**National Junior College**
**2016 – 2017 H2 Mathematics**
**Complex Numbers** **Lecture Questions**

**Key questions to answer:**

- What is a complex number?
    - *How is the complex number* i *defined*?
    - *How do we write a complex number in its Cartesian form?*
    - *What do we mean when we say that two complex numbers are equal?*

**Lecture Reading:** Section 1

**Definition (Imaginary Unit)**

The imaginary unit, i, is a number such that $\mathrm{i}^2 = -1$. Hence, $\mathrm{i} = \sqrt{-1}$.

**Definition (Complex Number in Cartesian form)**

A complex number, $z \in \mathbb{C}$, is a number of the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$.

***Question 1*** **(Real and Imaginary Parts of a Complex Number)**

Write down the real and imaginary parts of the following complex numbers:

(a) $3+\mathrm{i}$ (b) $-2\mathrm{i}$ (c) $7$ (d) $\mathrm{i}-6$

*Solution:*

**Note:**

1. The imaginary part of a complex number does not include ‘i'.
2. A complex number having only the imaginary part (i.e. real part equals zero**)** is known as a **purely imaginary** number.

***Question 2* (Equality of Complex Numbers)**

Find $x$ and $y$, where $x, y \in \mathbb{R}$, if $x+2y+3\mathrm{i} = x(2\mathrm{i}-1)-\mathrm{i}y$.

*Solution:*

**Alternatively**, we may group all the real and imaginary parts respectively on one side of the equation, giving us

$$(2x+2y)+(3-2x+y)\mathrm{i}=0.$$

Comparing real and imaginary parts on both sides, we get the equations $2x+2y=0$ and $3-2x+y=0$, which we can solve simultaneously.

***Question 3***

Given that $z = x+\mathrm{i}y$, where $x, y \in \mathbb{R}$, find the value of $x$ such that $2x+\mathrm{i}z = z-\mathrm{i}(x+1)$.

*Identify the mistake in the following proposed solution and correct it.*

Comparing real part on both sides: $2x = z$

Comparing imaginary part on both sides: $z = -x-1$ (*)

Substitute $z = 2x$ into equation (*): $2x = -x-1 \Rightarrow x = -\frac{1}{3}$.

**Correct solution:**

**Key questions to answer:**

- ☐ How do we carry out arithmetic operations (addition, subtraction, multiplication, division and taking square root) on complex numbers in Cartesian form?
    - o *How do we relate the addition and subtraction of complex numbers to addition and subtraction of vectors?*

**Prerequisite knowledge:** Complex number in Cartesian form ; Vector addition and subtraction.

**Lecture Reading:** Section 2

***Question 4*** **(Arithmetic Operations on Complex Numbers)**

Express the following complex numbers in the form $x+\mathrm{i}y$, where $x, y \in \mathbb{R}$.

(a) (3 + 3i) − (1 − 2i) (b) (2 + 4i)(2 – i) (c) $\dfrac{2+4\mathrm{i}}{2+\mathrm{i}}$ (d) $\dfrac{3}{\mathrm{i}}$

*Solution:*

(a) (3 + 3i) − (1 − 2i) = 3 − 1 + 3i + 2i = 2 + 5i

(b) (2 + 4i)(2 – i) = 2(2 – i) + 4i(2 – i) = 4 – 2i + 8i – 4i$^2$ = 4 + 6i + 4 = 8 + 6i

(c)

Recall: $(a+b)(a-b)=a^2-b^2$ for $a,b\in\mathbb{R}$. In fact, this property is true for $a,b\in\mathbb{C}$. Thus, in (c), $(2+\mathrm{i})(2-\mathrm{i})=2^2-\mathrm{i}^2=4-(-1)=5$.

(d)

**Note:**

1. In (c), $2-\mathrm{i}$ is the complex conjugate of $2+\mathrm{i}$, which we will further discuss in Question 7. Observe that the product of a complex number and its conjugate gives us a real number.
2. It is useful to know (and remember) that $\dfrac{1}{\mathrm{i}}=-\mathrm{i}$, as seen in (d).

***Question 5*** **(Solving Simultaneous Equations)**

Solve the simultaneous equations $2p-q\mathrm{i}=2$ and $p^2-q+8+2\mathrm{i}=0$, for the complex numbers $p$ and $q$ in Cartesian form.

*Solution:*

***Question 6*** **(Square Roots of Complex Numbers)**

Find the square roots of $3-4\mathrm{i}$.

*Solution:*
The question requires us to evaluate $\sqrt{3-4\mathrm{i}}$. Let $\sqrt{3-4\mathrm{i}}=x+\mathrm{i}y$.

Then $(x+\mathrm{i}y)^2=3-4\mathrm{i} \Rightarrow x^2-y^2+2\mathrm{i}xy=3-4\mathrm{i}$.

Comparing the real and imaginary parts on both sides, we have:

$x^2-y^2=3$ ----- (1) ; $2xy=-4 \Rightarrow xy=-2$ ----- (2)

Solving simultaneously (*write out the workings yourself*),
we have $x=2$, $y=-1$ or $x=-2$, $y=1$.

The square roots of 3 − 4i are $2-\mathrm{i}$ and $-2+\mathrm{i}$.


Can we answer Question 6 using the GC?

*Partly. Observed that the GC gives us only one of the two square roots that we obtained through an algebraic method.*

**Key questions to answer:**

- ❑ How do we find the conjugate of a complex number?
- ❑ What are the properties of complex conjugates and their applications?

**Prerequisite knowledge:** Arithmetic operations on complex numbers

**Lecture Readings:** Section 3

**Definition (Complex Conjugate)**

The **complex conjugate** of a complex number $z = x + \mathrm{i}y$ is defined to be the complex number $x - \mathrm{i}y$ and is denoted by $z^*$.

***Question 7*** **(Properties of Complex Conjugates)**

If $z_1 = 2 - \mathrm{i}$ and $z_2 = 1 + 3\mathrm{i}$, find

(a) $z_1 + z_1^*$ (b) $z_1 - z_1^*$ (c) $z_1 z_1^*$ (d) $(z_1 z_2)^*$ (e) $\left(\dfrac{2z_1 + z_2}{z_2 - 1}\right)^*$.

*Solution:*

**Note:**

1. In general, $z + z^* = 2\,\mathrm{Re}(z)$ and $z - z^* = 2\mathrm{i}\,\mathrm{Im}(z)$, as seen in (a) and (b).
2. The conjugate of a real number is the number itself.

***Question 8*** **(Properties of Complex Conjugates)**

Using Binomial Theorem, expand $(2+3\mathrm{i})^5$ in the form $a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. Hence, simplify $(2-3\mathrm{i})^5$ in a similar form.

*Solution:*

**Note:** In general, $\left(z^n\right)^* = \left(z^*\right)^n$, where $n\in\mathbb{Z}$.


Can we use the GC to perform these arithmetic operations on complex numbers?

*Yes, unless the question specifies that no calculator is allowed or an exact answer (or specific method) is required.*
<u>*Refer to Appendix I of the lecture notes on some basics of using GC in complex numbers.*</u>

**Key questions to answer:**

- ❑ What can we say about the roots of polynomial equations with real coefficients?

  - *How do we solve polynomial equations with real coefficients?*
  - *Understand that complex roots of a polynomial equation with real coefficients occur in conjugate pairs.*

**Prerequisite knowledge:** Solving polynomial equations (quadratic and cubic with real roots) ; Complex conjugates

**Lecture Reading:** Section 4

**Theorem (Fundamental Theorem of Algebra)**

Every polynomial of **degree *n*** with coefficients in $\mathbb{C}$ has **exactly *n* roots** in $\mathbb{C}$, including repeated roots.

**Theorem (Complex Conjugate Root Theorem)**

All non-real roots of a polynomial with **real coefficients** must occur in **conjugate pairs**.

That is, if $z\ (= a + \mathrm{i}b, b \neq 0)$ is a root of

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + .... + a_2 x^2 + a_1 x + a_0 = 0 \text{ where } a_i \in \mathbb{R}, \text{ for } i = 0,\ 1,\ 2,\ ...,\ n-1,\ n,$$

then $z^*\ (= a - \mathrm{i}b, b \neq 0)$ is also a root of the equation.

***Question 9*** **(Solving Quadratic/Cubic Equations containing Complex Roots)**

Solve $x^3 + x^2 + 3x - 5 = 0$.

*Solution:*

<u>Algebraic method:</u>
By Fundamental Theorem of Algebra, we expect 3 roots.

$$x^3 + x^2 + 3x - 5 = 0$$
$$\Rightarrow (x-1)(x^2 + 2x + 5) = 0$$
$$\Rightarrow x = 1 \text{ or } x^2 + 2x + 5 = 0$$

$$\begin{aligned} x^2 + 2x + 5 = 0 \Rightarrow x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= -1 + 2\mathrm{i} \text{ or } -1 - 2\mathrm{i} \end{aligned}$$

Thus, the 3 roots are 1, $-1+2\mathrm{i}$ and $-1-2\mathrm{i}$.

**Note:**
1. The 3 roots consist of a real root and a pair of complex conjugate roots.
2. We see that the Complex Conjugate Root Theorem holds true as the <u>**coefficients are real**</u>.

<u>GC method:</u> Try using the GC to solve the above equation. You may refer to Example 4.3 of the lecture notes for the steps.

***Question 10***

A quadratic equation $2x^2 + ax + b = 0$ has a root $1 - 2\text{i}$. Find the real constants $a$ and $b$.

*Solution:*

***Question 11***

Solve the equation $z^2 + 2\text{i}z - 3 = 0$.

*Solution:*

$$
\begin{aligned}
z^2 + 2\text{i}z - 3 &= 0 \\
z &= \frac{-2\text{i} \pm \sqrt{(2\text{i})^2 - 4(3)}}{2} \\
&= \frac{-2\text{i} \pm \sqrt{-16}}{2} \\
&= \frac{-2\text{i} \pm 4\text{i}}{2} \\
z &= \text{i} \text{ or } -3\text{i}
\end{aligned}
$$

**Note:** In this case, the 2 complex roots are not a conjugate pair as not all the coefficients are real.

**Key questions to answer:**

- ❑ How do we find the modulus and argument of a complex number given in Cartesian form?
- ❑ How do we represent a complex number in Cartesian form by a point in the Argand diagram?

  - *How do we interpret geometrically, the terms 'real part', 'imaginary part', 'modulus', 'argument' and 'conjugate' of a complex number?*

**Prerequisite knowledge:** Cartesian plane for real numbers ; Modulus function ; Trigonometric functions (sine/cosine/tangent)

**Lecture Reading:** Section 5

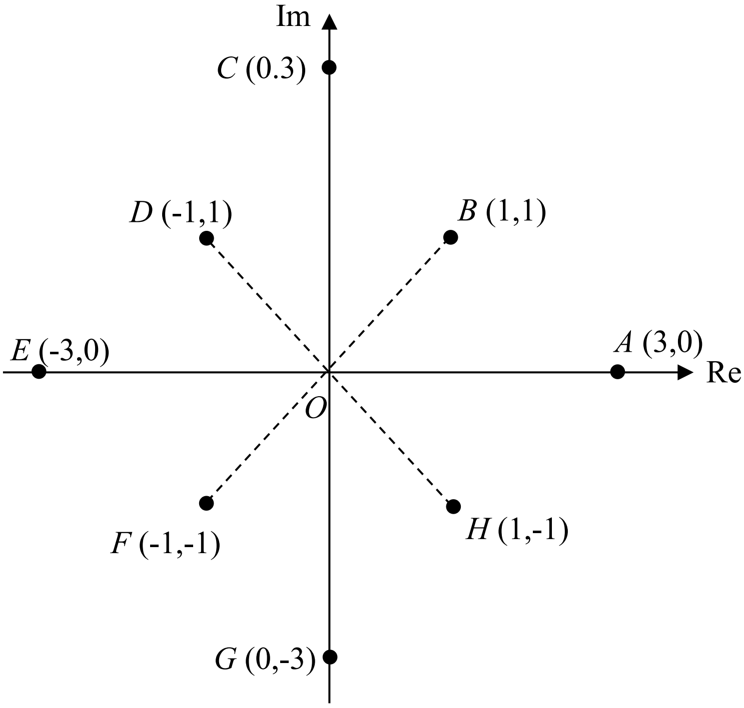
***Question 12*** **(Argand Diagram ; Modulus and Argument of Complex Numbers)**

Represent the following complex numbers in an Argand diagram, and find their corresponding modulus and argument:

$a=3,\ b=1+\mathrm{i},\ c=3\mathrm{i},\ d=-1+\mathrm{i},\ e=-3,\ f=-1-\mathrm{i},\ g=-3\mathrm{i}, h=1-\mathrm{i}$.

*Solution:*

Let *A*, *B*, …, *H* be the points representing *a*, *b*, …, *h* in the Argand diagram.



**Note:**

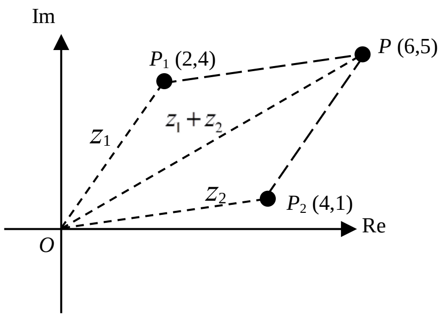
1. A dotted line is used to join the origin to the point representing the complex number in an Argand diagram.
2. The argument, $\theta$, of a complex number lies in the principal range, i.e. $-\pi < \theta \leq \pi$. The argument has a positive value if the complex number lies in the 1st or 2nd quadrant, and has a negative value if the complex number lies in the 3rd or 4th quadrant.
3. It is important to check the quadrant in which $z$ lies before computing its argument. In general, $\arg(x+\mathrm{i}y) \neq \tan^{-1}\left(\frac{y}{x}\right)$ for $x, y \in \mathbb{R}$.
4. arg(0) is undefined.
5. Complex numbers can alternatively be represented by position vectors e.g. the complex number $b$ in the above question can be represented by $\overrightarrow{OB}$.

***Question 13*** **(Vector Representation of Complex Numbers)**

Let $z_1 = 2+4\mathrm{i}$ and $z_2 = 4+\mathrm{i}$. Find $z_1+z_2$ and represent $z_1, z_2$ and $z_1+z_2$ by the points $P_1, P_2$ and $P$ in an Argand diagram.

*Solution:*

$z_1 + z_2 = (2+4\mathrm{i}) + (4+\mathrm{i}) = 6+5\mathrm{i}$




Comparing the above sum of two complex numbers with vector addition, we have:

$\overrightarrow{OP_1} + \overrightarrow{OP_2} = \overrightarrow{OP}$

(parallelogram law of vector addition)

**Note:**

1. Likewise, the difference between two complex numbers may be viewed as vector subtraction. E.g. how can we use vectors to represent $z_1 - z_2$?
2. Complex numbers can only be represented by position vectors and not displacement vectors. Thus, $z_2 - z_1$ cannot be represented by $\overrightarrow{P_1P_2}$.

***Question 14***

Let $z = x + \mathrm{i}y$, where $x > 0, y > 0$. Represent $z$ and $z^*$ in an Argand diagram. Write down the modulus and argument of these two complex numbers.

*Solution:*

**Note**:

1. $z$ and $z^*$ are reflections of each other in the real axis.
2. Observe that $\arg\left(z^*\right) = -\arg(z)$. Does this hold true for all $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$?
3. $zz^* = |z|^2 \quad \left(= x^2 + y^2\right)$

**Key questions to answer:**

- How do we convert a complex number from one of the following forms to another:

  (a) Cartesian form, (b) polar form and (c) exponential form?

- How do we multiply and divide two complex numbers given in polar and exponential forms?

- How do we represent a complex number in polar (or exponential) form by a point in the Argand diagram?

**Prerequisite knowledge:** Argand diagram ; Modulus and argument of complex numbers

**Lecture Reading:** Sections 6 to 7

| Cartesian form | Polar form | Exponential form |
|---|---|---|
| $z = x + \mathrm{i}y$ | $z = r(\cos\theta + \mathrm{i}\sin\theta)$,<br>where<br>$r = \lvert z \rvert = \sqrt{x^2 + y^2}$<br>$\theta = \arg(z), -\pi < \theta \le \pi$ | $z = r\mathrm{e}^{\mathrm{i}\theta}$,<br>where<br>$r = \lvert z \rvert = \sqrt{x^2 + y^2}$<br>$\theta = \arg(z), -\pi < \theta \le \pi$ |

***Question 15*** **(Complex Numbers in Polar Form and Exponential Form)**

Express the following complex numbers in polar form and exponential form:
(a) 3 (b) $-1+\mathrm{i}$ (c) $1-\mathrm{i}$

*Solution:*

| Cartesian form | Polar form | Exponential form |
|---|---|---|
| 3 | $3(\cos 0 + \mathrm{i}\sin 0)$ | $3\mathrm{e}^{\mathrm{i}(0)}$ |
| $-1+\mathrm{i}$ | $\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$ | $\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$ |
| $1-\mathrm{i}$ | $\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]$ | $\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}$ |

**Note:** $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ (Euler's Formula)

***Question 16*** **(Multiplication/Quotient of Complex Numbers in Exponential Form)**

Given that $z_1 = -1 + \mathrm{i}$ and $z_2 = 1 - \sqrt{3}\mathrm{i}$. Find

(a) $z_1 z_2$ (b) $\dfrac{z_1}{z_2}$ (c) $z_1^{\,3} z_2^{\,5}$,

leaving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$.

*Solution:*

(a) $$z_1 z_2 = \left[\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}\right]\left[2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}\right] = 2\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)} = 2\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{12}\right)}$$

From the above working, we observe that $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

(b) $$\frac{z_1}{z_2} = \frac{\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}}{2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}} = \frac{\sqrt{2}}{2}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)} = \frac{\sqrt{2}}{2}\mathrm{e}^{\mathrm{i}\left(\frac{13\pi}{12}\right)} \equiv \frac{\sqrt{2}}{2}\mathrm{e}^{\mathrm{i}\left(-\frac{11\pi}{12}\right)}$$

From the above working, we observe that $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ and $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

**Note:** We can extend our observations in (a) and (b) to the following results:

- $|z^n| = |z|^n$
- $\arg(z^n) = n\arg(z)$, where $n \in \mathbb{Z}$

(c)


Question 16 can be answered by first simplifying the given expressions in the Cartesian form $x+\mathrm{i}y$, where $x, y \in \mathbb{R}$, followed by conversion to the exponential form. Try doing it yourself, and compare the advantages and disadvantages of each method.