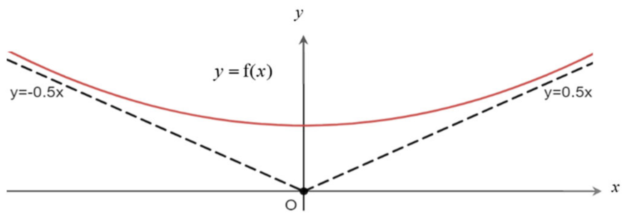
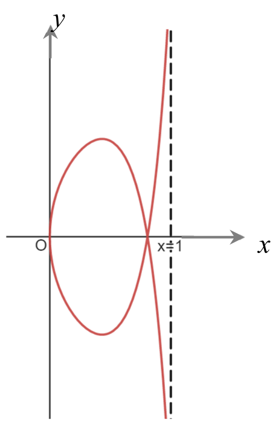
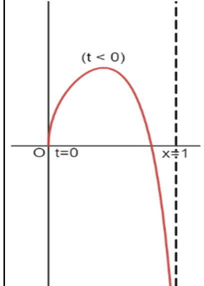
S/N	Solution
1	
2(i)	$u_{n+2}-(1+r)u_{n+1}+ru_n=0$ Auxiliary equation: $\lambda^2-(1+r)\lambda+r=0$ $(\lambda-1)(\lambda-r)=0$ $\lambda=1$ or r General solution is $u_n=A+Br^n$, A,B are arbitrary constants
(ii)	$u_0=0\Rightarrow A+B=0$ As $n\to\infty, u_n\to A=L$ provided $\lvert r\rvert<1$. $\therefore B=-L$ Hence, $u_n=L-Lr^n$.
5(i)	
(ii)	Curve intersects itself when $y=t^3-\lambda t=0$. $t=0$ or $t^2=\lambda$. $\Rightarrow x=\dfrac{\lambda}{1+\lambda}$
(iii)	The curve is symmetrical about the x-axis. $\therefore$ required area bounded by curve is $2\int_0^{\frac{\lambda}{1+\lambda}} y\ \mathrm{d}x$.  $x=\dfrac{t^2}{1+t^2}, \quad y=t^3-\lambda t,$ $=1-\dfrac{1}{1+t^2}$ $\mathrm{d}x=\dfrac{2t}{(1+t^2)^2}\mathrm{d}t$

$$2\int_0^{\frac{\lambda}{1+\lambda}} y\ \mathrm{d}x = 2\int_0^{-\sqrt{\lambda}} (t^3-\lambda t)\frac{2t}{(1+t^2)^2}\mathrm{d}t$$

$$=4\int_0^{-\sqrt{\lambda}} \frac{t^2(t^2-\lambda)}{(1+t^2)^2}\mathrm{d}t$$

$$\therefore \mathrm{f}(\lambda)=-\sqrt{\lambda},\quad \mathrm{g}(t^2)=\frac{t^2(t^2-\lambda)}{(1+t^2)^2}.$$

> $x=\dfrac{\lambda}{1+\lambda}$ is obtained from $t=\pm\sqrt{\lambda}$
>
> For $y>0$, we consider $t<0$.
>
> Hence, we take $t=-\sqrt{\lambda}$.

4

Let $\mathrm{f}(x)=1+\cos(\pi x)-2\sqrt{x}$.

$x=0,\quad \mathrm{f}(0)=2$

$x=1,\quad \mathrm{f}(1)=-2$

$$\therefore x_1=\frac{0(-2)-1(2)}{-2-2}=0.5$$

By Newton-Raphson method,

$$x_{n+1}=x_n-\frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$$

$$=x_n-\frac{1+\cos(\pi x_n)-2\sqrt{x_n}}{-\pi\sin(\pi x_n)-\frac{1}{\sqrt{x_n}}}$$

$$=x_n+\frac{1+\cos(\pi x_n)-2\sqrt{x_n}}{\pi\sin(\pi x_n)+\frac{1}{\sqrt{x_n}}}$$

Using $x_1=0.5$,

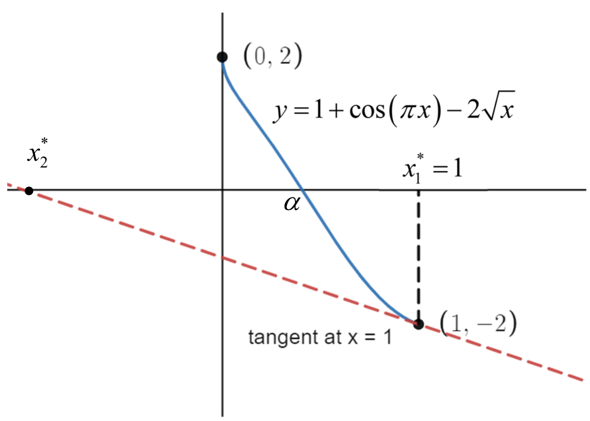
$x_2=0.4091$

$x_3=0.4096$

Since f(0.405) = 00212 > 0 and f(0.415) = –0.0245 < 0,

$\Rightarrow 0.405<\alpha<0.415$

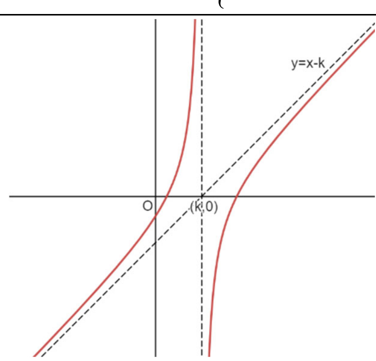
$\therefore \alpha=0.41$ (to 2 d.p)

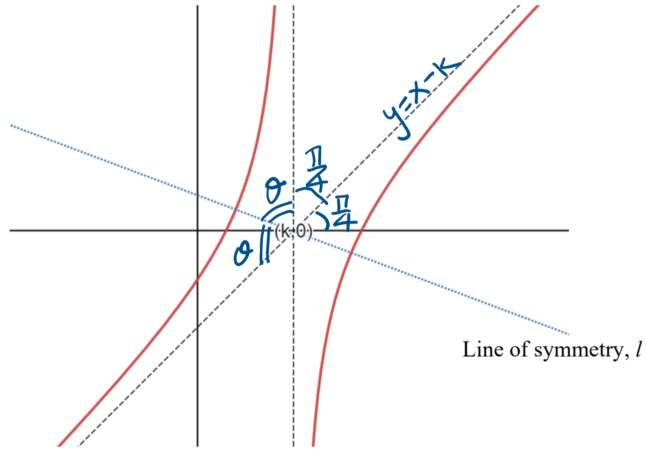


$x_2^*<0,\quad \sqrt{x_2^*}$ is undefined.

$\therefore \mathrm{f}(x_2^*)$ is undefined.

Hence Newton -Raphson method fails.

5(a)	$\int_{-1}^{0} \frac{1}{\sqrt{1-a^2x^2}}\,\mathrm{d}x = \frac{1}{a}\sin^{-1}(ax)\Big\vert_{-1}^{0}$ $= 0 - \frac{1}{a}\sin^{-1}(-a)$ $= \frac{1}{a}\sin^{-1}(a)$ $= \frac{1}{a}\left(\pi - \phi\right)$
(b)	$t = \tan\frac{x}{2}:\ \ \mathrm{d}t = \frac{1}{2}\sec^2\frac{x}{2}\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t$ $\sin x = \frac{2t}{1+t^2},\quad \cos t = \frac{1-t^2}{1+t^2},\quad t = \tan\frac{x}{2}:\ \ \sin x = \frac{2t}{1+t^2}$ $\int \frac{\cos x}{1+\cos x - \sin x}\mathrm{d}x = \int \frac{1-t^2}{1+t^2} \div \left(1 + \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}\right)\frac{2}{1+t^2}\mathrm{d}t$ $= \int \frac{(1-t^2)}{(1+t^2)(1-t)}\mathrm{d}t$ $= \int \frac{1+t}{1+t^2}\ \mathrm{d}t$ (shown) $= \int \frac{1}{1+t^2}\ \mathrm{d}t + \frac{1}{2}\int \frac{2t}{1+t^2}\ \mathrm{d}t$ $= \frac{1}{2}\tan^{-1}t + \frac{1}{2}\ln(1+t^2) + C$ $= \frac{1}{2}x + \frac{1}{2}\ln\left(1+\tan^2\frac{x}{2}\right) + C$ $= \frac{1}{2}x + \frac{1}{2}\ln\left(\sec^2\frac{x}{2}\right) + C$ or $= \frac{1}{2}x + \ln\left\vert\sec\frac{x}{2}\right\vert + C$
6(i)	$y = \frac{x^2 - 2kx + k}{x-k} = \frac{(x-k)^2 + k(1-k)}{x-k} = x - k + \frac{k(1-k)}{x-k}$ The asymptotes are $x = k$ and $y = x - k$.
(ii)	2 stationary points $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{k(1-k)}{(x+k)^2} = 0$ has 2 distinct real roots. $(x-k)^2 = k(1-k) > 0$ $\Rightarrow 0 < k < 1$ The set of k is $\{k \in \mathbb{R} : 0 < k < 1\}$.
(iii)	

(iv)	 l bisects the angle between the asymptotes. $2\theta+\frac{\pi}{4}=\pi \Rightarrow \theta=\frac{3\pi}{8}$ Acute angle between l and x-axis $=\frac{\pi}{2}-\frac{3\pi}{8}=\frac{\pi}{8}$ $\therefore$ gradient of $l=\ m=-\tan\frac{\pi}{8}.\ \left(\text{or }\tan\frac{7\pi}{8}\right)$
7(a)	$\mathrm{e}^{in\theta}-\mathrm{e}^{-in\theta}=(\cos n\theta+\mathrm{i}\sin n\theta)-(\cos(-n\theta)+\mathrm{i}\sin(-n\theta))$ $=(\cos n\theta+\mathrm{i}\sin n\theta)-(\cos n\theta-\mathrm{i}\sin n\theta)$ $=\cos n\theta+\mathrm{i}\sin n\theta-\cos n\theta+\mathrm{i}\sin n\theta$ $=2\mathrm{i}\sin n\theta$ (shown)
(b)	$\left(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}\right)^5=\mathrm{e}^{\mathrm{i}5\theta}-5\mathrm{e}^{\mathrm{i}4\theta}\cdot\mathrm{e}^{-\mathrm{i}\theta}+10\mathrm{e}^{\mathrm{i}3\theta}\cdot\mathrm{e}^{-\mathrm{i}2\theta}-10\mathrm{e}^{\mathrm{i}2\theta}\cdot\mathrm{e}^{-\mathrm{i}3\theta}+5\mathrm{e}^{\mathrm{i}\theta}\cdot\mathrm{e}^{-\mathrm{i}4\theta}$ $(2\mathrm{i}\sin\theta)^5=\left(\mathrm{e}^{\mathrm{i}5\theta}-\mathrm{e}^{-\mathrm{i}5\theta}\right)-5\left(\mathrm{e}^{\mathrm{i}3\theta}-\mathrm{e}^{-\mathrm{i}3\theta}\right)+10\left(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}\right)$ $32\mathrm{i}\sin^5\theta=2\mathrm{i}\sin 5\theta-10\mathrm{i}\sin 3\theta+20\mathrm{i}\sin\theta$ $\sin^5\theta=\frac{1}{16}\sin 5\theta-\frac{5}{16}\sin 3\theta+\frac{5}{8}\sin\theta$ $\cos^5\theta=\sin^5\left(\frac{\pi}{2}-\theta\right)$ $=\frac{1}{16}\sin\left(5\left(\frac{\pi}{2}-\theta\right)\right)-\frac{5}{16}\sin\left(3\left(\frac{\pi}{2}-\theta\right)\right)+\frac{5}{8}\sin\left(\frac{\pi}{2}-\theta\right)$ $=\frac{1}{16}\sin\left(\frac{5\pi}{2}-5\theta\right)-\frac{5}{16}\sin\left(\frac{3\pi}{2}-3\theta\right)+\frac{5}{8}\sin\left(\frac{\pi}{2}-\theta\right)$ $=\frac{1}{16}\cos 5\theta+\frac{5}{16}\cos 3\theta+\frac{5}{8}\cos\theta$
(c)	$\sin x+\sin 2x+\sin 3x+\cdots+\sin Nx$ $=\operatorname{Im}\left(\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{\mathrm{i}2x}+\mathrm{e}^{\mathrm{i}3x}+\cdots+\mathrm{e}^{\mathrm{i}Nx}\right)$ $=\operatorname{Im}\left[\frac{\mathrm{e}^{\mathrm{i}x}\left(\mathrm{e}^{\mathrm{i}Nx}-1\right)}{\mathrm{e}^{\mathrm{i}x}-1}\right]$

	$\dfrac{e^{ix}\left(e^{iNx}-1\right)}{e^{ix}-1}=\dfrac{e^{ix}\cdot e^{i\frac{Nx}{2}}\left(e^{i\frac{Nx}{2}}-e^{-i\frac{Nx}{2}}\right)}{e^{i\frac{x}{2}}\left(e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}\right)}$ $=\dfrac{e^{i\frac{(N+1)x}{2}}\left(2i\sin\frac{Nx}{2}\right)}{2i\sin\frac{x}{2}}$ $=\dfrac{e^{i\frac{(N+1)x}{2}}\sin\frac{Nx}{2}}{\sin\frac{x}{2}}$ $=\operatorname{cosec}\frac{x}{2}\left[\cos\frac{(N+1)x}{2}+i\sin\frac{(N+1)x}{2}\right]\sin\frac{Nx}{2}$ $\therefore \sin x+\sin 2x+\sin 3x+\cdots+\sin Nx$ $=\operatorname{cosec}\left(\frac{1}{2}x\right)\sin\left(\frac{N+1}{2}x\right)\sin\left(\frac{N}{2}x\right)$ (shown)
8(i)	cross-sectional area $\approx 2\times1+\frac{1}{2}\left[2+0+2(0.88+0.24+0.04)\right]$ or $\frac{1}{2}\left[2+0+2(2+0.88+0.24+0.04)\right]$ $=4.16\text{ m}^2$
(ii)	The builder will have enough cement as trapezium rule over-estimates the cross-sectional area since the curve is concave upwards.
(iii)	Note: Simpson's rule applies for an even number of strips. cross-sectional area $\approx 2\times1+\frac{1}{3}\left[2+0+2(0.24)+4(0.88+0.04)\right]$ $=4.05\text{ m}^2\,(3\text{sf})$
(iv)	$p(x)=\frac{1}{8}(x-5)^2$
(v)	Method I: Shell method Volume = $\pi(1)^2(2)+\int_1^5 2\pi xy\,dx$ $=2\pi+2\pi\int_1^5\frac{1}{8}x(x-5)^2\,dx$ ≈ 39.794 Method II: Disc method Volume = $\int_0^2 \pi x^2\,dy$ $=\pi\int_0^2\left[5-\sqrt{8y}\right]^2 dy$ ≈ 39.794 The builder needs to make at least 40 m^3 of cement.
9(i)	$I_2=\int_0^{\pi}\cos^2 2\theta\,d\theta$ $=\frac{1}{2}\int_0^{\pi}(\cos4\theta+1)\,d\theta$ $=\frac{1}{2}\left[\frac{\sin4\theta}{4}+\theta\right]_0^{\pi}$ $=\frac{\pi}{2}$

(ii)	$I_n = \int_0^{\pi} \cos^n(2\theta)\,\mathrm{d}\theta$ $= \int_0^{\pi} \cos(2\theta)\cos^{n-1}(2\theta)\,\mathrm{d}\theta$ $= \frac{\sin(2\theta)}{2}\cos^{n-1}(2\theta)\Big\|_0^{\pi} - \int_0^{\pi} \frac{\sin(2\theta)}{2}(n-1)\cos^{n-2}(2\theta)\left[-2\sin(2\theta)\right]\mathrm{d}\theta$ $= 0 + (n-1)\int_0^{\pi} \sin^2(2\theta)\cos^{n-2}(2\theta)\,\mathrm{d}\theta$ $= (n-1)\int_0^{\pi} [1-\cos^2(2\theta)]\cos^{n-2}(2\theta)\,\mathrm{d}\theta$ $= (n-1)I_{n-2} - (n-1)I_n$ $(1+n-1)I_n = (n-1)I_{n-2}$ $\therefore I_n = \frac{n-1}{n}I_{n-2}$ (shown)
(iii)	For odd integers n, I_n will be reduced to I_1. $I_1 = \int_0^{\pi} \cos 2\theta\,\mathrm{d}\theta = \frac{1}{2}\sin 2\theta\Big\|_0^{\pi} = 0$ $\therefore I_n = 0$ - independent of n
(iv)	$I_n = \frac{n-1}{n}I_{n-2}$ $= \frac{n-1}{n}\cdot\frac{n-3}{n-2}I_{n-4}$ $= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}I_{n-6}$ $= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdot\frac{3}{4}I_2$ $= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)\cdots 3.2.1}{n(n-2)(n-4)\cdots 4\; n(n-2)(n-4)\cdots 2}\left(\frac{\pi}{2}\right)$ $= \frac{n!}{2^{\frac{n}{2}-1}\frac{n}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots 2.2^{\frac{n}{2}}\frac{n}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots 1}\left(\frac{\pi}{2}\right)$ $= \frac{n!}{2^n\left[\left(\frac{n}{2}\right)!\right]^2}\pi$ (shown)
10(i)	$M_{n+1} = (1-p)M_n + k$
(ii)	$M_n = (1-p)\left[(1-p)M_{n-2} + k\right] + k$ $= (1-p)^2 M_{n-2} + k\left[(1-p)+1\right]$ $= \vdots$ $= (1-p)^{n-1}M_1 + k\left[(1-p)^{n-2} + (1-p)^{n-1} + \cdots + 1\right]$ $= (1-p)^{n-1}500 + k\frac{1-(1-p)^{n-1}}{p}$ $= \left(500 - \frac{k}{p}\right)(1-p)^{n-1} + \frac{k}{p}$

(iii)

$$M_6 = \left(500 - \frac{80}{p}\right)(1-p)^5 + \frac{80}{p} = 750$$

From GC, $p = 0.0495803$

$1 - p \approx 0.95042$

Hence the club needs to retain about 95% of its members each month.

(6v)

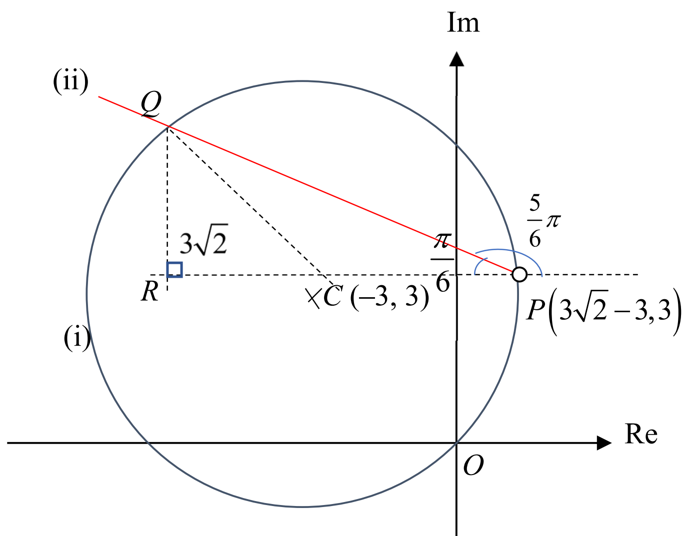
$$M_6 = \left(500 - \frac{k}{0.10}\right)(1-0.10)^5 + \frac{k}{0.10} \geq 750$$

From GC,

k	M_6	Y2
110	745.71	750
111	749.8	750
112	753.9	750
113	757.99	750
114	762.09	750

least value of $k = 112$.

11 (a)



$\angle QCR = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$ (angle at centre = $2 \times$ angle at circum)

$$QR = QC \sin\frac{\pi}{3} = \left(3\sqrt{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{6}}{2}$$

$$CR = QC \cos\frac{\pi}{3} = \left(3\sqrt{2}\right)\left(\frac{1}{2}\right) = \frac{3\sqrt{2}}{2}$$

Thus $Q\left(-3 - \frac{3\sqrt{2}}{2}, 3 + \frac{3\sqrt{6}}{2}\right)$

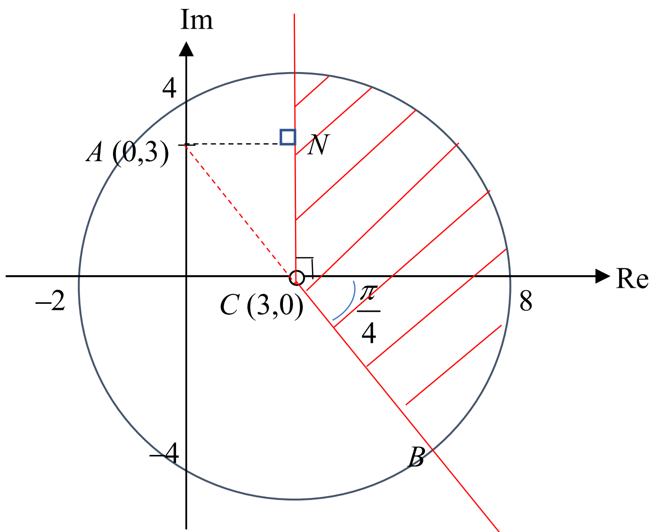
So, the complex number is $-3 - \frac{3\sqrt{2}}{2} + \mathrm{i}\left(3 + \frac{3\sqrt{6}}{2}\right)$

Alternatively,

$$PQ = 2\left(3\sqrt{2}\right)\cos\frac{\pi}{6} = 3\sqrt{6}$$

$$PR = PQ \cos\frac{\pi}{6} = 3\sqrt{6}\left(\frac{\sqrt{3}}{2}\right) = \frac{9\sqrt{2}}{2}$$

$$QR = PQ \sin\frac{\pi}{6} = 3\sqrt{6}\left(\frac{1}{2}\right) = \frac{3\sqrt{6}}{2}$$

	Thus $Q\left(-\frac{9\sqrt{2}}{2}+3\sqrt{2}-3, 3+\frac{3\sqrt{6}}{2}\right)$, i.e. $Q\left(-3-\frac{3\sqrt{2}}{2}, 3+\frac{3\sqrt{6}}{2}\right)$ So, the complex number is $-3-\frac{3\sqrt{2}}{2}+\mathrm{i}\left(3+\frac{3\sqrt{6}}{2}\right)$
(b)	$-\frac{3\pi}{4} \le \arg\left(\frac{z-3}{2\mathrm{i}}\right) \le 0$ $-\frac{3\pi}{4} \le \arg(z-3)-\arg(2\mathrm{i}) \le 0$ $-\frac{3\pi}{4} \le \arg(z-3)-\frac{\pi}{2} \le 0$ $-\frac{\pi}{4} \le \arg(z-3) \le \frac{\pi}{2}$  From the diagram, $AC=\sqrt{3^2+3^2}=3\sqrt{2}$ Greatest $\lvert z-3\mathrm{i}\rvert = AB = 3\sqrt{2}+5$ Least $\lvert z-3\mathrm{i}\rvert = AN = 3$