2023 Year 6 H2 Mathematics Preliminary Examination Paper 1: Solutions

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<tr><th>1</th><th>Solution</th></tr>
<tr><td>[4]</td><td>

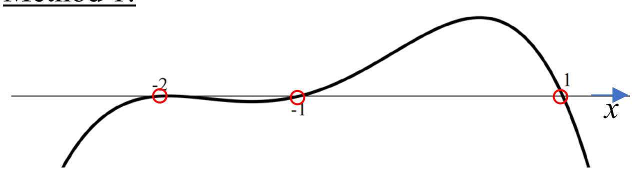
$\frac{9}{(1-x)(1+x)} < \frac{x+5}{x+1}, \quad x \neq \pm 1$

$\frac{9-(x+5)(1-x)}{(1-x)(1+x)} < 0$

$\frac{x^2+4x+4}{(1-x)(1+x)} < 0$

$(x+2)^2(1-x)(1+x) < 0 \cdots (*)$

<u>Method 1:</u>



$\therefore x < -2 \quad \text{or} \quad -2 < x < -1 \quad \text{or} \quad x > 1$

<u>Method 2:</u>

Since $(x+2)^2 \geq 0$ for $x \in \mathbb{R}$, (*) is equivalent to

$(1-x)(1+x) < 0$ and $x \neq -2$

$\therefore x < -1 \quad \text{or} \quad x > 1 \quad \text{and} \quad x \neq -2$

</td></tr>
</table>

<table>
<tr><th>2</th><th>Solution</th></tr>
<tr><td>[4]</td><td>

Let \$$x$, \$$y$ and \$$z$ denote the usual selling price of a small, medium and large bag of Griffles popcorn respectively.

To receive a total of 3 small, 7 medium and 1 large bag of Griffles popcorn, Beatrice bought 2 small, 5 medium and 1 large bag of popcorn.

$0.95(3x+7y+z) = 85.5 \quad \Rightarrow 3x+7y+z = 90 \cdots (1)$

$2x+5y+z = 85.5-18.50 \quad \Rightarrow 2x+5y+z = 67 \cdots (2)$

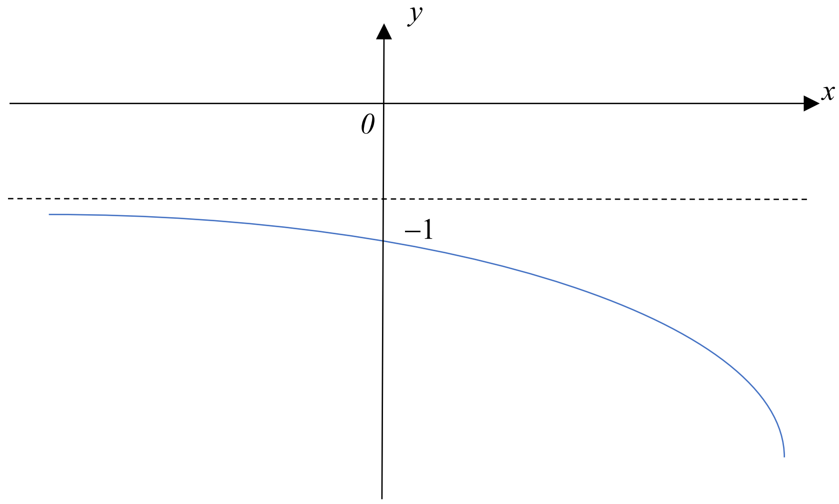
$z = 2.4x \quad \Rightarrow 2.4x - z = 0 \cdots (3)$

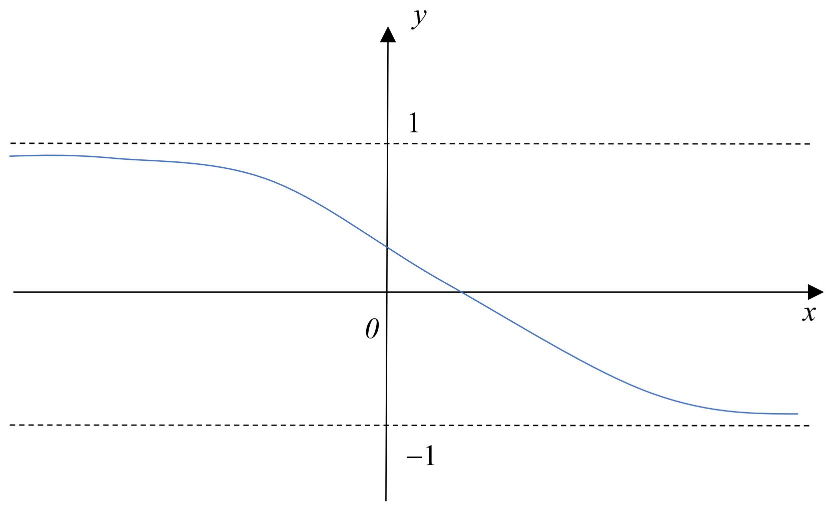
On solving, $x = 5$, $y = 9$ and $z = 12$

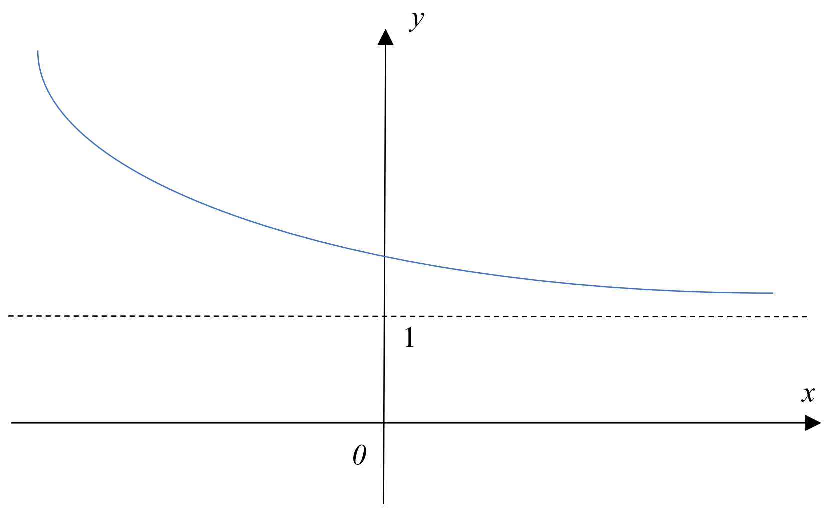
The usual selling price of a small, medium and large bag of Griffles popcorn is \$5, \$9 and \$12 respectively.

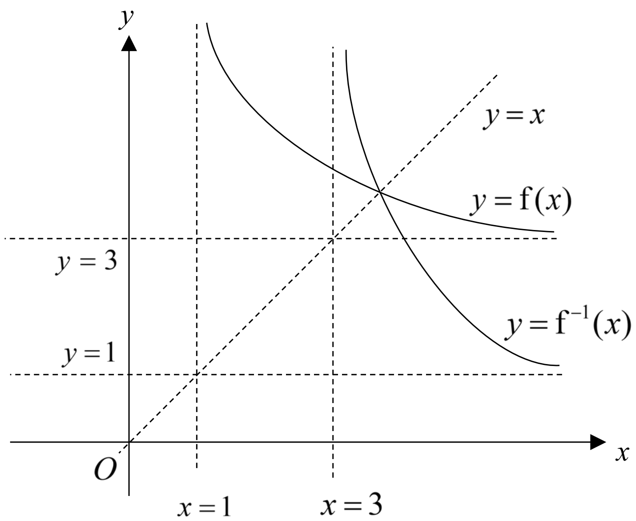
</td></tr>
</table>

3	**Solution**
(a) **[2]**	We have $A = 4\pi r^2 \Rightarrow \dfrac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r\dfrac{\mathrm{d}r}{\mathrm{d}t}$. Hence $-100 = 8\pi(5)\dfrac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow \dfrac{\mathrm{d}r}{\mathrm{d}t} = -0.796$ cm/s <u>Alternative</u> $A = 4\pi r^2 \Rightarrow \dfrac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$ $\dfrac{\mathrm{d}r}{\mathrm{d}t} = \dfrac{\mathrm{d}A}{\mathrm{d}t}\times\dfrac{\mathrm{d}r}{\mathrm{d}A}$ $= -100\times\dfrac{1}{8\pi(5)}$ $= -\dfrac{5}{2\pi}$ Hence the radius is decreasing at $\dfrac{5}{2\pi}$ cm/s.
(b) **[2]**	Volume of meteorite, $V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2\dfrac{\mathrm{d}r}{\mathrm{d}t}$ Since V decreases with t, we have $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -kA$ for proportionality constant $k > 0$. This means that $4\pi r^2\dfrac{\mathrm{d}r}{\mathrm{d}t} = -k\left(4\pi r^2\right) \Rightarrow \dfrac{\mathrm{d}r}{\mathrm{d}t} = -k$, which is a negative constant. Hence the radius is decreasing at a constant rate. <u>Alternative</u> We have $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -kA = -k(4\pi r^2)$ for proportionality constant $k > 0$. $V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$ $\dfrac{\mathrm{d}r}{\mathrm{d}t} = \dfrac{\mathrm{d}V}{\mathrm{d}t}\times\dfrac{\mathrm{d}r}{\mathrm{d}V}$ $= -k(4\pi r^2)\times\dfrac{1}{4\pi r^2}$ $= -k < 0$ Since k is a negative constant, thus the radius is decreasing at a constant rate.

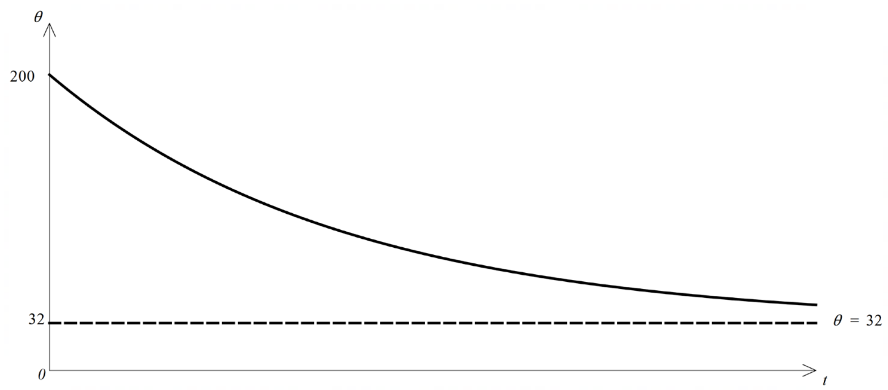
4	Solution
(a) **[3]**	**Method 1** Reflect the graph of $y=\mathrm{f}\left(x\right)$ about the *x*-axis, followed by scaling of the resulting graph by a factor of 2 parallel to the *y*-axis, and translating the resulting graph by 1 unit in the positive *y*-direction. **Method 2** Translate the graph of $y=\mathrm{f}\left(x\right)$ by $\frac{1}{2}$ unit in the negative *y*-direction, followed by reflecting the resulting graph about the *x*-axis, and scaling by a factor of 2 parallel to the *y*-axis. **Method 3:** Reflect the graph of $y=\mathrm{f}\left(x\right)$ about the *x*-axis, followed by translating the resulting graph by $\frac{1}{2}$ unit in the positive *y*-direction, and scaling by a factor of 2 parallel to the *y*-axis. **Method 4:** Scale the graph of $y=\mathrm{f}\left(x\right)$ by a factor of 2 parallel to the *y*-axis, followed by translating the resulting graph by 1 unit in the negative *y*-direction and then reflecting about the *x*-axis.
(b) **[2]**	A point is *R*-invariant if $\left(a,b\right)=\left(a,\frac{1}{b}\right)\Leftrightarrow b^2=1\Leftrightarrow b=\pm1$. Hence if there are no *R*-invariant points, the graph must not intersect the lines $y=\pm1$. Since $\mathrm{f}'(x)<0$, the graph is strictly decreasing for all real values of *x* and some possible graphs are thus 


y
1
0
x
−1

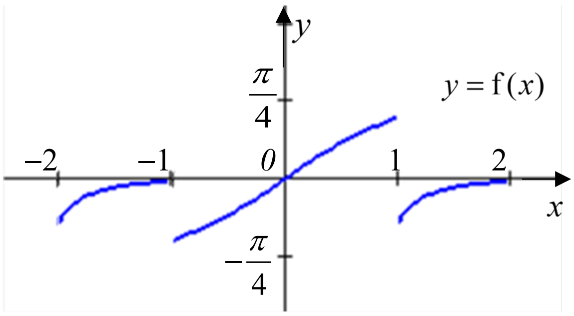

y
1
x
0

5	Solution
(a) **[3]**	$y=\dfrac{3x+2}{x-1}$ $yx-y=3x+2$ $(y-3)x=2+y$ $x=\dfrac{y+2}{y-3}$ $f^{-1}(x)=\dfrac{x+2}{x-3}$ $D_{f^{-1}}=R_f=(3,\infty)$ OR: $y=\dfrac{3x+2}{x-1}=3+\dfrac{5}{x-1}$ $y-3=\dfrac{5}{x-1}$ $x-1=\dfrac{5}{y-3}$ $x=1+\dfrac{5}{y-3}$ $f^{-1}(x)=1+\dfrac{5}{x-3}$, $D_{f^{-1}}=R_f=(3,\infty)$
(b) **[5]**	$D_f=(1,\infty)$ $D_{f^{-1}}=(3,\infty)$ $R_f=(3,\infty)$ $R_{f^{-1}}=(1,\infty)$  From graph, $f(x)=f^{-1}(x)$ intersects at $y=x$

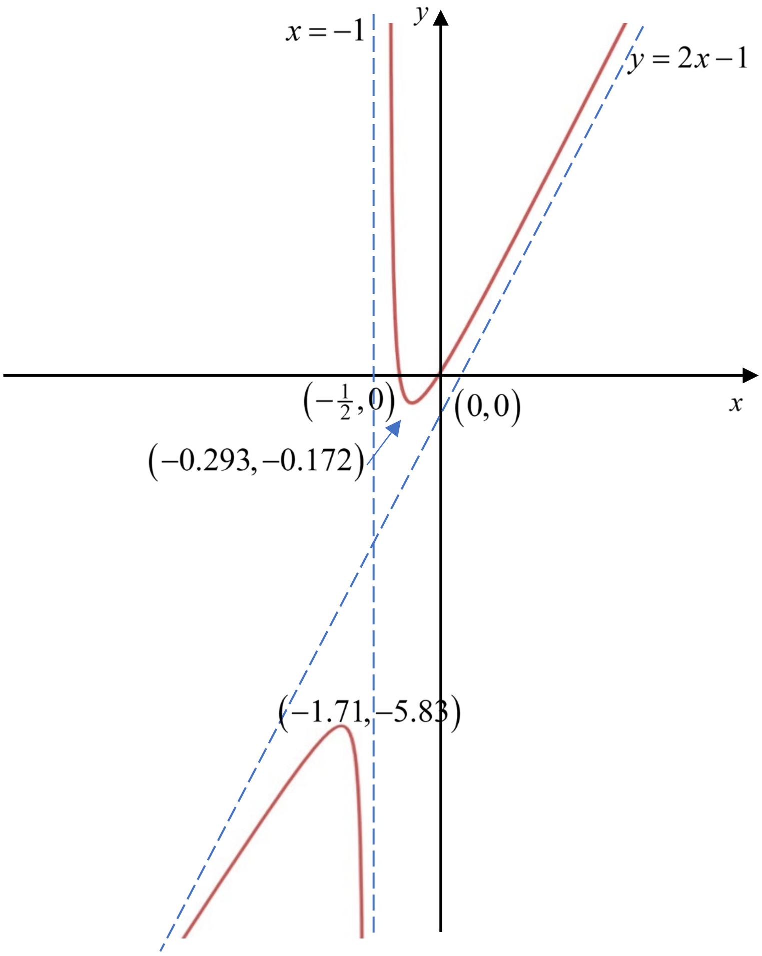
	$\mathrm{f}(x)=x$ $\dfrac{3x+2}{x-1}=x$ $x^2-4x-2=0$ $x=\dfrac{4\pm\sqrt{24}}{2}$ $=2\pm\sqrt{6}$ Since $x>3$, $x=2+\sqrt{6}$	OR: $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ $\dfrac{3x+2}{x-1}=\dfrac{x+2}{x-3}$ $(3x+2)(x-3)=(x+2)(x-1)$ $2x^2-8x-4=0$ $x^2-4x-2=0$

6	Solution
(a) **[5]**	$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k\left(\theta - 32\right)$ for some constant $k > 0$ $\int \frac{1}{\theta - 32}\,\mathrm{d}\theta = -k\int 1\ \mathrm{d}t$ Since pie is cooling down, $\theta \geq 32$. $\ln\left(\theta - 32\right) = -kt + C$ $\ln\left(\theta - 32\right) = -kt + C$ $\theta = 32 + A\mathrm{e}^{-kt}$, where $A = \mathrm{e}^{C}$ Alternative: $\ln\left\|\theta - 32\right\| = -kt + C$ $\theta - 32 = \pm\mathrm{e}^{-kt+C}$ $\theta = 32 + A\mathrm{e}^{-kt}$, where $A = \pm\mathrm{e}^{C}$ When $t = 0$, $\theta = 200 \Rightarrow 200 = 32 + A \Rightarrow A = 168$ When $t = 15$, $\theta = 180 \Rightarrow 180 = 32 + 168\mathrm{e}^{-15k}$ $\mathrm{e}^{-15k} = \frac{37}{42} \Rightarrow \mathrm{e}^{-k} = \left(\frac{37}{42}\right)^{1/15}$ $\therefore \theta = 32 + 168\left(\frac{37}{42}\right)^{t/15}$
(b) **[2]**	
(c) **[2]**	From GC, solution of $60 = 32 + 168\left(\frac{37}{42}\right)^{t/15}$ is t = 212.04 (5sf) = 212 mins (nearest min), equivalent to 3h 32 mins. Alternatively, $60 = 32 + 168\left(\frac{37}{42}\right)^{t/15} \Rightarrow t = \frac{15\ln\left(1/6\right)}{\ln\left(37/42\right)} \approx 212.04$ =212 mins (nearest min), equivalent to 3h 32 mins. To safely store the pie, t < 212 mins. Latest time to keep the pie is 4.32 pm.

7	**Solution**
(a) **[4]**	$\frac{1}{(2r-1)(2r+1)}=\frac{A}{2r-1}+\frac{B}{2r+1}$ $A=\frac{1}{2}$ and $B=-\frac{1}{2}$. $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}$ $=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$ $=\frac{1}{2}\begin{bmatrix} \frac{1}{1} & -\cancel{\frac{1}{3}} \\ +\cancel{\frac{1}{3}} & -\cancel{\frac{1}{5}} \\ +\cancel{\frac{1}{5}} & -\cancel{\frac{1}{7}} \\ +\cdots & \\ +\cancel{\frac{1}{2n-3}} & -\cancel{\frac{1}{2n-1}} \\ +\cancel{\frac{1}{2n-1}} & -\frac{1}{2n+1} \end{bmatrix}$ $=\frac{1}{2}\left[1-\frac{1}{2n+1}\right]$
(b) **[2]**	Since $\frac{1}{2n+1}\to 0$ when $n\to\infty$, the series converges and the sum to infinity is $\frac{1}{2}$.
(c) **[4]**	$\sum_{r=4}^{n}\frac{1}{(2r+3)(2r+5)}$ $=\sum_{m=6}^{n+2}\frac{1}{[2(m-2)+3][2(m-2)+5]}$ (replace r with $m-2$) $=\sum_{m=6}^{n+2}\frac{1}{[2m-1][2m+1]}$ $=\sum_{m=1}^{n+2}\frac{1}{[2m-1][2m+1]}-\sum_{m=1}^{5}\frac{1}{[2m-1][2m+1]}$ $=\frac{1}{2}\left[1-\frac{1}{2(n+2)+1}\right]-\frac{1}{2}\left[1-\frac{1}{2(5)+1}\right]$ $=\frac{1}{22}-\frac{1}{4n+10}$

8	Solution
(a) [4]	$u = 1-\mathrm{e}^x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = -\mathrm{e}^x$ $\displaystyle\int_1^2 \frac{\mathrm{e}^x}{\left(1-\mathrm{e}^x\right)^3}\,\mathrm{d}x = \int_{1-\mathrm{e}}^{1-\mathrm{e}^2} \frac{-1}{u^3}\,\mathrm{d}u$ $\displaystyle = \left[\frac{1}{2}u^{-2}\right]_{1-\mathrm{e}}^{1-\mathrm{e}^2}$ $\displaystyle = \frac{1}{2}\left[\left(1-\mathrm{e}^2\right)^{-2} - \left(1-\mathrm{e}\right)^{-2}\right]$ Otherwise, $\displaystyle\int_1^2 \frac{\mathrm{e}^x}{\left(1-\mathrm{e}^x\right)^3}\,\mathrm{d}x = -\int_1^2 (-\mathrm{e}^x)\left(1-\mathrm{e}^x\right)^{-3}\,\mathrm{d}x$ $\displaystyle = \left[-\frac{\left(1-\mathrm{e}^x\right)^{-2}}{-2}\right]_1^2$ $\displaystyle = \frac{1}{2}\left[\left(1-\mathrm{e}^2\right)^{-2} - \left(1-\mathrm{e}\right)^{-2}\right]$
(b) [3]	$\displaystyle\int_0^1 \tan^{-1}x\,\mathrm{d}x = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x$ $\displaystyle = \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ $\displaystyle = \frac{\pi}{4} - \frac{1}{2}\ln 2$
(c) [3]	

	$\int_{-2}^{2}\lvert f(x)\rvert\,dx=\int_{-2}^{-1}\lvert f(x)\rvert\,dx+\int_{-1}^{1}\lvert\tan^{-1}x\rvert\,dx+\int_{1}^{2}\left\lvert\frac{e^x}{(1-e^x)^3}\right\rvert dx$ $=\int_{1}^{2}\left\lvert\frac{e^x}{(1-e^x)^3}\right\rvert dx+\int_{-1}^{1}\lvert\tan^{-1}x\rvert\,dx+\int_{1}^{2}\left\lvert\frac{e^x}{(1-e^x)^3}\right\rvert dx$ $=2\int_{0}^{1}\tan^{-1}x\,dx-2\int_{1}^{2}\frac{e^x}{(1-e^x)^3}\,dx$ $=2\left(\frac{\pi}{4}-\frac{1}{2}\ln 2\right)-2\left[\frac{1}{2}\left((1-e^2)^{-2}-(1-e)^{-2}\right)\right]$ $=\frac{\pi}{2}-\ln 2-(1-e^2)^{-2}+(1-e)^{-2}$

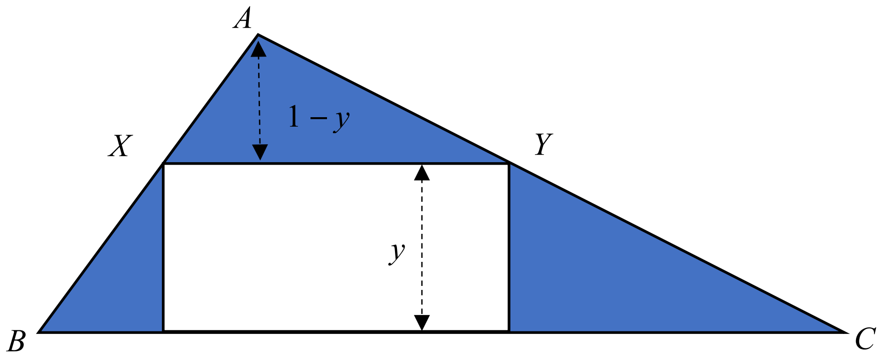
9	**Solution**
(a) **[4]**	$C: y=\dfrac{ax^2+x}{x+b},\ a\neq\dfrac{1}{b}$ and $a,\ b>0.$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+b)(2ax+1)-(ax^2+x)}{(x+b)^2}=\dfrac{ax^2+2abx+b}{(x+b)^2}$ Since C has no stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ has no real solutions. Hence, $ax^2+2abx+b=0$ has no real solutions. Thus the discriminant $(2ab)^2-4ab<0$. Since a and b are positive constants, on simplifying, the relationship is $0<ab<1$.
(b) **[4]**	$y=\dfrac{2x^2+x}{x+1}=2x-1+\dfrac{1}{x+1}$ 

(c) **[2]**	Volume = $\pi \int_{-\frac{1}{2}}^{0} \left(\dfrac{2x^2 + x}{x+1} \right)^2 \mathrm{d}x$ = 0.0245 (3.s.f)

10	Solution
(a) **[4]**	$z^3-4z^2+6z-4=0$ … (1) Checking z = 2: LHS = 8 – 16 + 12 – 4 = 0 $z-2$ is a factor. $z^3-4z^2+6z-4=(z-2)(z^2-2z+2)=0$ $z=2$ or $z=\dfrac{2\pm\sqrt{4-4(1)(2)}}{2}=\dfrac{2\pm 2\mathrm{i}}{2}=1\pm\mathrm{i}$
(b) **[4]**	$z^3-4z^2+6z-4=0$ $\mathrm{i}^4z^3+4\mathrm{i}^2z^2-6(\mathrm{i}^2z)-4=0$ $\mathrm{i}(\mathrm{i}z)^3+4(\mathrm{i}z)^2-6\mathrm{i}(\mathrm{i}z)-4=0$ $\therefore w=\mathrm{i}z$ Roots of $\mathrm{i}w^3+4w^2-6\mathrm{i}w-4=0$ are: $w=2\mathrm{i},\ -1+\mathrm{i},\ 1+\mathrm{i}$ $=2\mathrm{e}^{\mathrm{i}\frac{\pi}{2}},\ \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}},\ \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$
(c) **[3]**	$\left(\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}\right)^n=\sqrt{2^n}\mathrm{e}^{\mathrm{i}\frac{3n\pi}{4}}$ is a positive real number and n is positive, we have $\dfrac{3n\pi}{4}=2k\pi$ where $k\in\mathbb{Z}^+$ $n=\dfrac{8k}{3}$ Smallest integer n occur when k = 3. Therefore smallest n = 8

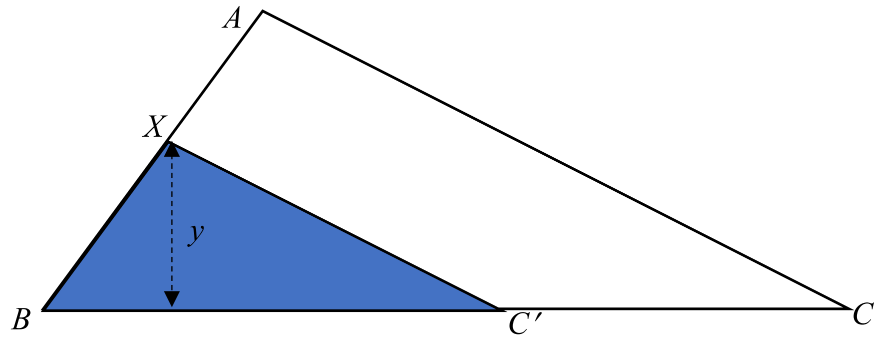
<table>
<tr><th>11</th><th>Solution</th></tr>
<tr><td>(a)
(i)
[1]</td><td>Using Pythagoras Theorem, we have the height of the rectangle to be $2\sqrt{r^2-x^2}$. Hence the area to be maximized is given by $A=4x\sqrt{r^2-x^2}$.</td></tr>
<tr><td>(a)
(ii)
[5]</td><td>Differentiating with respect to x, we have
$\frac{\mathrm{d}A}{\mathrm{d}x}=4\left[\sqrt{r^2-x^2}+x\frac{(\frac{1}{2})(-2x)}{\sqrt{r^2-x^2}}\right]$
$=4\frac{r^2-x^2-x^2}{\sqrt{r^2-x^2}}=4\frac{r^2-2x^2}{\sqrt{r^2-x^2}}$
A has a stationary point when $r^2-2x^2=0 \Leftrightarrow x=\frac{r}{\sqrt{2}}$.
Method 1 (1st Derivative Test)
$\frac{\mathrm{d}A}{\mathrm{d}x}=4\frac{r^2-2x^2}{\sqrt{r^2-x^2}}=8\frac{\left(\frac{r^2}{2}-x^2\right)}{\sqrt{r^2-x^2}}=8\frac{\left(\frac{r}{\sqrt{2}}-x\right)\left(\frac{r}{\sqrt{2}}+x\right)}{\sqrt{r^2-x^2}}$.
Since $\frac{r}{\sqrt{2}}+x>0$ and $\frac{8}{\sqrt{r^2-x^2}}>0$ for $0<x<r$, the sign of $\frac{\mathrm{d}A}{\mathrm{d}x}$ depends only on that of $\left(\frac{r}{\sqrt{2}}-x\right)$.
<table>
<tr><td></td><td>$x=\left(\frac{r}{\sqrt{2}}\right)^-$</td><td>$x=\frac{r}{\sqrt{2}}$</td><td>$x=\left(\frac{r}{\sqrt{2}}\right)^+$</td></tr>
<tr><td>$\left(\frac{r}{\sqrt{2}}-x\right)$</td><td>$>0$</td><td>0</td><td>$<0$</td></tr>
<tr><td>$\frac{\mathrm{d}A}{\mathrm{d}x}$</td><td>$>0$</td><td>0</td><td>$<0$</td></tr>
<tr><td>Shape</td><td>/</td><td>—</td><td>\</td></tr>
</table>
The corresponding maximum area is $A=4\left(\frac{\sqrt{2}r}{2}\right)\sqrt{\frac{r^2}{2}}=2r^2$.
Method 2 (2nd Derivative Test)
$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=4\frac{1}{\left(\sqrt{r^2-x^2}\right)^2}\left((-4x)\sqrt{r^2-x^2}-\frac{\left(r^2-2x^2\right)\left(\frac{1}{2}\right)(-2x)}{\sqrt{r^2-x^2}}\right)$
$=-\frac{4}{\left(r^2-x^2\right)^{\frac{3}{2}}}\left(4x(r^2-x^2)-x(r^2-2x^2)\right)$
At the stationary point $r^2-2x^2=0 \Leftrightarrow x=\frac{\sqrt{2}r}{2}$, so</td></tr>
</table>

	$\left.\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\right\|_{x=\frac{\sqrt{2}r}{2}}=-\frac{4}{\left(r^2-\frac{r^2}{2}\right)^{\frac{3}{2}}}\left(4\frac{r}{\sqrt{2}}\left(r^2-\frac{r^2}{2}\right)\right)=-16<0$ Alternatively, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=-\frac{4}{\left(r^2-x^2\right)^{\frac{3}{2}}}\left(4x(r^2-x^2)-x(r^2-2x^2)\right)$ $=-\frac{4}{\left(r^2-x^2\right)^{\frac{3}{2}}}\left(3x(r^2-x^2)+x^3\right)$ $=-\frac{12x}{\left(r^2-x^2\right)^{\frac{1}{2}}}-\frac{4x^3}{\left(r^2-x^2\right)^{\frac{3}{2}}}<0$ for all x since $\left(r^2-x^2\right)^{\frac{1}{2}}>0, x>0$, and thus each of the terms are negative. Hence the area is a maximum. The corresponding maximum area is $A=4\left(\frac{\sqrt{2}r}{2}\right)\sqrt{\frac{r^2}{2}}=2r^2$. **<u>Alternative Solution</u>** If we wish to avoid dealing with the square roots, we can work with $A^2=16x^2\left(r^2-x^2\right)$ since maximising A is equivalent to maximising A^2. We have $\frac{\mathrm{d}A^2}{\mathrm{d}x}=32x\left(r^2-x^2\right)+16x^2\left(-2x\right)=32x\left(r^2-x^2-x^2\right)$. Hence a stationary value occurs when $r^2-2x^2=0 \Leftrightarrow x=\frac{r}{\sqrt{2}}$. Using the 2nd derivative test, we see that $\frac{\mathrm{d}^2A^2}{\mathrm{d}x^2}=32\left(r^2-2x^2\right)+32x\left(-4x\right)=32\left(r^2-6x^2\right)$ $\Rightarrow \left.\frac{\mathrm{d}^2A^2}{\mathrm{d}x^2}\right\|_{x=\frac{r}{\sqrt{2}}}=32\left(r^2-3r^2\right)=-64r^2<0$ Hence A^2 is maximum and thus the required area $A=4\left(\frac{\sqrt{2}r}{2}\right)\sqrt{\frac{r^2}{2}}=2r^2$ is also maximum.
(iii) **[1]**	For a circle with area π unit^2, $r=1$. From part (ii), the area of the largest rectangle that can be inscribed on the circle is $2(1)^2=2$ units^2.

<table>
<tr><td>(b)(i)
[2]</td><td>


Let the rectangle touch the sides AB and AC at X and Y respectively.
Then triangle AXY is similar to triangle ABC and we have
Area of $AXY = (1-y)^2 T$

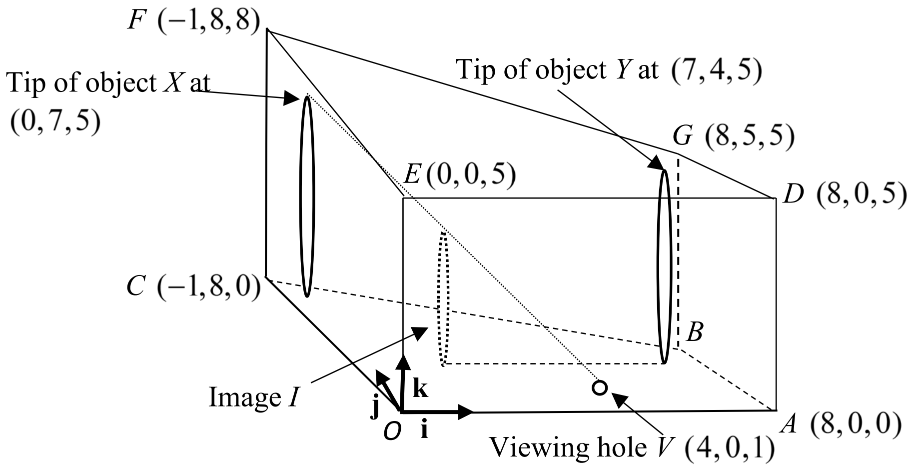
The other 2 smaller shaded triangles can be placed next to each other (translate Y to X parallel to the side of the rectangle, and let C be translated to C') to see that together they form another triangle similar to triangle ABC such that Area of $BXC' = y^2 T$.



Adding the two relations we have $S = \left(y^2 + (1-y)^2\right)T$.

Alternatively,
Let $XY = k$. Then we have $S = T - ky$. Since triangles AXY and ABC are similar, we have $\frac{1-y}{1} = \frac{k}{BC}$. However, considering the area of the triangle,
$T = \frac{1}{2}BC\left(1\right) \Rightarrow BC = 2T$. Therefore $k = 2T(1-y)$. Substituting this into the first relation we have
$S = T - ky$
$= T - \left(2T(1-y)\right)y$
$= T(1-2y+2y^2)$
$= \left(y^2 + (1-y)^2\right)T.$
</td></tr>
<tr><td>(ii)
[3]</td><td>We first note that thew result in (i) still holds if the height of the triangle is not 1. This is easily seen from the first solution, as the ratio is unaffected when we stretch the triangle to any other height.</td></tr>
</table>

	We consider $\frac{R}{T}$, where R denotes the area of the rectangle that can be placed in a triangle ABC. From **(i)**, we have $\frac{R}{T}=\frac{T-S}{T}=1-\frac{S}{T}$ $\quad =1-\left(y^2+(1-y)^2\right)=2y-2y^2$ $\quad =2\left(\frac{1}{4}-\left(y-\frac{1}{2}\right)^2\right)\le\frac{1}{2}$ since $\left(y-\frac{1}{2}\right)^2\ge 0.$ Hence $\frac{R}{T}$ attains maximum of $\frac{1}{2}$. Since the area of the triangle is π unit^2, the largest area of rectangle will be $\frac{\pi}{2}$ units^2. Alternatively, $\frac{\mathrm{d}S}{\mathrm{d}y}=\left(2y+2(1-y)(-1)\right)\pi=\left(4y-2\right)\pi=0\Leftrightarrow y=\frac{1}{2}$ and since $\frac{\mathrm{d}^2S}{\mathrm{d}y^2}=4\pi>0$, we conclude that S attains a minimum value of $\left(\left(\frac{1}{2}\right)^2+\left(1-\frac{1}{2}\right)^2\right)\pi=\frac{\pi}{2}$. When S is minimum, the area of the rectangle = $\pi-S=\frac{\pi}{2}$ units^2 is therefore maximum. **<u>Remark</u>**: The objective of (a)(iii) and (b)(ii) is to essentially show that rectangles can be more efficiently packed in circles than in triangles, since $2>\frac{\pi}{2}$.

12	Solution
(a) **[4]**	 $\overrightarrow{EF}=\begin{pmatrix}-1\\8\\8\end{pmatrix}-\begin{pmatrix}0\\0\\5\end{pmatrix}=\begin{pmatrix}-1\\8\\3\end{pmatrix}$, $\overrightarrow{EG}=\begin{pmatrix}8\\5\\5\end{pmatrix}-\begin{pmatrix}0\\0\\5\end{pmatrix}=\begin{pmatrix}8\\5\\0\end{pmatrix}$ normal of plane $EFG=\begin{pmatrix}-1\\8\\3\end{pmatrix}\times\begin{pmatrix}8\\5\\0\end{pmatrix}=\begin{pmatrix}-15\\24\\-69\end{pmatrix}=-3\begin{pmatrix}5\\-8\\23\end{pmatrix}$ $\mathbf{r}.\begin{pmatrix}5\\-8\\23\end{pmatrix}=\begin{pmatrix}0\\0\\5\end{pmatrix}.\begin{pmatrix}5\\-8\\23\end{pmatrix}=115\Rightarrow 5x-8y+23z=115$ (shown)
(b) **[4]**	Let the upper tip of the Object X be point $X(0,\ 7,\ 5)$, and upper tip of Image I be point I. Line of sight has direction vector $\overrightarrow{XV}=\begin{pmatrix}4\\0\\1\end{pmatrix}-\begin{pmatrix}0\\7\\5\end{pmatrix}=\begin{pmatrix}4\\-7\\-4\end{pmatrix}$, and given by the vector equation $\mathbf{r}=\begin{pmatrix}4\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}4\\-7\\-4\end{pmatrix}, \lambda\in\mathbb{R}$. Since the upper tip of Image I is also lying on this line and Image I is on the plane $y=4$, we have $\lambda=-\frac{4}{7}$. Hence $\overrightarrow{OI}=\begin{pmatrix}4\\0\\1\end{pmatrix}-\frac{4}{7}\begin{pmatrix}4\\-7\\-4\end{pmatrix}=\frac{1}{7}\begin{pmatrix}12\\28\\23\end{pmatrix}$. The ratio is $\frac{23}{7}:5\Rightarrow 23:35$ because image I has vertical height $\frac{23}{7}$ units and vertical height of object Y is 5 units.

(c) **[5]**	Line *BC* is given by the vector equation $\mathbf{r}=\begin{pmatrix}-1\\8\\0\end{pmatrix}+\mu\begin{pmatrix}-3\\1\\0\end{pmatrix}, \mu\in\mathbb{R}$. *S* is the foot of perpendicular from the upper tip of Object $Y(8,0,5)$ to line *BC* and therefore $\overrightarrow{OS}=\begin{pmatrix}-1\\8\\0\end{pmatrix}+\mu\begin{pmatrix}-3\\1\\0\end{pmatrix}$, for some $\mu\in\mathbb{R}$. Since $\overrightarrow{SY}$ is perpendicular to the line BC, $\left(\begin{pmatrix}-1\\8\\0\end{pmatrix}-\begin{pmatrix}7\\4\\5\end{pmatrix}+\mu\begin{pmatrix}-3\\1\\0\end{pmatrix}\right)\cdot\begin{pmatrix}-3\\1\\0\end{pmatrix}=0$ $(24+4)+\mu(9+1)=0$ $\mu=-2.8$ $\overrightarrow{OS}=\begin{pmatrix}-1\\8\\0\end{pmatrix}-2.8\begin{pmatrix}-3\\1\\0\end{pmatrix}=\begin{pmatrix}7.4\\5.2\\0\end{pmatrix}$ Alternatively, $\left\lvert\overrightarrow{SY}\right\rvert=\left\lvert\begin{pmatrix}-1\\8\\0\end{pmatrix}-\begin{pmatrix}7\\4\\5\end{pmatrix}+\mu\begin{pmatrix}-3\\1\\0\end{pmatrix}\right\rvert=\left\lvert\begin{pmatrix}-8-3\mu\\4+\mu\\-5\end{pmatrix}\right\rvert$ $=\sqrt{(-8-3\mu)^2+(4+\mu)^2+(-5)^2}$ By GC, minimum $\left\lvert\overrightarrow{SY}\right\rvert$ occurs at $\mu=-2.8$. Hence $\overrightarrow{OS}=\begin{pmatrix}-1\\8\\0\end{pmatrix}-2.8\begin{pmatrix}-3\\1\\0\end{pmatrix}=\begin{pmatrix}7.4\\5.2\\0\end{pmatrix}$.