.** [Maximum Mark: 9]

(a) Show that $\int x^3 e^x dx = e^x(x^3 - 3x^2 + 6x - 6) + C,$ where $C$ represents an arbitrary constant. [4]

(b) Hence, by using the substitution $x = \sqrt{t},$ evaluate $\int_0^2 te^{\sqrt{t}} dt.$ [5]

## Section B

Answer **all** questions in the answer sheets provided. Please start each question on a new page.

**10.** *[Maximum Mark: 15]

Consider the function $h(x) = \left(\sin(2x) + \cos(2x)\right)^2$.

(a) Show that $h(x) = \sin(4x) + 1$. [2]

(b) The graph of $y = h(x)$ can be obtained from the graph of $y = \sin x$ through two geometrical transformations.

(i) Describe the two geometrical transformations in order.

(ii) Sketch the graph of $y = h(x)$ for $0 \le x \le \pi$, clearly label all turning point(s) and axial intercept(s). [7]

(c) State

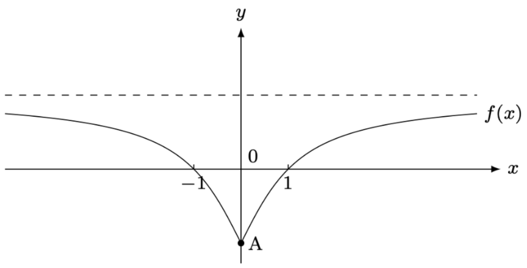
(i) the period of $h$ and;

(ii) the range of $h$. [2]

(d) Find the **exact** area bounded by the graph of $y = h(x)$, the $x-$axis and the $y-$axis. [4]

**11.** [Maximum Mark: 22]

A function $f$ is defined by $f(x) = \arcsin\left(\dfrac{x^2-1}{x^2+1}\right), x \in \mathbb{R}$. The graph of $y = f(x)$ is shown in the diagram below. The graph of $y = f(x)$ cuts the $x-$axis at $x = 1$ and $x = -1$, the $y-$intercept at point A is also the minimum point.




(a) Show that $f$ is an even function. [2]

(b) By considering the limit of $f$ as $x \to \infty$, show that the graph of $y = f(x)$ has a horizontal asymptote of $y = \dfrac{\pi}{2}$. [3]

(c) (i) Show that $f'(x) = \dfrac{2x}{|x|(x^2+1)}$ and state its domain clearly.

(ii) Hence, show that $f$ is decreasing at $x = -2$. [7]

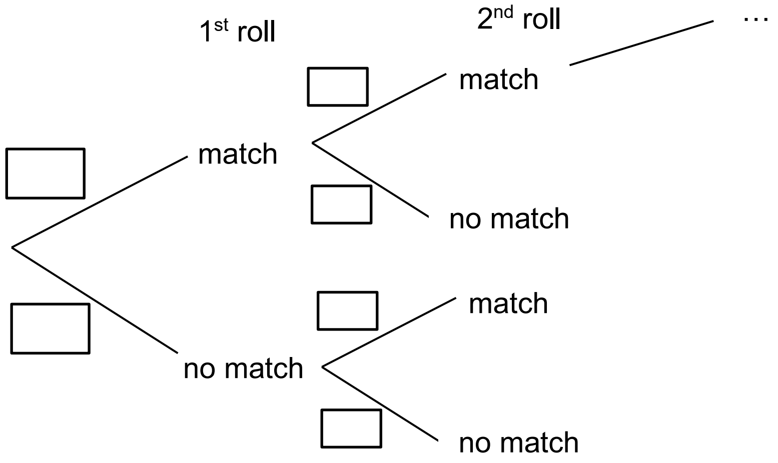
(d) Sketch the graph of $y = \dfrac{1}{f(x)}, x \in \mathbb{R}$, indicate any asymptote(s), turning point(s) and axial intercept(s) clearly. [4]

(e) A function $g$ is defined as $g(x) = \arcsin\left(\dfrac{x^2-1}{x^2+1}\right), x \in \mathbb{R}, x \ge 0$. Find an expression for $g^{-1}(x)$. [6]

**12.** [Maximum Mark: 18]

A fair four-sided die with faces labelled as 1,2,3 and 4 is rolled 4 times and the result recorded. A "match" is defined to be the occurrence of side $i$ on the $i$ th roll. For example, the die rolled a 1 on the first roll is a "match" and the die rolled a 2 on the second roll is also a "match".

(a) **Copy** and complete the following partial tree diagram **up to and including** the **second roll**. [3]




(b) Show that the probability of having at least one match on the 4th roll is $\frac{175}{256}$. [3]

A fair $n$ sided die with faces labelled 1, 2, 3, … and $n$ is used. Let $X$ be the number of matches obtained when the die is rolled $n$ times.

(c) Write down an expression of $P(X=0)$ in terms of $n$. [2]

(d) Hence or otherwise, show that $P(X \geq 1) = 1 - \left(1 - \frac{1}{n}\right)^n$. [2]

(e) Using Maclaurin expansion, show that $\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots$ . [2]

(f) (i) By expanding $\left(1-\frac{1}{n}\right)^n$ show that

$$\left(1-\frac{1}{n}\right)^n$$

$$= \frac{n^2\left(1-\frac{1}{n}\right)}{2!}\left(\frac{1}{n^2}\right) - \frac{n^3\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}\left(\frac{1}{n^3}\right) + \frac{n^4\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{4!}\left(\frac{1}{n^4}\right) - \ldots$$ [4]

(ii) Hence, show that as $n$ approaches infinity, $P(X \geq 1) = 1 - \frac{1}{e}$. [2]

**END OF PAPER**