National Junior College
2016 – 2017 H2 Further Mathematics
Topic F7: Matrices and Linear Spaces (Tutorial Set 1 Solutions)

Basic Mastery Questions

1

$$\begin{pmatrix} 2 & 3 & 3 & 25 \\ 3 & 2 & 3 & 24 \\ 4 & 1 & 2 & 21 \end{pmatrix} \to \begin{pmatrix} 1 & 1.5 & 1.5 & 12.5 \\ 3 & 2 & 3 & 24 \\ 4 & 1 & 2 & 21 \end{pmatrix} \to \begin{pmatrix} 1 & 1.5 & 1.5 & 12.5 \\ 0 & -2.5 & -1.5 & -13.5 \\ 0 & -5 & -4 & -29 \end{pmatrix} \to \begin{pmatrix} 1 & 1.5 & 1.5 & 12.5 \\ 0 & 1 & 0.6 & 5.4 \\ 0 & -5 & -4 & -29 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1.5 & 1.5 & 12.5 \\ 0 & 1 & 0.6 & 5.4 \\ 0 & 0 & -1 & -2 \end{pmatrix} \to \underline{\begin{pmatrix} 1 & 1.5 & 1.5 & 12.5 \\ 0 & 1 & 0.6 & 5.4 \\ 0 & 0 & 1 & 2 \end{pmatrix}} \to \begin{pmatrix} 1 & 0 & 0.6 & 4.4 \\ 0 & 1 & 0.6 & 5.4 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 3.2 \\ 0 & 1 & 0 & 4.2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

2

$$\mathbf{AB}^T = \begin{pmatrix} a & 3 & 2 \\ 0 & -1 & 6 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & -3 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} a+4 & -a+9 & -13 \\ 12 & -3 & -9 \end{pmatrix}. \ \mathbf{BA}^T = \left(\mathbf{AB}^T\right)^T = \begin{pmatrix} a+4 & 12 \\ -a+9 & -3 \\ -13 & -9 \end{pmatrix}.$$

4

$$\begin{vmatrix} 2016 & 2017 \\ 2018 & 2019 \end{vmatrix} = 2016 \times 2019 - 2017 \times 2018 = -2.$$

$$\begin{pmatrix} 2016 & 2017 \\ 2018 & 2019 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 2019 & -2017 \\ -2018 & 2016 \end{pmatrix} = \begin{pmatrix} -\frac{2019}{2} & \frac{2017}{2} \\ 1009 & -1008 \end{pmatrix}.$$

5

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & (a+c)-(b+c) & (a+b)-(b+c) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & a-b & a-c \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

6

(i)(a) $\det(\mathbf{A}) = 1\begin{vmatrix} 1 & -1 \\ 7 & 0 \end{vmatrix} - 3\begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} = 1(0+7) - 3(0+2) = 1.$

(i)(b) $\det(\mathbf{A}) = 1\times1\times0 + 0\times7\times0 + 2\times3\times(-1) - 2\times1\times0 - 7\times(-1)\times1 - 0\times0\times3 = 1.$

(i)(c) $\begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 2 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1\times1\times1 = 1.$

(ii)(a) $\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 7 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right) \to$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 0 & -3 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right)$. Thus $\mathbf{A}^{-1} = \begin{pmatrix} 7 & 0 & -3 \\ -2 & 0 & 1 \\ -2 & -1 & 1 \end{pmatrix}$.

(ii)(b) $C_{11} = \begin{vmatrix} 1 & -1 \\ 7 & 0 \end{vmatrix} = 7$, $C_{12} = -\begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} = -2$, $C_{13} = \begin{vmatrix} 0 & 1 \\ 2 & 7 \end{vmatrix} = -2$,

$C_{21} = -\begin{vmatrix} 3 & 0 \\ 7 & 0 \end{vmatrix} = 0$, $C_{22} = \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$, $C_{23} = -\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = -1$,

$C_{31} = \begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix} = -3$, $C_{32} = -\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = 1$, $C_{33} = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$.

Thus, $\mathbf{A}_{adj} = \begin{pmatrix} 7 & -2 & -2 \\ 0 & 0 & -1 \\ -3 & 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 7 & 0 & -3 \\ -2 & 0 & 1 \\ -2 & -1 & 1 \end{pmatrix}$, $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{A}_{adj} = \begin{pmatrix} 7 & 0 & -3 \\ -2 & 0 & 1 \\ -2 & -1 & 1 \end{pmatrix}$.

7(a) From **Q1**,

we know that $\begin{matrix} 2x & +3y & +3z & =25 \\ 3x & +2y & +3z & =24 \\ 4x & +y & +2z & =21 \end{matrix}$ can be reduced to $\begin{matrix} x & +1.5y & +1.5z & =12.5 \\ & y & +0.6z & =5.4 \\ & & z & =2 \end{matrix}$.

By backward substitutions, we have

$$z = 2,\ y = 5.4 - 0.6\times 2 = 4.2,\ x = 12.5 - 1.5\times 4.2 - 1.5\times 2 = 3.2.$$

7(b) From **Q1**, we know the linear system can be further reduced to

$\begin{matrix} x & & & =3.2 \\ & y & & =4.2 \\ & & z & =2 \end{matrix}$, which is the solution.

7(c) $\begin{matrix} 2x & +3y & +3z & =25 \\ 3x & +2y & +3z & =24 \\ 4x & +y & +2z & =21 \end{matrix}$ can be rewritten as $\begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 25 \\ 24 \\ 21 \end{pmatrix}$. Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 25 \\ 24 \\ 21 \end{pmatrix} = \begin{pmatrix} 3.2 \\ 4.2 \\ 2 \end{pmatrix}.$$

7(d) $\begin{vmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 4 & 1 & 2 \end{vmatrix} = 5$, $\begin{vmatrix} 25 & 3 & 3 \\ 24 & 2 & 3 \\ 21 & 1 & 2 \end{vmatrix} = 16$, $\begin{vmatrix} 2 & 25 & 3 \\ 3 & 24 & 3 \\ 4 & 21 & 2 \end{vmatrix} = 21$, $\begin{vmatrix} 2 & 3 & 25 \\ 3 & 2 & 24 \\ 4 & 1 & 21 \end{vmatrix} = 10$.

By Cramer's Rule, $x = \frac{16}{5}$, $y = \frac{21}{5}$, $z = \frac{10}{5} = 2$.

The three planes intersect at exactly one point, with coordinates $(3.2, 4.2, 2)$.

8(a)

$$\begin{matrix} x & +2y & +3z & +4w & =5 \\ x & +3y & +5z & +7w & =11 \\ x & & -3z & -2w & =-7 \end{matrix} \Rightarrow \begin{matrix} x & +2y & +3z & +4w & =5 \\ & y & +2z & +3w & =6 \\ & -2y & -6z & -6w & =-12 \end{matrix} \Rightarrow \begin{matrix} x & +2y & +3z & +4w & =5 \\ & y & +2z & +3w & =6 \\ & & -2z & & =0 \end{matrix}$$

$$\Rightarrow \begin{matrix} x & +2y & +3z & +4w & =5 \\ & y & +2z & +3w & =6 \\ & & z & & =0 \end{matrix} \Rightarrow \begin{matrix} x & & -z & -2w & =-7 \\ & y & +2z & +3w & =6 \\ & & z & & =0 \end{matrix} \Rightarrow \begin{matrix} x & & & -2w & =-7 \\ & y & & +3w & =6 \\ & & z & & =0 \end{matrix}$$

Let $w=t$, we have $x=-7+2t, y=6-3t, z=0, w=t, t\in\mathbb{R}$..

9 $\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \xrightarrow{R_2 \times \frac{1}{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The corresponding elementary matrices are $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$. Then

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}.$$

Thus, $\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}^{-1}\mathbf{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

Practice Questions

10

$$\begin{vmatrix} 2 & -5 & -1 \\ 1 & a & 1 \\ -3 & 10 & 2a \end{vmatrix} = 2(a)(2a)+1(10)(-1)+(-3)(-5)1-(-3)a(-1)-1(-5)(2a)-2(10)(1)$$

$$= 4a^2 - 10 + 15 - 3a + 10a - 20$$
$$= 4a^2 + 7a - 15$$
$$= (4a-5)(a+3)$$

When the matrix is singular, $a=-3$ or $a=\dfrac{5}{4}$.

When $b \neq -3$ and $b \neq \dfrac{5}{4}$, the coefficient matrix is invertible and there is exactly one solution, i.e. the three planes intersect at exactly one common point.

When $b=-3$, the augmented matrix is $\begin{pmatrix} 2 & -5 & -1 & 0 \\ 1 & -3 & 1 & -2 \\ -3 & 10 & -6 & 10 \end{pmatrix}$, whose reduced row-echelon form is $\begin{pmatrix} 1 & 0 & -8 & 10 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. There are infinitely many solutions, the three planes intersect in a common line.

When $b=\dfrac{5}{4}$, the augmented matrix is $\begin{pmatrix} 2 & -5 & -1 & 0 \\ 1 & \frac{5}{4} & 1 & -2 \\ -3 & 10 & \frac{5}{2} & 10 \end{pmatrix}$, whose reduced row-echelon form is $\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. There is no solution, the three planes have no point in common. Since the planes are parallel pairwise, they form a triangular prismatic surface.

11

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ {a_1}^2 & {a_2}^2 & {a_3}^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & a_2-a_1 & a_3-a_1 \\ 0 & {a_2}^2-{a_1}^2 & {a_3}^2-{a_1}^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & a_2-a_1 & a_3-a_1 \\ 0 & {a_2}^2-{a_1}^2 & {a_3}^2-{a_1}^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & a_2-a_1 & a_3-a_1 \\ 0 & 0 & (a_3-a_1)(a_3+a_1)-(a_3-a_1)(a_2+a_1) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & a_2-a_1 & a_3-a_1 \\ 0 & 0 & (a_3-a_1)(a_3-a_2) \end{vmatrix}$$

$$=1(a_2-a_1)(a_3-a_1)(a_3-a_2)=(a_2-a_1)(a_3-a_2)(a_3-a_1)$$

12(a)

$$\begin{vmatrix} a & a+b & a+2b \\ a+b & a+2b & a \\ a+2b & a & a+b \end{vmatrix} = 3a(a+2b)(a+b)-a^3-(a+b)^3-(a+2b)^3$$

$$=3a^3-9a^2b-6ab^2-a^3-a^3-3a^2b-3ab^2-b^3-a^3-6a^2b-12ab^2-8b^3$$

$$=-9ab^2-9b^3$$

$$=-9b^2(a+b)$$

If $a=-b$, then $a+b=0$, the determinant is 0.

If the determinant $-9b^2(a+b)=0$, then $a+b=0$ since $b\neq 0$, i.e. $a=-b$.

(b)

$$\begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & b_1 \\ b_2 & c_2 & b_2 \\ b_3 & c_3 & b_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_1 & a_1 \\ b_2 & a_2 & a_2 \\ b_3 & a_3 & a_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_1 & b_1 \\ b_2 & a_2 & b_2 \\ b_3 & a_3 & b_3 \end{vmatrix}$$

$$+\begin{vmatrix} c_1 & c_1 & a_1 \\ c_2 & c_2 & a_2 \\ c_3 & c_3 & a_3 \end{vmatrix} + \begin{vmatrix} c_1 & c_1 & b_1 \\ c_2 & c_2 & b_2 \\ c_3 & c_3 & b_3 \end{vmatrix} + \begin{vmatrix} c_1 & a_1 & a_1 \\ c_2 & a_2 & a_2 \\ c_3 & a_3 & a_3 \end{vmatrix} + \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$$

$$=(-1)^2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}+0+0+0+0+0+0+(-1)^2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$=2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

13 $n\det(\mathbf{A})=\det(\mathbf{A}^n)=\det(\mathbf{O})=0$ for some positive integer *n*, so $\det(\mathbf{A})=0$ and **A** is not invertible.

14 $\det(\mathbf{A})=\det(\mathbf{A}^T)=\begin{vmatrix}a & d & g\\ b & e & h\\ c & f & i\end{vmatrix}=\begin{vmatrix}a & d & g\\ b & e & h\\ a+c & d+f & g+i\end{vmatrix}=\begin{vmatrix}a & d & g\\ b & e & h\\ a+b+c & d+e+f & g+h+i\end{vmatrix}$

$=\begin{vmatrix}a & d & g\\ b & e & h\\ 0 & 0 & 0\end{vmatrix}=0$. Thus **A** is not invertible.

15 $\det(\mathbf{A}^2)=\left[\det(\mathbf{A})\right]^2$, $\det(-\mathbf{I})=(-1)^n\det(\mathbf{I})=-1$ when *n* is odd.

Since $\left[\det(\mathbf{A})\right]^2\geq 0>-1$, $\det(\mathbf{A}^2)\neq\det(-\mathbf{I})$.

So $\mathbf{A}^2\neq-\mathbf{I}$, i.e. $\mathbf{A}^2+\mathbf{I}=\mathbf{O}$ has no solution when *n* is odd.

16 When **A** is singular,

$$0=\begin{vmatrix}1 & 4 & 3\\ -2 & 8 & a\\ 1 & b & 3\end{vmatrix}=24-6b+4a-24+24-ab=24-6b+4a-ab=(6+a)(4-b),$$

so $a=-6$ or $b=4$.

17 **(i)** Show by induction starting from $m=1$.

(ii) Using the result that $\mathbf{D}^m\mathbf{D}^{-m}=\mathbf{I}$ if **D** is invertible to explain how **(i)** holds.

18

$$\mathbf{A}=\begin{pmatrix}2 & 1 & 3\\ -1 & 0 & 4\\ 3 & 1 & 0\end{pmatrix}\xrightarrow{R1\leftrightarrow R2}\begin{pmatrix}-1 & 0 & 4\\ 2 & 1 & 3\\ 3 & 1 & 0\end{pmatrix}\xrightarrow{R1\times(-1)}\begin{pmatrix}1 & 0 & -4\\ 2 & 1 & 3\\ 3 & 1 & 0\end{pmatrix}\xrightarrow{R2+(-2)R1}\begin{pmatrix}1 & 0 & -4\\ 0 & 1 & 11\\ 3 & 1 & 0\end{pmatrix}$$

$$\xrightarrow{R3+(-3)R1}\begin{pmatrix}1 & 0 & -4\\ 0 & 1 & 11\\ 0 & 1 & 12\end{pmatrix}\xrightarrow{R3+(-1)R2}\begin{pmatrix}1 & 0 & -4\\ 0 & 1 & 11\\ 0 & 0 & 1\end{pmatrix},$$

$$\text{so}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\xrightarrow{R1\leftrightarrow R2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\xrightarrow{R1\times(-1)}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\xrightarrow{R2+(-2)R1}\begin{pmatrix}0 & -1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\xrightarrow{R3+(-3)R1}\begin{pmatrix}0 & -1 & 0\\ 1 & 2 & 0\\ 0 & 3 & 1\end{pmatrix}\xrightarrow{R3+(-1)R2}\begin{pmatrix}0 & -1 & 0\\ 1 & 2 & 0\\ -1 & 1 & 1\end{pmatrix}=\mathbf{P}$$

$$\mathbf{B}=\begin{pmatrix}2 & 1 & 3\\ -1 & 1 & 12\\ 3 & 1 & 0\end{pmatrix}\xrightarrow{R1\leftrightarrow R2}\begin{pmatrix}-1 & 1 & 12\\ 2 & 1 & 3\\ 3 & 1 & 0\end{pmatrix}\xrightarrow{R1\times(-1)}\begin{pmatrix}1 & -1 & -12\\ 2 & 1 & 3\\ 3 & 1 & 0\end{pmatrix}\xrightarrow{R2+(-2)R1}\begin{pmatrix}1 & -1 & -12\\ 0 & 3 & 27\\ 3 & 1 & 0\end{pmatrix}$$

$$\xrightarrow{R3+(-3)R1}\begin{pmatrix}1 & -1 & -12\\ 0 & 3 & 27\\ 0 & 4 & 36\end{pmatrix}\xrightarrow{R2\times\left(\frac{1}{3}\right)}\begin{pmatrix}1 & -1 & -12\\ 0 & 1 & 9\\ 0 & 4 & 36\end{pmatrix}\xrightarrow{R3-4R2}\begin{pmatrix}1 & -1 & -12\\ 0 & 1 & 9\\ 0 & 0 & 0\end{pmatrix},$$

so $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{R1\leftrightarrow R2}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\xrightarrow{R1\times(-1)}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}\xrightarrow{R2+(-2)R1}\begin{pmatrix}0&-1&0\\1&2&0\\0&0&1\end{pmatrix}$

$\xrightarrow{R3+(-3)R1}\begin{pmatrix}0&-1&0\\1&2&0\\0&3&1\end{pmatrix}\xrightarrow{R2\times\left(\frac{1}{3}\right)}\begin{pmatrix}0&-1&0\\\frac{1}{3}&\frac{2}{3}&0\\0&3&1\end{pmatrix}\xrightarrow{R3-4R2}\begin{pmatrix}0&-1&0\\\frac{1}{3}&\frac{2}{3}&0\\-\frac{4}{3}&\frac{1}{3}&1\end{pmatrix}=\mathbf{Q}$.

A has an inverse, **B** has no inverse.

From the row-echelon form $\begin{pmatrix}1&0&-4\\0&1&11\\0&0&1\end{pmatrix}\xrightarrow{R2+(-11)R3,R1+4R3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. Performing the same elementary row operations on **P**, $\begin{pmatrix}0&-1&0\\1&2&0\\-1&1&1\end{pmatrix}\xrightarrow{R2+(-11)R3,R1+4R3}\begin{pmatrix}-4&3&4\\12&-9&-11\\-1&1&1\end{pmatrix}=\mathbf{A}^{-1}$

(a) $\mathbf{x}=\mathbf{A}^{-1}\begin{pmatrix}4\\1\\5\end{pmatrix}=\begin{pmatrix}-4&3&4\\12&-9&-11\\-1&1&1\end{pmatrix}\begin{pmatrix}4\\1\\5\end{pmatrix}=\begin{pmatrix}7\\-16\\2\end{pmatrix}$

(b) $\mathbf{x}=\begin{pmatrix}3t+1\\-9t+2\\t\end{pmatrix}$ by finding the row-echelon form of the augmented matrix.

19

$\mathbf{AA}^{-1}=\mathbf{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, so the first column of $\mathbf{A}^{-1}$ must satisfy the equation $\mathbf{Ax}=\begin{pmatrix}1\\0\\0\end{pmatrix}$.

The second and third columns of $\mathbf{A}^{-1}$ must satisfy the equations $\mathbf{Ay}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\mathbf{Az}=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

Let $\mathbf{P}^{-1}=\begin{pmatrix}\mathbf{x}&\mathbf{y}&\mathbf{z}\end{pmatrix}$. then $\mathbf{Px}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{Py}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{Pz}=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

Let $\mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, then $x_3=0, x_2+cx_3=0, x_1+ax_2+bx_3=1$, so $\mathbf{x}=\begin{pmatrix}1\\0\\0\end{pmatrix}$.

Let $\mathbf{y}=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$, then $y_3=0, y_2+cy_3=1, y_1+ay_2+by_3=0$, so $\mathbf{y}=\begin{pmatrix}-a\\1\\0\end{pmatrix}$.

Let $\mathbf{z}=\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$, then $z_3=1, z_2+cz_3=0, z_1+az_2+bz_3=0$, so $\mathbf{z}=\begin{pmatrix}ac-b\\-c\\1\end{pmatrix}$.

Thus, $\mathbf{P}^{-1}=\begin{pmatrix}1 & -a & ac-b\\0 & 1 & -c\\0 & 0 & 1\end{pmatrix}$. $\mathbf{BP}=\begin{pmatrix}3 & -1 & 2\\1 & 0 & 5\end{pmatrix}\Rightarrow \mathbf{BPP}^{-1}=\begin{pmatrix}3 & -1 & 2\\1 & 0 & 5\end{pmatrix}\begin{pmatrix}1 & -a & ac-b\\0 & 1 & -c\\0 & 0 & 1\end{pmatrix}$

$$\mathbf{B}=\begin{pmatrix}3 & -3a-1 & 3ac-3b+c+2\\1 & -a & ac-b+5\end{pmatrix}$$

20 (i) $(a_4-a_3)(a_4-a_2)(a_4-a_1)(a_3-a_2)(a_3-a_1)(a_2-a_1)$.

(ii) By Factor Theorem, we can verify the expression in **(i)** is a factor of the determinant as substituting a_i with a_j will always result in two identical columns, making the determinant 0. From PQ11, $\begin{vmatrix}1 & 1 & 1 & 1\\a_1 & a_2 & a_3 & a_4\\a_1^2 & a_2^2 & a_3^2 & a_4^2\\a_1^3 & a_2^3 & a_3^3 & a_4^3\end{vmatrix}$

$$=-a_1^3\begin{vmatrix}1 & 1 & 1\\a_2 & a_3 & a_4\\a_2^2 & a_3^2 & a_4^2\end{vmatrix}+a_2^3\begin{vmatrix}1 & 1 & 1\\a_1 & a_3 & a_4\\a_1^2 & a_3^2 & a_4^2\end{vmatrix}-a_3^3\begin{vmatrix}1 & 1 & 1\\a_1 & a_2 & a_4\\a_1^2 & a_2^2 & a_4^2\end{vmatrix}+a_4^3\begin{vmatrix}1 & 1 & 1\\a_1 & a_2 & a_3\\a_1^2 & a_2^2 & a_3^2\end{vmatrix}$$

$$=-a_1^3(a_4-a_3)(a_4-a_2)(a_3-a_2)+a_2^3(a_4-a_3)(a_4-a_1)(a_3-a_1)$$
$$-a_3^3(a_4-a_2)(a_4-a_1)(a_2-a_1)+a_4^3(a_3-a_2)(a_3-a_1)(a_2-a_1)$$

The degree of the determinant is at most 6 and the degree of the expression in **(i)** is 6. Thus,the determinant $=k(a_4-a_3)(a_4-a_2)(a_4-a_1)(a_3-a_2)(a_3-a_1)(a_2-a_1)$.

To find the value of *k*, we can substitute $a_1=-1, a_2=0, a_3=1, a_4=2$:

$\text{LHS}=1(1)(2)(1)+0-1(2)(3)(1)+8(1)(2)(1)=12$

$\text{RHS}=k(1)(2)(3)(1)(2)(1)=12k$, so $k=1$.

(iii) The determinant is $\prod_{1\le i<j\le n}(a_j-a_i)$. Similar to **(ii)**, we can first verify the factors and the degree, to conclude that the determinant must be $\lambda\prod_{1\le i<j\le n}(a_j-a_i)$.

To find the value of λ, we substitute $a_1=0$.

The determinant now, assuming it is true of $(n-1)\times(n-1)$, is

$$\begin{vmatrix}a_2 & a_3 & \cdots & a_n\\a_2^2 & a_3^2 & \cdots & a_n^2\\\vdots & \vdots & & \vdots\\a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1}\end{vmatrix}=a_2a_3\ldots a_n\begin{vmatrix}1 & 1 & \cdots & 1\\a_2 & a_3 & \cdots & a_n\\\vdots & \vdots & & \vdots\\a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2}\end{vmatrix}=a_2a_3\ldots a_n\prod_{2\le i<j\le n}(a_j-a_i)$$

At the same time,

$$\lambda\prod_{1\le i<j\le n}(a_j-a_i)=\lambda\prod_{2\le i<j\le n}(a_j-a_i)(a_2-0)(a_3-0)\ldots(a_n-0)=\lambda a_2a_3\ldots a_n\prod_{2\le i<j\le n}(a_j-a_i)$$

We have $\lambda=1$.

Since we have verified some initial cases, by mathematical induction, the determinant of such $n\times n$ matrix is $\prod_{1\le i<j\le n}(a_j-a_i)$.

21 **(a)** Since $\mathbf{AB} = \mathbf{BA}$

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^3 &= (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) \\ &= (\mathbf{AA}+\mathbf{AB}+\mathbf{BA}+\mathbf{BB})(\mathbf{A}+\mathbf{B}) \\ &= (\mathbf{AA}+2\mathbf{AB}+\mathbf{BB})(\mathbf{A}+\mathbf{B}) \\ &= \mathbf{AAA}+2\mathbf{ABA}+\mathbf{BBA}+\mathbf{AAB}+2\mathbf{ABB}+\mathbf{BBB} \\ &= \mathbf{AAA}+2\mathbf{AAB}+\mathbf{BAB}+\mathbf{AAB}+2\mathbf{ABB}+\mathbf{BBB} \\ &= \mathbf{AAA}+2\mathbf{AAB}+\mathbf{ABB}+\mathbf{AAB}+2\mathbf{ABB}+\mathbf{BBB} \\ &= \mathbf{A}^3+3\mathbf{A}^2\mathbf{B}+3\mathbf{AB}^2+\mathbf{B}^3\end{aligned}$$

(b) When $\mathbf{PC} = \mathbf{CP}$,

$$\begin{aligned}\mathbf{P}^2-\mathbf{PC}-6\mathbf{C}^2 &= \mathbf{P}^2-3\mathbf{PC}+2\mathbf{PC}-6\mathbf{C}^2 \\ &= \mathbf{P}(\mathbf{P}-3\mathbf{C})+2\mathbf{CP}-6\mathbf{C}^2 \\ &= \mathbf{P}(\mathbf{P}-3\mathbf{C})+2\mathbf{C}(\mathbf{P}-3\mathbf{C}) \\ &= (\mathbf{P}+2\mathbf{C})(\mathbf{P}-3\mathbf{C})\end{aligned}$$

Observe that the product is $\mathbf{O}$ if $\mathbf{P} = -2\mathbf{C}$ or $\mathbf{P} = 3\mathbf{C}$.
(Note that this does not imply the matrix equation has only two solutions)

(c) (Note that verification is NOT a valid approach to show all the solutions are of the given form!)

Let $\mathbf{Q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\mathbf{QD} = \begin{pmatrix} a+b & a \\ c+d & c \end{pmatrix}$, $\mathbf{DQ} = \begin{pmatrix} a+c & b+d \\ a & c \end{pmatrix}$.

Now we have $\begin{cases} a+b = a+c \\ a = b+d \\ c+d = a \\ c = c \end{cases} \Rightarrow \begin{cases} a = c+d \\ b = c \end{cases}$.

$\mathbf{Q} = \begin{pmatrix} c+d & c \\ c & d \end{pmatrix} = \alpha\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \beta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $\alpha = c$ and $\beta = d$.

22 (First suggest the general forms for symmetric and skew-symmetric matrices)

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be symmetric, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, so $b = c$. Thus a 2×2 symmetric matrix takes the form $\begin{pmatrix} a & c \\ c & d \end{pmatrix}$.

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be skew-symmetric, $-\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, so $a = d = 0$ and $b = -c$. Thus a 2×2 skew-symmetric matrix takes the form $\begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}$.

For $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ to be skew-symmetric, $-\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$, so

$a = e = i = 0, b = -d, f = -h, c = -g$. Thus a 3×3 skw-symmetric matrix takes the form $\begin{pmatrix} 0 & -d & -g \\ d & 0 & -h \\ g & h & 0 \end{pmatrix}$.

We can use this results to verify **(i)(iii)(iv)(v)** are true.

(ii) is false with a possible counterexample $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0$.

23 When $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$,

$$\text{LHS} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \mathbf{O}$$

Now $\mathbf{A}^2 = 4\mathbf{A} + 5\mathbf{I}$ ($b_2 = 4, c_2 = 5$)

Assuming $\mathbf{A}^k = b_k\mathbf{A} + c_k\mathbf{I}$, we have

$$\mathbf{A}^{k+1} = (b_k\mathbf{A} + c_k\mathbf{I})\mathbf{A} = b_k\mathbf{A}^2 + c_k\mathbf{A} = b_k(4\mathbf{A} + 5\mathbf{I}) + c_k\mathbf{A} = (4b_k + c_k)\mathbf{A} + 5b_k\mathbf{I} = b_{k+1}\mathbf{A} + c_{k+1}\mathbf{I}$$

By induction, the statement is true for all $n \geq 2$.

Since $\mathbf{A}^1 = 1\mathbf{A} + 0\mathbf{I}$, the statement is true for all positive integer *n*.

$\mathbf{A}^3 = (4\times 4 + 5)\mathbf{A} + (5\times 4)\mathbf{I} = 21\mathbf{A} + 20\mathbf{I}$, $\mathbf{A}^4 = (4\times 21 + 20)\mathbf{A} + (5\times 21)\mathbf{I} = 104\mathbf{A} + 105\mathbf{I}$.

$$\begin{aligned}\mathbf{B} &= \mathbf{A}^4 - 3\mathbf{A}^3 - 7\mathbf{A}^2 - 10\mathbf{A} - 6\mathbf{I} \\ &= 104\mathbf{A} + 105\mathbf{I} - 63\mathbf{A} - 60\mathbf{I} - 28\mathbf{A} - 35\mathbf{I} - 10\mathbf{A} - 6\mathbf{I} \\ &= 3\mathbf{A} + 4\mathbf{I} \\ &= \begin{pmatrix} 7 & 6 \\ 12 & 13 \end{pmatrix}\end{aligned}$$

24 **(i)** By performing the same row operations on **I**, $\mathbf{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}$.

(ii) $$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & -3 & -6 \\ 4 & -9 & -13 & -25 \end{pmatrix} \xrightarrow{R2-2R1, R3-4R1} \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & -7 & -7 & -14 \\ 0 & -21 & -21 & -41 \end{pmatrix}$$

$$\xrightarrow{R3-3R2} \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & -7 & -7 & -14 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R2\times\left(-\frac{1}{7}\right)} \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$. Performing the same row operations on **I**:

$$\mathbf{I}\xrightarrow{R2-R1,R3-4R1}\begin{pmatrix}1&0&0\\-2&1&0\\-4&0&1\end{pmatrix}\xrightarrow{R3-3R2}\begin{pmatrix}1&0&0\\-2&1&0\\2&-3&1\end{pmatrix}\xrightarrow{R2\times\left(-\frac{1}{7}\right)}\begin{pmatrix}1&0&0\\\frac{2}{7}&-\frac{1}{7}&0\\2&-3&1\end{pmatrix}.$$

Thus $\mathbf{P}=\begin{pmatrix}1&0&0\\\frac{2}{7}&-\frac{1}{7}&0\\2&-3&1\end{pmatrix}, x=1, y=2, s=0, t=1.$

$$\mathbf{Bx}=\begin{pmatrix}5\\3\\-1\end{pmatrix}\Rightarrow\mathbf{PBx}=\mathbf{P}\begin{pmatrix}5\\3\\-1\end{pmatrix}\Rightarrow\begin{pmatrix}1&3&2&4\\0&1&1&2\\0&0&0&1\end{pmatrix}\mathbf{x}=\begin{pmatrix}5\\1\\0\end{pmatrix}.$$

Let $\mathbf{x}=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$, then $\begin{cases}a+3b+2c+4d=5\\b+c+2d=1\\d=0\end{cases}$. By letting $c=\lambda$, we have $\mathbf{x}=\begin{pmatrix}2+\lambda\\1-\lambda\\\lambda\\0\end{pmatrix}$.

25 $\mathbf{A}^{-1}=\begin{pmatrix}2&-1&-1\\-1&2&0\\-4&0&3\end{pmatrix}$ (working omitted).

B is obtained by dividing the first, second and third rows of **A** by 6, 6 and 24 respectively, so $\mathbf{B}^{-1}$ can be obtained by multiplying the first, second and third columns of $\mathbf{A}^{-1}$ by 6, 6 and 24 respectively.

$$\mathbf{B}^{-1}=\begin{pmatrix}12&-6&-24\\-6&12&0\\-24&0&72\end{pmatrix}.$$

(Note that verification is NOT a valid approach to show all the solutions are of the given form!)

Since **B** is invertible, $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\mathbf{B}^{-1}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$. Now $\mathbf{x}=\mathbf{B}^{-1}\begin{pmatrix}c_1+\delta\\c_2-\delta\\c_3-\delta\end{pmatrix}=\mathbf{B}^{-1}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}+\delta\mathbf{B}^{-1}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$

$$=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\delta\begin{pmatrix}12&-6&-24\\-6&12&0\\-24&0&72\end{pmatrix}\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\delta\begin{pmatrix}42\\-18\\-96\end{pmatrix}=\begin{pmatrix}x_1+42\delta\\x_2-18\delta\\x_3-96\delta\end{pmatrix}$$

26 $\mathbf{A}^{-1}=\begin{pmatrix}-1&2&3\\2&1&0\\-4&2&5\end{pmatrix}$ (working omitted)

(a) $\mathbf{XA}=\mathbf{K}\Rightarrow\mathbf{XAA}^{-1}=\mathbf{KA}^{-1}\Rightarrow\mathbf{X}=\begin{pmatrix}-7&6&12\end{pmatrix}.$

(b) $\begin{pmatrix}x&y&z&t\end{pmatrix}\begin{pmatrix}-1&2&3\\-5&4&3\\10&-7&-6\\-8&6&5\end{pmatrix}\mathbf{A}^{-1}=\begin{pmatrix}2&-2&1\end{pmatrix}\mathbf{A}^{-1}$

From **(a)**, $\begin{pmatrix} x & y & z & t \end{pmatrix}\begin{pmatrix} -7 & 6 & 12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \end{pmatrix}\mathbf{A}^{-1} = \begin{pmatrix} -10 & 4 & 11 \end{pmatrix}$

$\begin{cases} -7x + y = -10 \\ 6x + z = 4 \\ 12x + t = 11 \end{cases}$. By letting $x = \lambda$, the solution is

$x = \lambda, y = 7\lambda - 10, z = 4 - 6\lambda, t = 11 - 12\lambda$.

27 $\mathbf{M}^2 = \mathbf{I} \Rightarrow \det(\mathbf{M}^2) = \det(\mathbf{I}) = 1 \Rightarrow \det(\mathbf{M}) \neq 0$, so $\mathbf{M}^{-1}$ exists.

$\mathbf{M}^2 = \mathbf{I} \Rightarrow \mathbf{M}^{-1}(\mathbf{MM}) = \mathbf{M}^{-1}\mathbf{I} \Rightarrow (\mathbf{M}^{-1}\mathbf{M})\mathbf{M} = \mathbf{M}^{-1} \Rightarrow \mathbf{IM} = \mathbf{M}^{-1} \Rightarrow \mathbf{M} = \mathbf{M}^{-1}$.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{M}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$

$\begin{cases} a^2 + bc = 1 \\ b(a + d) = 0 \\ c(a + d) = 0 \\ bc + d^2 = 1 \end{cases}$.

Case 1: When $b = 0$, $a^2 = d^2 = 1$.

Case 1.1: $a = d = 1$, then $c = 0$ and $\mathbf{M} = \mathbf{I}$.

Case 1.2: $a = d = -1$, then $c = 0$ and $\mathbf{M} = -\mathbf{I}$.

Case 1.3: $a = -d = \pm 1$, then $a + d = 0$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -1$.

Case 2: When $b \neq 0$, $a = -d$, $1 = a^2 + bc = -ad + bc$. Then $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -1$.

So $\mathbf{M} = \mathbf{I}$ or $\mathbf{M} = -\mathbf{I}$ or ($\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -1$ and $a + d = 0$).

Application Problems

28 **(i)** By considering the traffic flow at each of the 6 junctions:

$$\begin{array}{rrrrrrrl} x & +y & & & & & & = 800 \\ x & -y & & +u & & & & = 400 \\ & y & -z & & & & & = 600 \\ & & z & & & & -t & = -1200 \\ & & & u & & +w & -t & = 0 \\ & & & & v & +w & & = 1000 \end{array}.$$

(ii) By assigning $w = \lambda$ and $t = \mu$, the solution is

$(x, y, z, u, v, w, t) = (\lambda - 200, \mu - 600, \mu - 1200, \mu - \lambda, 1000 - \lambda, \lambda, \mu)$, $\lambda, \mu \in \mathbb{R}$.

(iii) By letting $x = y = 0$, then $\lambda = 200, \mu = 600$. $z = -600$. The direction of the traffic flow should be from D to C instead.

29

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x & y & z & 1 \end{vmatrix} = 0 \Rightarrow -\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} x + \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} y - \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} z = 0$$

is the equation of a plane.

When we substitute $(x, y, z) = (x_i, y_i, z_i)$, $i = 1, 2, 3$, the equation holds as the matrix will have two identical rows. Thus the plane passes through the three points.

For the fourth point to be coplanar, we need $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$.

30 **(a)** **(i)** The corresponding uncoded row matrices of 'HELLO WORLD' are $\begin{pmatrix} -6 & -9 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 10 & 2 & 5 \end{pmatrix}, \begin{pmatrix} -2 & -10 & 0 \end{pmatrix}$. Multiplying them by **A** to the right, we obtain the cryptogram,

$$-79 \quad -44 \quad -27 \quad -6 \quad -2 \quad -2 \quad 106 \quad 54 \quad 37 \quad -42 \quad -26 \quad -14.$$

(ii) Multiplying the row matrices from the cryptogram, $\begin{pmatrix} -46 & -23 & -16 \end{pmatrix}, \begin{pmatrix} -33 & -12 & -14 \end{pmatrix}, \begin{pmatrix} -32 & -20 & -11 \end{pmatrix}$, by $\mathbf{A}^{-1}$ to the right, we obtain the uncoded row matrices,

$$\begin{pmatrix} -5 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 9 & -9 \end{pmatrix}, \begin{pmatrix} 0 & -8 & -1 \end{pmatrix},$$

which correspond to the text "I LOVE FM"

The matrix must be invertible, otherwise the cryptogram cannot be decoded.

(b) **(i)** There are 9 unknown entries in the matrix **B**, every uncoded row matrices of 3 entries and its corresponding row matrices in the cryptogram will generate 3 equations. Since we need to solve for 9 unknowns, there must be at least 9 equations to get the specific values.

(ii) A system of 9 equations in 9 unknowns may not have a unique solution.

A larger $n \times n$ matrix can be used to enhance security as the hacker needs to have the cryptogram of a text message of at least n^2 to find the matrix.

Using more than one matrix to encrypt the message, for example, multiply by two different $n \times n$ matrices in order. Though there are only $2n^2$ unknowns, the equations are non-linear and much harder to solve.