H2 2017 Preliminary Exam Paper 2 Question
Section A: Pure Matheatics [40 marks].

1

The cubic equation $az^3 - 31z^2 + 212z + b = 0$, where a and b are real numbers, has a complex root $z = 1 - 3\text{i}$.

(i) Explain why the equation must have a real root. [2]

(ii) Find the values of a and b and the real root, showing your working clearly. [5]

2

Relative to the origin O, the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{a}+\mathbf{c}$ and $\mathbf{c}$ respectively. The point X is on AC produced such that $AC:CX$ is $2:3$ and the point Y is such that $AXYB$ is a parallelogram.

(i) The lines AY and BX intersect at the point N. Show that $\overrightarrow{ON} = \frac{1}{4}(\mathbf{7c} - \mathbf{a})$. [3]

(ii) Given that the area of triangle OAB is 4 square units, find the area of triangle OAN. [4]

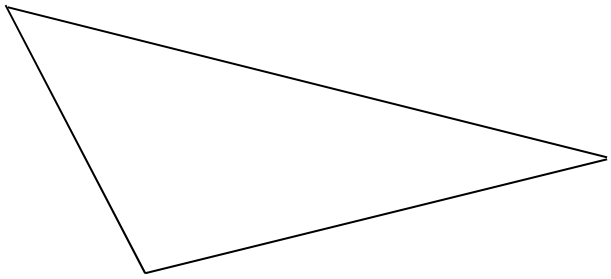
(iii) Give a geometrical interpretation of $\left|\overrightarrow{OA} \times \frac{\overrightarrow{AN}}{|\overrightarrow{AN}|}\right|$. Use the results from part **(ii)** to show that

$$\left|\overrightarrow{OA} \times \frac{\overrightarrow{AN}}{|\overrightarrow{AN}|}\right| = \frac{56}{|7\mathbf{c} - 5\mathbf{a}|}.$$ [3]

3

(a) Find the series expansion of $\text{e}^{2x}\ln(1+3x)$, where $-\frac{1}{3} < x \leq \frac{1}{3}$, in ascending powers of x, up to and including the term in x^3. [3]

(b) In the triangle PQR as shown in the diagram below, $PR = 1$, angle $QPR = \frac{3\pi}{4}$ radians and angle $PRQ = 2\theta$ radians.



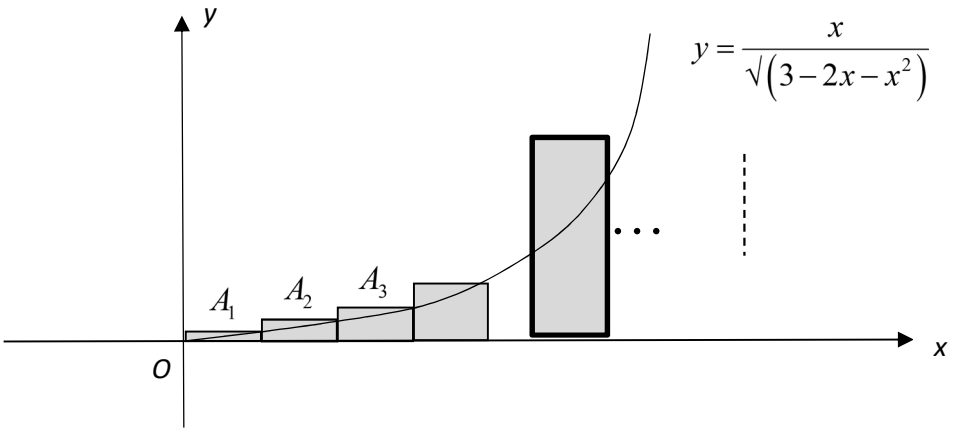
(i) Show that $QR = \frac{1}{\cos 2\theta - \sin 2\theta}$. [4]

(ii) Given that θ is sufficiently small angle, show that $QR \approx 1 + a\theta + b\theta^2$, for constants a and b to be determined. [4]

4

(a) Find $\int \text{e}^x \sin x \, \text{d}x$. [3]

(b)



The diagram shows the curve with equation $y = \frac{x}{\sqrt{(3-2x-x^2)}}$ for $0 \le x < 1$.

The region bounded by the curve, the x-axis and the line $x = k$, $0 < k < 1$ is denoted by S.

It is given that n rectangles of equal width are drawn between $x = 0$ and $x = k$.

(i) Show that the area of the first rectangle, $A_1 = \frac{k^2}{n\sqrt{3n^2 - 2nk - k^2}}$. [1]

(ii) Show that the total area of all the n rectangles is

$$\sum_{r=1}^{n} \frac{rk^2}{n\sqrt{(3n^2 - anrk - br^2k^2)}},$$

where a and b are constants to be determined. [2]

It is now given that $k = (\sqrt{3}) - 1$.

(iii) Use integration to find the actual area of region S. Hence state the exact value of

$$\sum_{r=1}^{\infty} \frac{rk^2}{n\sqrt{(3n^2 - anrk - br^2k^2)}}.$$ [6]

Section B: Probability and Statistics [60 marks]

5 An unbiased six-sided die is rolled twice. The random variable X represents the higher of the two values if they are different, and their common value if they are the same. The probability distribution of X is given by the formula

$$P(X = r) = k(2r - 1) \quad \text{for } r = 1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

(i) Find the exact value of k, giving your answer as a fraction in its simplest form. [2]

(ii) Find the expectation of X. [2]

A round of the game consists of rolling the unbiased six-sided die twice, and X is taken as the score for the round. A player plays three rounds of the game.

(iii) Find the probability that the total score for the three rounds is 16. [2]

6 A geologist splits rocks to look for fossils. On average 7% of the rocks selected from a particular area contain fossils.

The geologist selects a random sample of 20 rocks from this area.

(i) Find the probability that at least three of the rocks contain fossils. [2]

A random sample of n rocks is selected from this area.

(ii) The geologist wants to have a probability of 0.8 or greater of finding fossils in at least three of these rocks. Find the least possible value of n. [3]

In early 2017, geologists found the fossils of *zilantophis schuberti*, a new discovered species of winged serpent. On average, the proportion of rocks that contain fossils of *zilantophis schuberti* in this area is p. It is known that the modal number of fossils of *zilantophis schuberti* in a random sample of 10 rocks is 3.

(iii) Use this information to find exactly the range of values that p can take. [4]

7 A pilot records the take-off distance, S metres, for his private aircraft on runways at various altitudes of h metres. The data are shown in the table below.

h	0	300	600	900	1200	1500	1800
S	635	690	750	840	950	1080	1250

(i) Plot a scatter diagram on graph paper for these values, labelling the axes, using a scale of 2 cm to represent a take-off distance of 100 metres on the y-axis and an appropriate scale for the x-axis. [2]

It is thought that the take-off distance S can be modelled by one of the formulae

$$S = ah + b \quad \text{or} \quad S = ch^2 + d,$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) h and S,

(b) h^2 and S. [2]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $S = ah + b$ or $S = ch^2 + d$ is the better model. [2]

(iv) Find the equation of the least-square regression line for the model you have chosen in part **(iii)**. [1]

(v) Use the equation of your regression line to estimate the take-off distance for altitude of 2200 metres. Comment on the reliability of your estimate when $h = 2200$. [2]

8 A manufacturing plant processes raw material for a supplier. An order placed with the plant is considered to be a bulk order when a worker is expected to process more than 300 kg (kilograms) of raw material.

Albert uses a machine to process X kg of raw material and Bob uses a separate machine to process Y kg of raw material on a working day. X and Y are independent random variables with the distributions $\mathrm{N}(296,\ 8^2)$ and $\mathrm{N}(290,\ 12^2)$ respectively.

(i) Find the probability that Albert processes more than 300 kg of raw material on a randomly selected working day. [2]

(ii) Find the probability that, over a period of 15 independent working days, there are exactly four working days on which Albert processes more than 300 kg of raw material. [2]

(iii) Find the probability that the total amount of raw material Bob processes over two working days exceeds twice the amount of raw material Albert processes on one working day. [4]

The plant receives a bulk order and Albert wants to have a probability of at least 0.95 of meeting the order.

(iv) This can be done by changing the value of μ, the mean amount of raw material Albert processes using the machine, but the standard deviation remains unchanged. Find the least value of μ. [3]

9 The town council is investigating the mass of rubbish in domestic dustbins. In 2016, the mean mass of rubbish in domestic dustbins was 20.0 kg per household per week. The town council starts a recycling initiative and wishes to determine whether there has been a reduction in the mass of rubbish in domestic dustbins.

The mass of rubbish in a domestic dustbin is denoted by X kg. A random sample of 50 domestic dustbins is selected and the results are summarised as follows.

$$n = 50 \qquad \sum x = 924.5 \qquad \sum x^2 = 18249.2$$

(i) Explain what is meant in this context by the term 'a random sample'. [2]

(ii) Explain why the town council is able to carry out a hypothesis test without knowing anything about the distribution of the mass of rubbish in domestic dustbins. [2]

(iii) Find the unbiased estimates of the population mean and variance and carry out the test at the 1% level of significance for the town council. [6]

(iv) Use your results in part **(iii)** to find the range of values of n for which the result of the test would be that the null hypothesis is rejected at the 1% level of significance. [3]

10 The number of employees of a statutory board, classified by department and years of working experience, is shown below.

	5 years or less	5 to 10 years	10 years or more	Total
Human Resource Department	20	50	30	100
Legal Department	15	60	45	120
Finance Department	25	30	45	100
Total	60	140	120	320

The Managing Director of the statutory board wishes to select three employees to participate in an overseas conference. The Managing Director selects one employee from each department to participate in the conference.

	(i)	Find the probability that two of the selected employees have years of working experience '10 years or more' and the remaining one has years of working experience '5 years or less'. [3]
	(ii)	Given that exactly one of the selected employees has years of working experience '5 years or less', find the probability that one of the selected employees is from the Legal Department and has years of working experience '5 to 10 years'. [3]

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