RAFFLES INSTITUTION
H2 Mathematics (9758)
2022 Year 5

You do not need to print this set of lecture notes as it will be included (with solutions) in the next mathematics package (Book 5) issued in Year 6.

Chapter 8A: Integration Techniques

SYLLABUS INCLUDES

- Integration of $\mathrm{f}'(x)\left[\mathrm{f}(x)\right]^n$, $\mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$, $\sin^2 x$, $\cos^2 x$, $\tan^2 x$, $\sin mx \cos nx$, $\cos mx \cos nx$ and $\sin mx \sin nx$, $\dfrac{1}{a^2+x^2}$, $\dfrac{1}{\sqrt{a^2-x^2}}$, $\dfrac{1}{a^2-x^2}$ and $\dfrac{1}{x^2-a^2}$
- Integration by a given substitution
- Integration by parts

PRE-REQUISITES

- Basic Trigonometry
- Partial Fractions
- Differentiation Techniques

CONTENT

1 INTRODUCTION

We have seen that differentiating a function with respect to its variable gives us its derivative, i.e. the function that tells us the instantaneous rate of change of the original function with respect to its variable. This raises the question of whether it is possible to recover the original function, given its derivative. Just as differentiation is a process that gives you the derivative of a function, integration as introduced here will be a process that seeks to give you the original function given its derivative, albeit with some additional information required. Here, integration is viewed as **anti-differentiation**, and the functions obtained after integration are known as **anti-derivatives**.

In the next chapter, we will then see how the processes of differentiation and integration are linked via the fundamental ideas of limits and use these principles to show how, just as differentiation may be used to find gradients of curves, integration may also be used to find areas and volumes.

1.1 Integration as the Reverse Process of Differentiation

One basic way to perform integration is to modify the integrand (the expression that is to be integrated), such that it represents the derivative of a common function.

The ability to integrate a function in such a manner depends primarily on recognizing it as the derivative of another function.

Exercise : Given the following expression for $\mathrm{f}(x)$, find an expression $\mathrm{F}(x)$ such that $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x)=\mathrm{f}(x)$.

$\mathrm{f}(x)$	$\mathrm{F}(x)$	$\mathrm{f}(x)$	$\mathrm{F}(x)$
1	x	e^x	e^x
$2x$	x^2	$\cos x$	$\sin x$
$3x^2$	x^3	$\sec^2 x$	$\tan x$
x^3	$\frac{x^4}{4}$	$\frac{1}{x}$	$\ln x$

1.2 Indefinite Integral

1.2.1 Notation

Suppose f and F are two functions related as follows: $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{F}(x)) = \mathrm{f}(x)$.

Then, for any constant c, $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{F}(x)+c\right] = \mathrm{f}(x)$.

Hence, we have $\quad \mathrm{F}(x)+c = \int \mathrm{f}(x)\,\mathrm{d}x$.

We call	$\int \mathrm{f}(x)\ \mathrm{d}x$	the indefinite integral of f with respect to x,
	c	the arbitrary constant of integration,
	$\mathrm{f}(x)$	the integrand,
	$\int$	the integral sign.

For example, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}(x-3)^2+3\right) = x-3 \text{ and } \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}(x-3)^2+300\right) = x-3,$$

therefore $\int (x-3)\ \mathrm{d}x = \dfrac{1}{2}(x-3)^2+c$.

1.2.2 Basic Rules of Integration

(i) $\int \left[\mathrm{f}(x) \pm \mathrm{g}(x)\right]\mathrm{d}x = \int \mathrm{f}(x)\ \mathrm{d}x \pm \int \mathrm{g}(x)\ \mathrm{d}x$

(ii) $\int k\mathrm{f}(x)\ \mathrm{d}x = k\int \mathrm{f}(x)\ \mathrm{d}x$, where k is a constant, $k \neq 0$

(iii) $\int \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{f}(x)\right)\ \mathrm{d}x = \mathrm{f}(x)+c$

(iv) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\int \mathrm{f}(x)\ \mathrm{d}x\right) = \mathrm{f}(x)$

Note : $\int \left[\mathrm{f}(x)\mathrm{g}(x)\right]\mathrm{d}x \neq \int \mathrm{f}(x)\ \mathrm{d}x \bullet \int \mathrm{g}(x)\ \mathrm{d}x$

$$\int \frac{\mathrm{f}(x)}{\mathrm{g}(x)}\mathrm{d}x \neq \frac{\int \mathrm{f}(x)\ \mathrm{d}x}{\int \mathrm{g}(x)\ \mathrm{d}x}$$

1.3 Definite Integral

If f is a function **defined** and **continuous** on an interval $I=[u,v]$ and $\int \mathrm{f}(x)\ \mathrm{d}x=\mathrm{F}(x)+c$, then for $a, b \in I$ such that $u \le a < b \le v$, and the ***definite integral*** of $\mathrm{f}(x)$ from a to b w.r.t. x is denoted by

$$\int_a^b \mathrm{f}(x)\ \mathrm{d}x=\left[\mathrm{F}(x)\right]_a^b=\mathrm{F}(b)-\mathrm{F}(a)$$

where ***a*** and ***b*** are called the ***lower*** and ***upper limits of the integral*** respectively.

1.3.1 Basic Properties of Definite Integrals

(i) $\int_a^a \mathrm{f}(x)\ \mathrm{d}x=0$

(ii) $\int_a^b \mathrm{f}(x)\ \mathrm{d}x=-\int_b^a \mathrm{f}(x)\ \mathrm{d}x$

(iii) $\int_a^b \mathrm{f}(x)\ \mathrm{d}x=\int_a^c \mathrm{f}(x)\ \mathrm{d}x+\int_c^b \mathrm{f}(x)\ \mathrm{d}x$, where c is such that $a \le c \le b$

(iv) $\int_a^b \left[\mathrm{f}(x)\pm \mathrm{g}(x)\right]\mathrm{d}x=\int_a^b \mathrm{f}(x)\ \mathrm{d}x \pm \int_a^b \mathrm{g}(x)\ \mathrm{d}x$

(v) $\int_a^b k\mathrm{f}(x)\ \mathrm{d}x=k\int_a^b \mathrm{f}(x)\ \mathrm{d}x$, where k is any constant, $k \neq 0$

2 Integration of Standard Functions

2.1 Integration of the form x^n and Variations, where $n \neq -1$

Recall that $\frac{\mathrm{d}}{\mathrm{d}x}x^{n+1}=(n+1)x^n$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]^{n+1}=(n+1)\left[\mathrm{f}(x)\right]^n \mathrm{f}'(x)$.

For example,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x)^5=5(x)^4$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x+5)^5=5(2x+5)^4(2) \qquad \mathrm{f}(x)=2x+5,\quad \mathrm{f}'(x)=2,\quad n=4$$

$\int x^n \mathrm{d}x=\frac{x^{n+1}}{n+1}+c,\quad n\neq -1$	$\int \mathrm{f}'(x)\left[\mathrm{f}(x)\right]^n \mathrm{d}x=\frac{\left[\mathrm{f}(x)\right]^{n+1}}{n+1}+c,\quad n\neq -1$

Note: In particular, when $\mathrm{f}(x)=ax+b$, $a\in\mathbb{R}\setminus\{0\}$, $b\in\mathbb{R}$ are constants, then $\mathrm{f}'(x)=a$ and

$$\int a(ax+b)^n\ \mathrm{d}x=\frac{(ax+b)^{n+1}}{n+1}+c \quad\Rightarrow\quad \int (ax+b)^n\ \mathrm{d}x=\frac{(ax+b)^{n+1}}{a(n+1)}+d,\quad n\neq -1$$

Example 1

Find

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(a) $\int\left(\frac{1}{x^3}-\sqrt{x}\right)\mathrm{d}x$

$=\int\left(x^{-3}-x^{\frac{1}{2}}\right)\mathrm{d}x$

$=\frac{x^{-2}}{-2}-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c=-\frac{1}{2x^2}-\frac{2}{3}x^{\frac{3}{2}}+c$

Check:

$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{2x^2}-\frac{2}{3}x^{\frac{3}{2}}+c\right)=\frac{1}{x^3}-\sqrt{x}$

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(b) $\int(5x+4)^2\ \mathrm{d}x$

$\mathrm{f}(x)=5x+4$
$\mathrm{f}'(x)=5$
$n=2$

$=\frac{1}{5}\int5(5x+4)^2\ \mathrm{d}x$

$=\frac{1}{5}\frac{(5x+4)^3}{3}+c$

$=\frac{1}{15}(5x+4)^3+c$

Check:

$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{15}(5x+4)^3+c\right]=(5x+4)^3$

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(c) $\int x\left(2x^2-3\right)^3\ \mathrm{d}x$

$\mathrm{f}(x)=2x^2-3$
$\mathrm{f}'(x)=4x, n=3$

$=\frac{1}{4}\int4x\left(2x^2-3\right)^3\ \mathrm{d}x$

$=\frac{1}{4}\frac{\left(2x^2-3\right)^4}{4}+c$

$=\frac{1}{16}\left(2x^2-3\right)^4+c$

Check:

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{16}\left(2x^2-3\right)^4+c\right)=x(2x^2-3)^3$

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(d) $\int\frac{2x+3}{(x^2+3x)^4}\ \mathrm{d}x$

$\mathrm{f}(x)=x^2+3x$
$\mathrm{f}'(x)=2x+3$
$n=-4$

$=\int(2x+3)\left(x^2+3x\right)^{-4}\ \mathrm{d}x$

$=-\frac{1}{3\left(x^2+3x\right)^3}+c$

Check:

$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{3\left(x^2+3x\right)^3}+c\right)=\frac{2x+3}{(x^2+3x)^4}$

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2.2 Integration of the form x^{-1} and Variations

$$\ln|x|=\begin{cases}\ln x, & x>0\\ \ln(-x), & x<0,\end{cases}\qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln|x|=\begin{cases}\frac{\mathrm{d}}{\mathrm{d}x}\ln x=\frac{1}{x}, & x>0\\ \frac{\mathrm{d}}{\mathrm{d}x}\ln(-x)=\frac{-1}{-x}=\frac{1}{x}, & x<0.\end{cases}$$

When the domain is not specified, we write $\int\frac{1}{x}\mathrm{d}x=\ln|x|+c$.

In general, we have $\int \frac{f'(x)}{f(x)} dx = \ln\left|f(x)\right| + c$.

$\int \frac{1}{x} dx = \ln\|x\| + c$	$\int \frac{f'(x)}{f(x)} dx = \ln\|f(x)\| + c$

Example 2

Find

(a) $\int \frac{3}{x} dx$ $= 3\int \frac{1}{x} dx$ $= 3\ln\|x\| + c$	**(b)** $\int \frac{1}{1-5x} dx$ $= -\frac{1}{5}\int \frac{-5}{1-5x} dx$ $= -\frac{1}{5}\ln\|1-5x\| + c$ $f(x) = 1-5x$ $f'(x) = -5$
(c) $\int \frac{2x-1}{(2x^2-2x+1)} dx$ $= \frac{1}{2}\int \frac{4x-2}{(2x^2-2x+1)} dx$ $= \frac{1}{2}\ln\|2x^2-2x+1\| + c$ $f(x) = 2x^2-2x+1$ $f'(x) = 4x-2$	**(d)** $\int \frac{1}{1-2e^{-x}} dx$ $= \int \frac{e^x}{e^x-2} dx$ $= \ln\|e^x-2\| + c$ $f(x) = e^x - 2$ $f'(x) = e^x$

2.3 Integration of Exponential Function and Variations

Since $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$, we have

$\int e^x\, dx = e^x + c$	$\int f'(x)\, e^{f(x)}\, dx = e^{f(x)} + c$

Example 3

Find

(a) $\int_0^1 e^{3x+2} dx$ $= \frac{1}{3}\int_0^1 3e^{3x+2} dx$ $= \left[\frac{1}{3}e^{3x+2}\right]_0^1 = \frac{1}{3}(e^5 - e^2)$ $f(x) = 3x+2$ $f'(x) = 3$	**(b)** $\int xe^{x^2}\, dx$ $= \frac{1}{2}\int (2x)e^{x^2}\, dx$ $= \frac{1}{2}e^{x^2} + c$ $f(x) = x^2$ $f'(x) = 2x$

(c) $\int 4^{3x+2}\ \mathrm{d}x$ $=\int \mathrm{e}^{(3x+2)\ln 4}\ \mathrm{d}x$ $=\frac{1}{3\ln 4}\mathrm{e}^{(3x+2)\ln 4}+c$ $=\frac{1}{3\ln 4}4^{3x+2}+c$ *note: we can write a^x as $\mathrm{e}^{\ln a^x}$	**(d)** $\int (\sin x + x\cos x)\mathrm{e}^{x\sin x}\ \mathrm{d}x$ $=\mathrm{e}^{x\sin x}+c$ $\mathrm{f}(x)=x\sin x$ $\mathrm{f}'(x)$ $=\sin x+x\cos x$

2.4 Integration of Standard Trigonometric Functions and Variations

$\int \cos x\ \mathrm{d}x=\sin x+c$	$\int \mathrm{f}'(x)\cos(\mathrm{f}(x))\ \mathrm{d}x=\sin(\mathrm{f}(x))+c$
$\int \sin x\ \mathrm{d}x=-\cos x+c$	$\int \mathrm{f}'(x)\sin(\mathrm{f}(x))\ \mathrm{d}x=-\cos(\mathrm{f}(x))+c$
$\int \sec^2 x\ \mathrm{d}x=\tan x+c$	$\int \mathrm{f}'(x)\sec^2(\mathrm{f}(x))\mathrm{d}x=\tan(\mathrm{f}(x))+c$

Example 4

Find

(a) $\int (3\sin x+\cos 4x)\ \mathrm{d}x$ $=-3\cos x+\frac{1}{4}\sin 4x+c$	**(b)** $\int 2\sin(1-2x)\ \mathrm{d}x$ $=-\int -2\sin(1-2x)\ \mathrm{d}x$ $=\cos(1-2x)+c$
(c) $\int 3\sec^2(5x-2)\ \mathrm{d}x$ $=\frac{3}{5}\int 5\sec^2(5x-2)\ \mathrm{d}x$ $=\frac{3}{5}\tan(5x-2)+c$	**(d)** $\int x\cos(3x^2)\ \mathrm{d}x$ $=\frac{1}{6}\int 6x\cos(3x^2)\ \mathrm{d}x$ $=\frac{1}{6}\sin(3x^2)+c$

2.4.1 Integration using Trigonometric Identities

Commonly used trigonometry identities (to be used for simplification)

- Basic trigonometric identities
  - $\sin^2 x + \cos^2 x = 1$
  - $\tan^2 x + 1 = \sec^2 x$
  - $\cot^2 x + 1 = \operatorname{cosec}^2 x$

- Double angle formula: (in MF 26)
  - $\sin 2x = 2\sin x \cos x$
  - $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x$

Example 5

Find

(a) $\int \tan^2 x \, dx$

$= \int \left(\sec^2 x - 1\right) dx$

$= \tan x - x + c$

(b) $\int \cos^2 x \, dx$

$= \frac{1}{2}\int \left(1 + \cos 2x\right) dx$

$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + c$

(c) $\int \sin 3x \cos 3x \, dx$

$= \frac{1}{2}\int 2\sin 3x \cos 3x \, dx$

$= \frac{1}{2(6)}\int 6\sin 6x \, dx$

$= -\frac{1}{12}\cos 6x + c$

(d) $\int \frac{1}{1+\cos 2x} \, dx$

$= \int \frac{1}{1+\left(2\cos^2 x - 1\right)} dx$

$= \frac{1}{2}\int \sec^2 x \, dx$

$= \frac{1}{2}\tan x + c$

While we now know how to integrate expressions such as $\sin x\cos x$ or other products of the sine and cosine functions where the angles are identical, how do we integrate expressions such as $\sin x\cos 3x$ with respect to x? The answer lies in the factor formulae, which allow us to re-express such products as sums or differences of individual sine or cosine functions.

The following four identities are found in the MF26 list and constitute what are known as the factor formulae.

$$\sin P+\sin Q=2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$
$$\sin P-\sin Q=2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$
$$\cos P+\cos Q=2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$
$$\cos P-\cos Q=-2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

Let $A=\dfrac{P+Q}{2}$ and $B=\dfrac{P-Q}{2}$.

Observe that $P=\left(\dfrac{P+Q}{2}\right)+\left(\dfrac{P-Q}{2}\right)=A+B$ while $Q=\left(\dfrac{P+Q}{2}\right)-\left(\dfrac{P-Q}{2}\right)=A-B$,

so the above identities become

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B$$
$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$
$$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$$

Example 6

Find

(a) $\int \sin 3x\cos x\,\mathrm{d}x$

$$=\frac{1}{2}\int\left[\sin 4x+\sin 2x\right]\mathrm{d}x$$
$$=\frac{1}{2}\left(-\frac{1}{4}\cos 4x-\frac{1}{2}\cos 2x\right)+c$$
$$=-\frac{1}{8}\cos 4x-\frac{1}{4}\cos 2x+c$$

(b) $\int \sin x\sin 3x\,\mathrm{d}x$

$$=-\frac{1}{2}\int\left[\cos 4x-\cos 2x\right]\mathrm{d}x$$
$$=-\frac{1}{2}\left(\frac{1}{4}\sin 4x-\frac{1}{2}\sin 2x\right)+c$$
$$=\frac{1}{4}\sin 2x-\frac{1}{8}\sin 4x+c$$

Sometimes, we may also need to apply the results in sections **2.1** and **2.2**; namely

$$\int \mathrm{f}'(x)\left[\mathrm{f}(x)\right]^n \mathrm{d}x = \frac{\left[\mathrm{f}(x)\right]^{n+1}}{n+1} + c, \quad n \neq -1 \text{ and } \int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \mathrm{d}x = \ln\left|\mathrm{f}(x)\right| + c.$$

Example 7

Find

(a) $\int \frac{\sec^2\theta}{1+\tan\theta}\,\mathrm{d}\theta$ $= \ln\lvert 1+\tan\theta\rvert + c$	**(b)** $\int \sec x\,\mathrm{d}x$ $= \int \sec x \times \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)\mathrm{d}x$ $= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,\mathrm{d}x$ $= \ln\lvert \sec x + \tan x\rvert + c$ (in MF 26)
(c) $\int \tan x\,\mathrm{d}x$ $= -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x$ $= -\ln\lvert\cos x\rvert + c$ $= \ln\left\lvert\frac{1}{\cos x}\right\rvert + c$ $= \ln\lvert\cos x\rvert^{-1} + c$ $= \ln\lvert\sec x\rvert + c$	**(d)** $\int \sin x \cos^2 x\,\mathrm{d}x$ $= -\int(-\sin x)(\cos x)^2\,\mathrm{d}x$ $= -\frac{\cos^3 x}{3} + c$

3 Other Integration Techniques

3.1 Integration of $\dfrac{1}{\sqrt{a^2-x^2}}$, $\dfrac{1}{a^2+x^2}$ $(a>0)$

Recall that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}x\right)=\dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}x\right)=\dfrac{1}{1+x^2}$.

Therefore, $\displaystyle\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\sin^{-1}x+c$ and $\displaystyle\int\frac{1}{1+x^2}\,\mathrm{d}x=\tan^{-1}x+c$.

This result can be extended to find $\displaystyle\int\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x$ and $\displaystyle\int\frac{1}{a^2+x^2}\,\mathrm{d}x$.

$\displaystyle\int\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\sin^{-1}\frac{x}{a}+c$, $\lvert x\rvert<a$	$\displaystyle\int\frac{\mathrm{f}'(x)}{\sqrt{a^2-\left[\mathrm{f}(x)\right]^2}}\,\mathrm{d}x=\sin^{-1}\frac{\mathrm{f}(x)}{a}+c$
$\displaystyle\int\frac{1}{a^2+x^2}\,\mathrm{d}x=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$	$\displaystyle\int\frac{\mathrm{f}'(x)}{a^2+\left[\mathrm{f}(x)\right]^2}\,\mathrm{d}x=\frac{1}{a}\tan^{-1}\frac{\mathrm{f}(x)}{a}+c$

Example 8

Find

(a) $\displaystyle\int\frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x$ $\displaystyle=\int\frac{1}{\sqrt{2^2-x^2}}\,\mathrm{d}x$ $=\sin^{-1}\left(\dfrac{x}{2}\right)+c$	**(b)** $\displaystyle\int\frac{2}{3+x^2}\,\mathrm{d}x$ $\displaystyle=2\int\frac{1}{\left(\sqrt{3}\right)^2+x^2}\,\mathrm{d}x$ $=\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right)+c$
(c) $\displaystyle\int\frac{1}{(2x+1)^2+9}\,\mathrm{d}x$ $\displaystyle=\frac{1}{2}\int\frac{2}{(2x+1)^2+3^2}\,\mathrm{d}x$ $=\dfrac{1}{6}\tan^{-1}\left(\dfrac{2x+1}{3}\right)+c$ $\mathrm{f}(x)=2x+1$ $\mathrm{f}'(x)=2$	**(d)** $\displaystyle\int\frac{1}{x^2+2x+5}\,\mathrm{d}x$ $\displaystyle=\int\frac{1}{(x+1)^2+2^2}\,\mathrm{d}x$ $=\dfrac{1}{2}\tan^{-1}\left(\dfrac{x+1}{2}\right)+c$ $\mathrm{f}(x)=x+1$ $\mathrm{f}'(x)=1$

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(e) $\displaystyle\int \frac{1}{\sqrt{-12+16x-4x^2}}\,\mathrm{d}x$

$\displaystyle= \int \frac{1}{2\sqrt{-x^2+4x-3}}\,\mathrm{d}x$

$\displaystyle= \int \frac{1}{2\sqrt{-\left(x^2-4x+3\right)}}\,\mathrm{d}x$

$\displaystyle= \int \frac{1}{2\sqrt{-\left(x^2-4x+4-1\right)}}\,\mathrm{d}x$

$\displaystyle= \int \frac{1}{2\sqrt{1-(x-2)^2}}\,\mathrm{d}x$

$\mathrm{f}(x) = x-2$

$\mathrm{f}'(x) = 1$

$\displaystyle= \frac{1}{2}\sin^{-1}(x-2)+c$

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(f) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,\mathrm{d}x$

$\displaystyle= \frac{1}{2}\int \frac{2x}{\sqrt{1-\left(x^2\right)^2}}\,\mathrm{d}x$

$\displaystyle= \frac{1}{2}\sin^{-1}\left(x^2\right)+c$

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3.2 Integration Using Partial Fractions

Consider the integral $\displaystyle\int \frac{3x+4}{(x+2)(x+1)}\,\mathrm{d}x$.

By partial fractions, $\displaystyle\frac{3x+4}{(x+2)(x+1)} = \frac{2}{x+2} + \frac{1}{x+1}$

Thus

$$\int \frac{3x+4}{(x+2)(x+1)}\,\mathrm{d}x = \int \left[\frac{2}{x+2} + \frac{1}{x+1}\right]\mathrm{d}x$$
$$= 2\ln|x+2| + \ln|x+1| + c$$

The following formulae for decomposition of partial fractions are given in MF 26.

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)\left(x^2+c^2\right)} = \frac{A}{(ax+b)} + \frac{Bx+C}{\left(x^2+c^2\right)}$$

Example 9

Find

(a)	**For non-repeated linear factors:** $\int \frac{1}{(x+1)(2x-1)} \mathrm{d}x$ $= \int \left[-\frac{1}{3(x+1)} + \frac{2}{3(2x-1)} \right] \mathrm{d}x$ $= -\frac{1}{3}\ln\lvert x+1\rvert + \frac{1}{3}\ln\lvert 2x-1\rvert + c$	Let $\frac{1}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}$ $1 = A(2x-1) + B(x+1)$ When $x=-1$, $A = -\frac{1}{3}$ When $x = \frac{1}{2}$, $B = \frac{2}{3}$ $\therefore \frac{1}{(x+1)(2x-1)} = -\frac{1}{3(x+1)} + \frac{2}{3(2x-1)}$
(b)	**For repeated linear factors:** $\int \frac{1}{(x+1)(x+2)^2} \mathrm{d}x$ $= \int \left[\frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{(x+2)^2} \right] \mathrm{d}x$ $= \ln\lvert x+1\rvert - \ln\lvert x+2\rvert + \frac{1}{x+2} + c$	Let $\frac{1}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ $1 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$ When $x=-1$, $A = 1$ When $x=-2$, $C=-1$ By comparing coefficient of x^2, $0 = A + B$, so $B = -1$. $\therefore \frac{1}{(x+1)(x+2)^2} = \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{(x+2)^2}$

(c)	**For non-repeated quadratic factors:** $\int \frac{5}{(x+1)(x^2+4)}\, dx$ $= \int \left[\frac{1}{(x+1)} + \frac{-x+1}{(x^2+4)} \right] dx$ $= \int \left[\frac{1}{(x+1)} + \frac{-x}{(x^2+4)} + \frac{1}{(x^2+4)} \right] dx$ $= \ln\lvert x+1\rvert - \frac{1}{2}\int \frac{2x}{x^2+4}\, dx + \frac{1}{2}\tan^{-1}\frac{x}{2}$ $= \ln\lvert x+1\rvert - \frac{1}{2}\ln\left(x^2+4\right) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$	Let $\frac{5}{(x+1)\left(x^2+4\right)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$ $5 = A\left(x^2+4\right) + (Bx+C)(x+1)$ When $x=-1$, $A=1$ When $x=0$, $C=1$ By comparing coefficient of x, $0 = B+1 \Rightarrow B=-1$ $\frac{5}{(x+1)\left(x^2+4\right)} = \frac{1}{(x+1)} + \frac{-x+1}{\left(x^2+4\right)}$

Example 10 Find $\int \frac{2x^2-7x-1}{(x+1)(x-2)}\, dx$.

$\int \frac{2x^2-7x-1}{(x+1)(x-2)}\, dx$ $= \int 2 - \frac{8}{3(x+1)} - \frac{7}{3(x-2)}\, dx$ $= 2x - \frac{8}{3}\ln\lvert x+1\rvert - \frac{7}{3}\ln\lvert x-2\rvert + c$	Let $\frac{2x^2-7x-1}{(x+1)(x-2)} = A + \frac{B}{(x+1)} + \frac{C}{(x-2)}$ By observation, $A=2$ $2x^2-7x-1 = 2(x+1)(x-2) + A(x-2) + B(x+1)$ When $x=-1$, $B=-\frac{8}{3}$ When $x=2$, $C=-\frac{7}{3}$ $\frac{2x^2-7x-1}{(x+1)(x-2)} = 2 - \frac{8}{3(x+1)} - \frac{7}{3(x-2)}$

3.2.1 Integration of $\frac{1}{x^2-a^2}$ and $\frac{1}{a^2-x^2}$ $(a>0)$

Using partial fraction, $\frac{1}{x^2-a^2}=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$. Hence

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x=\int \frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)\mathrm{d}x=\frac{1}{2a}\left(\ln|x-a|-\ln|x+a|\right)+c=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c\,.$$

Hence we have the following formulae in MF 26.

$\int \frac{1}{x^2-a^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c\,,\ x>a$	$\int \frac{1}{a^2-x^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+c\,,\ \lvert x\rvert<a$

Note : For integrals similar to the forms $\int \frac{1}{x^2-a^2}\,\mathrm{d}x$ and $\int \frac{1}{a^2-x^2}\,\mathrm{d}x$ $(a>0)$, the above result may be quoted without proof.

In general, $\int \frac{\mathrm{f}'(x)}{[\mathrm{f}(x)]^2-a^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left|\frac{\mathrm{f}(x)-a}{\mathrm{f}(x)+a}\right|+c$ and

$$\int \frac{\mathrm{f}'(x)}{a^2-[\mathrm{f}(x)]^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left|\frac{a+\mathrm{f}(x)}{a-\mathrm{f}(x)}\right|+c.$$

Example 11

Find

(a) $\int \frac{1}{2x^2-4}\,\mathrm{d}x$ $=\frac{1}{2}\int \frac{1}{x^2-2}\,\mathrm{d}x$ $=\frac{1}{2}\int \frac{1}{x^2-\left(\sqrt{2}\right)^2}\,\mathrm{d}x$ $=\frac{1}{2}\times\frac{1}{2(\sqrt{2})}\ln\left\lvert\frac{x-\sqrt{2}}{x+\sqrt{2}}\right\rvert+c$ $=\frac{1}{4\sqrt{2}}\ln\left\lvert\frac{x-\sqrt{2}}{x+\sqrt{2}}\right\rvert+c$	**(b)** $\int \frac{1}{x^2+4x-5}\,\mathrm{d}x$ $=\int \frac{1}{x^2+4x+4-9}\,\mathrm{d}x$ $=\int \frac{1}{(x+2)^2-3^2}\,\mathrm{d}x$ $=\frac{1}{2(3)}\ln\left\lvert\frac{x+2-3}{x+2+3}\right\rvert+c$ $=\frac{1}{6}\ln\left\lvert\frac{x-1}{x+5}\right\rvert+c$

Example 12 By considering $\frac{\mathrm{d}}{\mathrm{d}x}(4x^2+4x-8)$, find $\int \frac{8x+5}{4x^2+4x-8}\,\mathrm{d}x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(4x^2+4x-8)=8x+4$$

$$\begin{aligned}\int \frac{8x+5}{4x^2+4x-8}\,\mathrm{d}x &= \int \frac{8x+4+1}{4x^2+4x-8}\,\mathrm{d}x\\ &= \int \frac{8x+4}{4x^2+4x-8}+\frac{1}{4x^2+4x-8}\,\mathrm{d}x\\ &= \ln\left|4x^2+4x-8\right|+\frac{1}{2}\int \frac{2}{(2x+1)^2-3^2}\,\mathrm{d}x\\ &= \ln\left|4x^2+4x-8\right|+\frac{1}{12}\ln\left|\frac{2x+1-3}{2x+1+3}\right|+c\\ &= \ln\left|4x^2+4x-8\right|+\frac{1}{12}\ln\left|\frac{x-1}{x+2}\right|+c\end{aligned}$$

The following table of integrals is given in MF 26.

(Arbitrary constants are omitted; a denotes a positive constant.)

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$	
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$	$(\lvert x\rvert<a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$ *	$(x>a)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$ *	$(\lvert x\rvert<a)$
$\tan x$	$\ln(\sec x)$ *	$\left(\lvert x\rvert<\frac{1}{2}\pi\right)$
$\cot x$	$\ln(\sin x)$ *	$(0<x<\pi)$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x+\cot x)$ *	$(0<x<\pi)$
$\sec x$	$\ln(\sec x+\tan x)$ *	$\left(\lvert x\rvert<\frac{1}{2}\pi\right)$

* Observe that the given formulae do not take the moduli of the expressions within the logarithm function. The formulae are given for specific domains, indicated within the brackets at the side, which ensure that the expressions within the logarithms are always positive. **If domain is not specified, then you should take moduli of the expressions within the logarithm function.**

3.3 Integration by Substitution (Change of Variable)

Consider the integral $\int \frac{(\ln x)^3}{x}\,\mathrm{d}x$.

Here we introduce "something extra" and this "something extra" is a new variable. We transform the integral involving the variable x to an integral involving a new variable, say u. The idea is illustrated below.

Let $u = \ln x$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \Rightarrow \mathrm{d}u = \frac{1}{x}\mathrm{d}x$

$$\begin{aligned}
&\therefore \int \frac{(\ln x)^3}{x}\,\mathrm{d}x \\
&= \int (\ln x)^3 \cdot \frac{1}{x}\,\mathrm{d}x \\
&= \int u^3\,\mathrm{d}u \\
&= \frac{1}{4}u^4 + c \\
&= \frac{(\ln x)^4}{4} + c
\end{aligned}$$

In general, this method works when we can write the integral in the form $\int \mathrm{f}(\mathrm{g}(x))\mathrm{g}'(x)\,\mathrm{d}x$.

In the example above, $\mathrm{f}(x) = x^3,\ \mathrm{g}(x) = \ln x$.

Substitution Rule

$$\int \mathrm{f}(\mathrm{g}(x))\mathrm{g}'(x)\,\mathrm{d}x = \int \mathrm{f}(u)\,\mathrm{d}u \qquad \text{where } u = \mathrm{g}(x)$$

Step 1: Substitute $u = \mathrm{g}(x)$ and $\mathrm{d}u = \mathrm{g}'(x)\,\mathrm{d}x$ to obtain the integral $\int \mathrm{f}(u)\,\mathrm{d}u$

Step 2: Integrate with respect to u

Step 3: Replace u by $\mathrm{g}(x)$ in the result

For definite integrals,

$$\int_a^b \mathrm{f}(\mathrm{g}(x))\mathrm{g}'(x)\,\mathrm{d}x = \int_{\mathrm{g}(a)}^{\mathrm{g}(b)} \mathrm{f}(u)\,\mathrm{d}u \qquad \text{where } u = \mathrm{g}(x)$$

Example 13
Using the given substitution, find

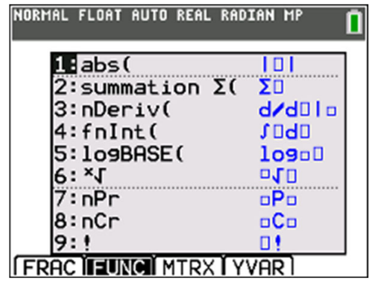
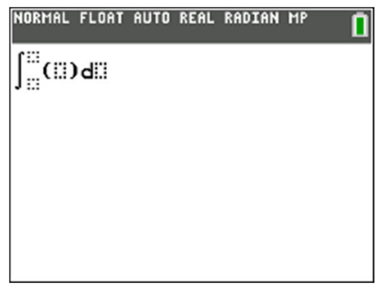
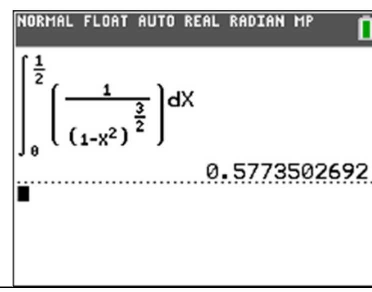
(a)	$\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ $\displaystyle= 2\int e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\,dx$ $\displaystyle= 2\int e^{u}\,du$ $= 2e^{u} + c$ $= 2e^{\sqrt{x}} + c$ Remember to substitute back the original variable for an indefinite integral	Let $u = \sqrt{x}$ $\displaystyle\frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow du = \frac{1}{2\sqrt{x}}dx$
(b)	$\displaystyle\int \frac{e^{x}}{e^{x}+e^{-x}}\,dx$ $\displaystyle= \int \frac{1}{e^{x}+e^{-x}} \cdot e^{x}dx$ $\displaystyle= \int \frac{1}{u+u^{-1}}du$ $\displaystyle= \int \frac{u}{u^{2}+1}\,du$ $\displaystyle= \frac{1}{2}\int \frac{2u}{u^{2}+1}\,du$ $\displaystyle= \frac{1}{2}\ln(u^{2}+1) + c$ $\displaystyle= \frac{1}{2}\ln(e^{2x}+1) + c$	Let $u = e^{x}$ $\displaystyle\frac{du}{dx} = e^{x} \Rightarrow du = e^{x}dx$
(c)	$\displaystyle\int x\sqrt{x-2}\,dx$ $\displaystyle= \int \left(u^{2}+2\right)(u)\,2u\,du$ $\displaystyle= 2\int u^{4} + 2u^{2}\,du$ $\displaystyle= 2\left[\frac{u^{5}}{5} + \frac{2u^{3}}{3}\right] + c$ $\displaystyle= \frac{2\sqrt{(x-2)^{5}}}{5} + \frac{4\sqrt{(x-2)^{3}}}{3} + c$	Let $u = \sqrt{x-2}$ $\displaystyle\frac{du}{dx} = \frac{1}{2\sqrt{x-2}} \Rightarrow 2u\,du = dx$ $u = \sqrt{x-2} \Rightarrow x = u^{2} + 2$

Example 14
Using the given substitution, evaluate the following definite integrals, leaving your answer in the exact form.

(a)	$\int_{\frac{2}{\sqrt{3}}}^{2} \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x$ (Substitute $x=\frac{1}{y}$) $=\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{y}{\sqrt{\frac{1}{y^2}-1}}\left(-\frac{1}{y^2}\right)\mathrm{d}y$ $=\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{y^2}{\sqrt{1-y^2}}\left(-\frac{1}{y^2}\right)\mathrm{d}y$ $=-\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y$ $=-\left[\sin^{-1} y\right]_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}}$ $=-\left[\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$ $=-\left(\frac{\pi}{6}-\frac{\pi}{3}\right)=\frac{\pi}{6}$	Let $x=\frac{1}{y}$ $\frac{\mathrm{d}x}{\mathrm{d}y}=-\frac{1}{y^2}\Rightarrow \mathrm{d}x=-\frac{1}{y^2}\mathrm{d}y$ When $x=2$, $y=\frac{1}{2}$ When $x=\frac{2}{\sqrt{3}}$, $y=\frac{\sqrt{3}}{2}$
(b)	$\int_{2}^{3} \frac{1}{\sqrt{(3-x)(x-2)}}\,\mathrm{d}x$ (Substitute $x=2\cos^2\theta+3\sin^2\theta$) $=\int_{0}^{\frac{\pi}{2}} \frac{1}{\sqrt{(1-\sin^2\theta)(\sin^2\theta)}}(2\sin\theta\cos\theta)\,\mathrm{d}\theta$ $=\int_{0}^{\frac{\pi}{2}} \frac{1}{\cos\theta\sin\theta}(2\sin\theta\cos\theta)\,\mathrm{d}\theta$ $=\int_{0}^{\frac{\pi}{2}} 2\,\mathrm{d}\theta=\left[2\theta\right]_0^{\frac{\pi}{2}}=\pi$	$x=2\cos^2\theta+3\sin^2\theta$ $=2+\sin^2\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=2\sin\theta\cos\theta$ $\Rightarrow \mathrm{d}x=2\sin\theta\cos\theta\,\mathrm{d}\theta$ When $x=2$, $\theta=0$ When $x=3$, $\theta=\frac{\pi}{2}$
(c)	$\int_{0}^{\frac{1}{2}} \frac{1}{\left(1-x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x$ $\int_{0}^{\frac{1}{2}} \frac{1}{\left(1-x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x$ $=\int_{0}^{\frac{\pi}{6}} \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}}\cos\theta\,\mathrm{d}\theta$ $=\int_{0}^{\frac{\pi}{6}} \frac{1}{\cos^3\theta}\cos\theta\,\mathrm{d}\theta$ $=\int_{0}^{\frac{\pi}{6}} \sec^2\theta\,\mathrm{d}\theta$ $=\left[\tan\theta\right]_0^{\frac{\pi}{6}}=\left[\tan\left(\frac{\pi}{6}\right)-\tan(0)\right]=\frac{1}{\sqrt{3}}$	Let $x=\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\cos\theta\Rightarrow \mathrm{d}x=\cos\theta\,\mathrm{d}\theta$ When $x=\frac{1}{2}$, $\theta=\frac{\pi}{6}$ When $x=0$, $\theta=0$

For questions involving definite integrals, you can verify the accuracy of your answers using the numerical integration function of the GC.

With reference to Example 14(c).

1. Press [ALPHA] [WINDOW] and select 4	
2. Key in $\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}} \mathrm{d}x$	
3. Press ENTER , you would obtain the numerical answer 0.5773502692	

3.4 Integration by Parts

This method is usually used when the integrands are products of two distinct types of functions such as $\ln x$, e^{x}, $\sin x$ etc. It is based on the product rule of differentiation,
and its purpose is to express the integral of a product in terms of a second integral that is usually easier to find.

From the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Rewriting, we get

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}.$$

Integrating with respect to x, we obtain

$$\boxed{\int u\frac{dv}{dx}\ dx = uv - \int v\frac{du}{dx}\ dx}$$

To use this formula we need to choose appropriate u and $\frac{dv}{dx}$, and find $\frac{du}{dx}$ and v.

Consider the following table and compare the two ways of assigning u and $\frac{dv}{dx}$ for $\int xe^{2x}\ dx$.

	u	$\frac{dv}{dx}$	v	$\frac{du}{dx}$	Integration by parts formula
Option 1	x	e^{2x}	$\frac{1}{2}e^{2x}$	1	$\int xe^{2x}\ dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\ dx$
Option 2	e^{2x}	x	$\frac{1}{2}x^2$	$2e^{2x}$	$\int xe^{2x}dx = \frac{1}{2}x^2e^{2x} - \int x^2e^{2x}dx$

The first option gives rise to $\int \frac{1}{2}e^{2x}\ dx$, which can be found easily.

The second option leads to $\int x^2e^{2x}\ dx$, which is even harder to find than $\int xe^{2x}\ dx$.

Hence, using the first option,

$$\begin{aligned}\int xe^{2x}\ dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\ dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c.\end{aligned}$$

Example 15

Find

(a)

$$\int x\sin x \, dx$$
$$= x(-\cos x) - \int (1)(-\cos x) \, dx$$
$$= -x\cos x + \sin x + c$$

$u = x$	$\frac{dv}{dx} = \sin x$
$\frac{du}{dx} = 1$	$v = -\cos x$

(b)

$$\int_0^1 \tan^{-1} x \, dx$$
$$= \int_0^1 (1)\left(\tan^{-1} x\right) dx$$
$$= \left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$
$$= \left[x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right)\right]_0^1$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

$u = \tan^{-1} x$	$\frac{dv}{dx} = 1$
$\frac{du}{dx} = \frac{1}{1+x^2}$	$v = x$

Important Technique:
Insert "1" so as to complete the product of two terms.

(c)

$$\int x^2 e^{2x} \, dx$$
$$= \frac{1}{2}x^2 e^{2x} - \int (2x)\left(\frac{1}{2}e^{2x}\right) dx$$
$$= \frac{1}{2}x^2 e^{2x} - \int xe^{2x} \, dx$$
$$= \frac{1}{2}x^2 e^{2x} - \left(\frac{1}{2}xe^{2x} - \int (1)\left(\frac{1}{2}e^{2x}\right) dx\right)$$
$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \int \frac{1}{2}e^{2x} \, dx$$
$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$$

$u = x^2$	$\frac{dv}{dx} = e^{2x}$
$\frac{du}{dx} = 2x$	$v = \frac{1}{2}e^{2x}$

$u = x$	$\frac{dv}{dx} = e^{2x}$
$\frac{du}{dx} = 1$	$v = \frac{1}{2}e^{2x}$

(d)

$$\int x^3 e^{x^2} \, dx$$
$$= \int \left(x^2\right)\left(xe^{x^2}\right) dx$$
$$= \frac{1}{2}x^2 e^{x^2} - \int (2x)\left(\frac{1}{2}e^{x^2}\right) dx$$
$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$$

$u = x^2$	$\frac{dv}{dx} = xe^{x^2}$
$\frac{du}{dx} = 2x$	$v = \frac{1}{2}e^{x^2}$

(e) $\int e^x \cos x \, dx$

$= e^x \cos x - \int e^x(-\sin x) \, dx$

$= e^x \cos x + \int e^x(\sin x) \, dx$

$= e^x \cos x + \left[e^x \sin x - \int e^x \cos x \, dx \right]$

$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$

$\therefore 2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$

$\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + c$

$u = \cos x$	$\frac{dv}{dx} = e^x$
$\frac{du}{dx} = -\sin x$	$v = e^x$

$u = \sin x$	$\frac{dv}{dx} = e^x$
$\frac{du}{dx} = \cos x$	$v = e^x$

Note that the integral recurs on the RHS after doing integration by parts twice.

CONCLUSION

In this chapter, we learnt about the various techniques of integration. Integration is essentially the reverse process of differentiation, and that provides a means of checking of answers. Integration can be applied to obtain areas and volumes, and summing the areas of rectangles is a way which enables us to approximate a definite integral, and this will be discussed in the next chapter.

SUMMARY

RAFFLES INSTITUTION
H2 Mathematics (9758)
2023 Year 6

You do not need to print this set of tutorial questions as it will be included in the next mathematics package (Book 5) issued in Year 6. Please check with your tutor regarding the timeline for completing this tutorial.

Tutorial 8A: Integration Techniques

The following table of integrals is given in MF26.

(Arbitrary constants are omitted; a denotes a positive constant.)		
$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{x-a}{x+a}\right)$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right)$	$(\lvert x\rvert < a)$
$\tan x$	$\ln(\sec x)$	$\left(\lvert x\rvert < \dfrac{1}{2}\pi\right)$
$\cot x$	$\ln(\sin x)$	$(0 < x < \pi)$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x + \cot x)$	$(0 < x < \pi)$
$\sec x$	$\ln(\sec x + \tan x)$	$\left(\lvert x\rvert < \dfrac{1}{2}\pi\right)$

Section A (Discussion Questions)

Evaluate the following integrals

1. (a) $\int (5x-2)^6 \, \mathrm{d}x$ (b) $\int \frac{4}{3-2x} \, \mathrm{d}x$ (c) $\int \mathrm{e}^{1-2x} \, \mathrm{d}x$ (d) $\int \cos(3x+1) \, \mathrm{d}x$

2. (a) $\int \frac{1}{(1-4x)^{\frac{5}{3}}} \, \mathrm{d}x$ (b) $\int \frac{x}{2x^2+3} \, \mathrm{d}x$ (c) $\int (\mathrm{e}^x - 3\mathrm{e}^{2x})^2 \, \mathrm{d}x$ (d) $\int \sec^2 \frac{x}{3} \, \mathrm{d}x$

3. (a) $\int \frac{x^2-1}{\sqrt{x^3-3x}} \, \mathrm{d}x$ (b) $\int (x^2-1)(x^3-3x)^5 \, \mathrm{d}x$ (c) $\int \frac{x}{x+3} \mathrm{d}x$ (d) $\int \frac{1}{\sqrt{x}(1-\sqrt{x})} \, \mathrm{d}x$

(e) $\int \sin^2 3x \, \mathrm{d}x$ (f) $\int \tan 3x \, \mathrm{d}x$ (g) $\int \cot\theta \, \mathrm{d}\theta$ (h) $\int \sin\frac{x}{2}\cos\frac{x}{2} \, \mathrm{d}x$

(i) $\int \cos(3x)\sin^3(3x)\,\mathrm{d}x$

4. (a) $\int \frac{\sec^2 x}{(1+\tan x)^3} \, \mathrm{d}x$ (b) $\int \frac{1}{2x^2-12x-14} \, \mathrm{d}x$ (c) $\int \frac{1}{x^2+2x} \, \mathrm{d}x$

(d) $\int \frac{x^3+2}{x^2-1} \, \mathrm{d}x$ (e) $\int \frac{2x^2-x+9}{(x+1)(x-3)^2} \, \mathrm{d}x$

5. (a) $\int \frac{1}{\sqrt{4-(x-3)^2}} \, \mathrm{d}x$ (b) $\int \frac{2}{x^2-6x+8} \, \mathrm{d}x$ (c) $\int \frac{x}{x^4-4} \, \mathrm{d}x$

(d) $\int \frac{1}{x^2+2x+3} \, \mathrm{d}x$ (e) $\int \frac{1}{\sqrt{6-4x-x^2}} \mathrm{d}x$

6. (a) $\int \frac{1}{1-\cos 2x} \, \mathrm{d}x$ (b) $\int \frac{1}{\sqrt{3-4x^2}} \, \mathrm{d}x$ (c) $\int \frac{1}{1+\cos x} \mathrm{d}x$ (d) $\int \frac{1}{4+81x^2} \, \mathrm{d}x$

(e) $\int \sin\frac{5x}{2}\cos\frac{3x}{2}\mathrm{d}x$ (f) $\int \cos\left(\frac{a\theta}{2}\right)\cos\left(\frac{3a\theta}{2}\right)\mathrm{d}\theta$ (g) $\int \sin(\pi+1)\theta\sin(\pi-1)\theta \, \mathrm{d}\theta$

7. Integration using a Substitution (Change of Variables)
By using the given substitution, or otherwise, determine the following integrals, leaving your answers in the exact form whenever applicable.

(a) $\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^x+2} \, \mathrm{d}x$	$[u=\mathrm{e}^x+2]$	(b) $\int_3^7 x\sqrt{x-3} \, \mathrm{d}x$	$[x=u^2+3]$
(c) $\int \frac{x}{\sqrt{1-x^4}} \, \mathrm{d}x$	$[u=x^2]$	(d) $\int \frac{1-x^2}{(1+x^2)^2} \, \mathrm{d}x$	$[x=\tan\theta]$
(e) $\int_0^3 \sqrt{9-x^2} \, \mathrm{d}x$	$[x=3\sin\theta]$		

8. Evaluate the following integrals

(a) $\int x\mathrm{e}^{4x}\ \mathrm{d}x$ **(b)** $\int \sqrt{x}\ln x\ \mathrm{d}x$ **(c)** $\int \frac{x+2}{x^2-x}\ \mathrm{d}x$ **(d)** $\int x\cos 5x\ \mathrm{d}x$

(e) $\int x^2\cos x\ \mathrm{d}x$ **(f)** $\int x\sec x\tan x\ \mathrm{d}x$

9. 9233/1999/01/Q19

(i) Prove that $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\sec x+\tan x\right)=\sec x$.

(ii) Find $\int x\sin x\ \mathrm{d}x$.

(iii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} x\sin x\ln\left(\sec x+\tan x\right)\mathrm{d}x$.

10. 9233/1995/01/Q16 (modified)

Use the substitution $u=\mathrm{e}^x$ to show that $\int_0^1 \frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}\ \mathrm{d}x=\int_1^{\mathrm{e}} \frac{u-1}{u\left(u+1\right)}\ \mathrm{d}u$. Hence, by using partial fractions, find the exact value of $\int_0^1 \frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}\ \mathrm{d}x$. Suggest an alternative method.

11. Evaluate the following integrals

(a) $\int \frac{x}{\sqrt{x-1}}\ \mathrm{d}x$ **(b)** $\int \cos^3 x\,\mathrm{d}x$ **(c)** $\int \mathrm{e}^x\sin 2x\ \mathrm{d}x$ **(d)** $\int\ x^3\cos x^2\ \mathrm{d}x$

(e) $\int \tan^4 x\ \mathrm{d}x$ **(f)** $\int \frac{1}{x\left(\ln x\right)^3}\mathrm{d}x$

Answers to Section A

1. (a) $\frac{1}{35}(5x-2)^7+c$, (b) $-2\ln|3-2x|+c$, (c) $-\frac{1}{2}\mathrm{e}^{1-2x}+c$, (d) $\frac{1}{3}\sin(3x+1)+c$

2. (a) $\frac{3}{8}(1-4x)^{-\frac{2}{3}}+c$, (b) $\frac{1}{4}\ln|2x^2+3|+c$, (c) $\frac{1}{2}\mathrm{e}^{2x}-2\mathrm{e}^{3x}+\frac{9}{4}\mathrm{e}^{4x}+c$, (d) $3\tan\frac{x}{3}+c$

3. (a) $\frac{2}{3}\left(x^3-3x\right)^{\frac{1}{2}}+c$, (b) $\frac{1}{18}\left(x^3-3x\right)^6+c$, (c) $x-3\ln|x+3|+c$, (d) $-2\ln\left|1-\sqrt{x}\right|+c$

(e) $\frac{1}{2}\left(x-\frac{\sin 6x}{6}\right)+c$, (f) $-\frac{1}{3}\ln|\cos 3x|+c$, (g) $\ln|\sin\theta|+c$, (h) $-\frac{1}{2}\cos x+c$,

(i) $\frac{1}{12}\sin^4(3x)+c$

4. (a) $-\frac{1}{2(1+\tan x)^2}+c$, (b) $\frac{1}{16}\ln\left|\frac{x-7}{x+1}\right|+c$, (c) $\frac{1}{2}\ln\left|\frac{x}{x+2}\right|+c$,

(d) $\frac{x^2}{2}-\frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x-1|+c$, (e) $\frac{3}{4}\ln|x+1|+\frac{5}{4}\ln|x-3|-\frac{6}{x-3}+c$

5. (a) $\sin^{-1}\frac{x-3}{2}+c$, (b) $\ln\left|\frac{x-4}{x-2}\right|+c$, (c) $\frac{1}{8}\ln\left|\frac{x^2-2}{x^2+2}\right|+c$, (d) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}+c$,

(e) $\sin^{-1}\frac{x+2}{\sqrt{10}}+c$

6. (a) $-\frac{1}{2}\cot x+c$, (b) $\frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{3}}+c$, (c) $\tan\frac{x}{2}+c$, (d) $\frac{1}{18}\tan^{-1}\frac{9x}{2}+c$

(e) $-\frac{1}{8}(\cos 4x+4\cos x)+c$, (f) $\frac{1}{4a}\left[\sin 2a\theta+2\sin a\theta\right]+C$, (g) $\frac{1}{4\pi}\left[\pi\sin 2\theta-\sin 2\pi\theta\right]+C$

7. (a) $\mathrm{e}^x-2\ln|\mathrm{e}^x+2|+c$, (b) $\frac{144}{5}$, (c) $\frac{1}{2}\sin^{-1}x^2+c$, (d) $\frac{x}{1+x^2}+C$, (e) $\frac{9\pi}{4}$

8. (a) $\frac{1}{4}x\mathrm{e}^{4x}-\frac{1}{16}\mathrm{e}^{4x}+c$, (b) $\frac{2}{3}x^{\frac{3}{2}}\left(\ln x-\frac{2}{3}\right)+c$, (c) $-2\ln|x|+3\ln|x-1|+c$,

(d) $\frac{1}{5}x\sin 5x+\frac{1}{25}\cos 5x+c$, (e) $x^2\sin x+2x\cos x-2\sin x+c$,

(f) $x\sec x-\ln|\sec x+\tan x|+c$

9. (ii) $-x\cos x+\sin x+c$, (iii) $\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)\ln\left(\sqrt{2}+1\right)+\frac{\pi^2}{32}-\ln\sqrt{2}$

10. $2\ln(\mathrm{e}+1)-2\ln 2-1$

11. (a) $2(x-1)^{\frac{1}{2}}\left(\frac{1}{3}x+\frac{2}{3}\right)+c$, (b) $\sin x-\frac{\sin^3 x}{3}+c$, (c) $\frac{1}{5}(\mathrm{e}^x\sin 2x-2\mathrm{e}^x\cos 2x)$,

(d) $\frac{1}{2}x^2\sin x^2+\frac{1}{2}\cos x^2+c$, (e) $\frac{1}{3}\tan^3 x-\tan x+x+c$, (f) $-\frac{1}{2(\ln x)^2}+c$

Section B (Assignment Questions)

1. IJC Prelim 9758/2017/01/Q2

(i) Find $\int n\cos^{-1}(nx)\,\mathrm{d}x$, where n is a positive constant. **[3]**

(ii) Hence find the exact value of $\int_0^{\frac{1}{2n}} n\cos^{-1}(nx)\,\mathrm{d}x$. **[2]**

2. 9740/2016/02/Q2(a)

(i) Find $\int x^2\cos nx\,\mathrm{d}x$, where n is a positive integer. **[3]**

(ii) Hence find $\int_{\pi}^{2\pi} x^2\cos nx\,\mathrm{d}x$, giving your answers in the form $a\dfrac{\pi}{n^2}$, where the possible values of a are to be determined. **[2]**

3. 9233/2001/01/Q15

(a) Use the fact that $7\cos x-4\sin x=\frac{3}{2}(\cos x+\sin x)+\frac{11}{2}(\cos x-\sin x)$ to find the exact value of $\int_0^{\frac{\pi}{2}}\dfrac{7\cos x-4\sin x}{\cos x+\sin x}\,\mathrm{d}x$. **[4]**

(b) Use integration by parts to find the exact value of $\int_1^{\mathrm{e}}(\ln x)^2\,\mathrm{d}x$. **[4]**

(c) Using the substitution $x=\dfrac{1}{y}$ or otherwise, find $\int\dfrac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x$ for $x>1$. **[4]**