RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2018 Year 5

Chapter 7C: Complex Numbers III – De Moivre's Theorem and its Applications

SYLLABUS INCLUDES

H2 Further Mathematics:

- Use of De Moivre's Theorem to find the powers and nth roots of a complex number, and to derive trigonometric identities.

PRE-REQUISITES

- Trigonometry,
- Indices and algebraic manipulation

CONTENT

1 De Moivre's Theorem and its Applications

1.1 De Moivre's Theorem for Rational Index

Previously, we learnt that if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, where $r_1,\ r_2 > 0$, then $z_1 z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

So if $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$, where $n \in \mathbb{Z}^+$.

Extending this result to the set of integers, we have the following theorem:

De Moivre's Theorem for Integer Index

For $n \in \mathbb{Z}$,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \qquad \text{[Trigonometric version]}$$

$$\left(e^{i\theta}\right)^n = e^{in\theta} \qquad \text{[Exponential version]}$$

[Refer to Appendix for the proof of this result]

Thus, if $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, then $z^n = r^n[\cos(n\theta) + i\sin(n\theta)] = r^n e^{in\theta}$ for $n \in \mathbb{Z}$.

1.2 Application 1: Evaluating Powers of a Complex Number

The De Moivre's Theorem is useful in simplifying a complex number raised to some power, since if $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, then $z^n = r^n[\cos(n\theta) + i\sin(n\theta)] = r^n e^{in\theta}$, $n \in \mathbb{Z}$.

This theorem has a delightful representation in an Argand diagram. Consider the complex number $z = \cos\theta + i\sin\theta$. Then, $|z| = 1$ and $\arg(z) = \theta$.

By De Moivre's Theorem, $z^2 = \cos 2\theta + i\sin 2\theta \qquad |z^2| = 1,\ \arg(z^2) = 2\theta$

$z^3 = \cos 3\theta + i\sin 3\theta \qquad |z^3| = 1,\ \arg(z^3) = 3\theta$

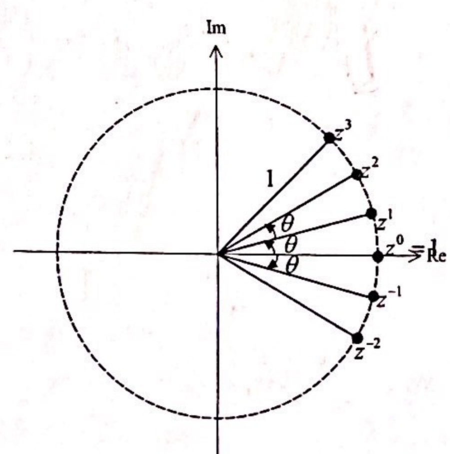
$\vdots$

Also, $z^{-1} = \cos(-\theta) + i\sin(-\theta) \qquad |z^{-1}| = 1,\ \arg(z^{-1}) = -\theta$

$z^{-2} = \cos(-2\theta) + i\sin(-2\theta) \qquad |z^{-2}| = 1,\ \arg(z^{-2}) = -2\theta$

$\vdots$

This means that all the integral powers of z will correspond to points on the unit circle, and the angle between each consecutive powers of z are separated by θ.



Example 1

Find $(\sqrt{3}-\mathrm{i})^8$ in the form $a+\mathrm{i}b$, giving a and b in exact form.

Solution

$\left|\sqrt{3}-\mathrm{i}\right|=2$ and $\arg(\sqrt{3}-\mathrm{i})=-\dfrac{\pi}{6}$ $\Rightarrow$ $\sqrt{3}-\mathrm{i}=2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$

$(\sqrt{3}-\mathrm{i})^8=\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}\right)^8$

$= 2^8 \mathrm{e}^{\mathrm{i}\left(-\frac{4\pi}{3}+2\pi\right)}$ need to fit into $-\pi<\theta\le\pi$ for agument

$=256\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)$

$=256\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)$

$(\sqrt{3}-\mathrm{i})^8=2^8\mathrm{e}^{\mathrm{i}8\left(-\frac{\pi}{6}\right)}$

$=256\mathrm{e}^{-\frac{4\pi}{3}\mathrm{i}}$

$=256\left(\cos\left(-\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{4\pi}{3}\right)\right)$

$=256\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)$

$=-128+128\sqrt{3}\mathrm{i}$

1.3 Application 2: Finding the n^{th} Roots of a Complex Number

Recall that the Fundamental Theorem of Algebra tells us that a polynomial equation of degree n will have n roots (real or non-real).

In this section, we use the De Moivre's Theorem to find all the roots of the polynomial equation $z^n=w$, where $n\in\mathbb{Z}^+$ and w is a given complex number.

The steps involved are as follows:

$w=r_0\mathrm{e}^{\mathrm{i}\alpha}$ [Express w in polar form]

Let $z=r\mathrm{e}^{\mathrm{i}\theta}$, then

$$z^n=w$$
$$\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^n=r_0\mathrm{e}^{\mathrm{i}\alpha}$$
$$r^n\mathrm{e}^{\mathrm{i}n\theta}=r_0\mathrm{e}^{\mathrm{i}\alpha}$$

Thus, $r^n=r_0$ and $n\theta=\alpha+2k\pi$

$\Rightarrow$ $r=\sqrt[n]{r_0}$ $\Rightarrow$ $\theta=\dfrac{\alpha}{n}+\dfrac{2k\pi}{n}$

$\therefore z=\sqrt[n]{r_0}\,\mathrm{e}^{\mathrm{i}\left(\frac{\alpha}{n}+\frac{2k\pi}{n}\right)}$, where k will take n consecutive values such that $-\pi<\arg(z)\le\pi$.

$\underbrace{k=0,1,\dots,n-1}_{n \text{ roots}}$

Example 2

Find the eighth roots of unity, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [i.e. find the 8 roots of the equation $z^8=1$]

Solution

Example 3

Find the 5 roots of the equation $z^5=-32\mathrm{i}$, giving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. Represent all the roots on an Argand diagram.

Solution

$z^5 = -32i$

$r^5 e^{i5\theta} = 32e^{-i\frac{\pi}{2}}$

$r^5 = 32$, $5\theta = -\frac{\pi}{2} + 2k\pi$

$r = 2$ $\qquad \theta = -\frac{\pi}{10} + \frac{2}{5}k\pi$,

$k = \pm 2, \pm 1, 0$

$\cancel{2e^{\frac{\pi}{10}i}}$, $2e^{-\frac{\pi}{10}i}$, $\cancel{2e}$

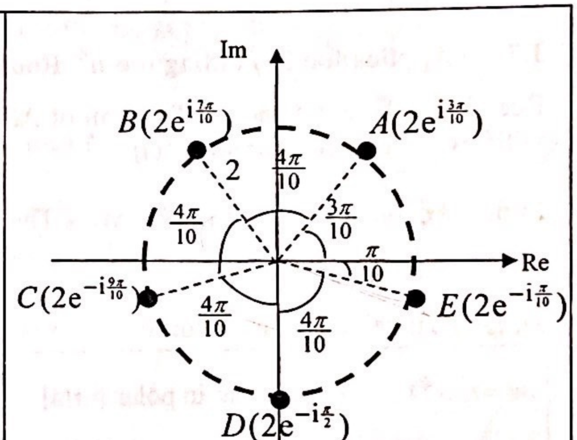
The 5 roots are

diff = $\frac{4\pi}{10}$

$2\mathrm{e}^{-\mathrm{i}\frac{9\pi}{10}}$, $2\mathrm{e}^{-\mathrm{i}\frac{5\pi}{10}}$, $2\mathrm{e}^{-\mathrm{i}\frac{\pi}{10}}$, $2\mathrm{e}^{\mathrm{i}\frac{3\pi}{10}}$ and $2\mathrm{e}^{\mathrm{i}\frac{7\pi}{10}}$.

Remarks:

There is no need to substitute the values of k to find the 5 roots unless we need them for subsequent parts (e.g. to represent them on an Argand diagram)



Remarks:

- The points representing the 5 roots lie on a circle with centre at the origin and radius 2.
- The points representing the 5 roots also form the vertices of a regular pentagon

Example 4

Without the use of a GC, solve the equation $z^6+z^5+\cdots+z+1=0$.

$\sum_{n=0}^{6} z^n$

Solution:

$1+(\sqrt[6]{r}+\sqrt[5]{r}+\ldots\sqrt{r})[\cos(\theta$

$= \dfrac{1(1-e^{7i\theta})}{1-e^{i\theta}}$ C-S-C-S z^7-z^0

$z=e^{i\theta}$

$= \dfrac{e^{\frac{7i\theta}{2}}(e^{-\frac{7i\theta}{2}}-e^{\frac{7i\theta}{2}})}{(\ldots)}$

$= \dfrac{e^{\frac{7\theta}{2}}(-2\sin\frac{7}{2}i\theta)}{(\ldots)}$

$= \ldots = 0$

solve

1.4 Application 3: Proving Trigonometric Identities

De Moivre's Theorem, together with the Binomial Theorem, can be used to express $\cos n\theta$, $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$, where n is a positive integer.

Let $z=\cos\theta+\mathrm{i}\sin\theta$, then $z^n=(\cos\theta+\mathrm{i}\sin\theta)^n=\cos n\theta+\mathrm{i}\sin n\theta$, by De Moivre's Theorem.

Also, for $n\in\mathbb{Z}^+$, by Binomial Theorem,

$$z^n=(\cos\theta+\mathrm{i}\sin\theta)^n=(\cos\theta)^n+\binom{n}{1}(\cos\theta)^{n-1}(\mathrm{i}\sin\theta)+\binom{n}{2}(\cos\theta)^{n-2}(\mathrm{i}\sin\theta)^2+\ldots+(\mathrm{i}\sin\theta)^n$$

Comparing Real and Imaginary parts,

0, 2, 4, 6 $$\cos n\theta=\mathrm{Re}\left[(\cos\theta+\mathrm{i}\sin\theta)^n\right]=(\cos\theta)^n-\binom{n}{2}(\cos\theta)^{n-2}(\sin\theta)^2+\binom{n}{4}(\cos\theta)^{n-4}(\sin\theta)^4+\ldots$$

1, 3, 5 $$\sin n\theta=\mathrm{Im}\left[(\cos\theta+\mathrm{i}\sin\theta)^n\right]=\binom{n}{1}(\cos\theta)^{n-1}(\sin\theta)-\binom{n}{3}(\cos\theta)^{n-3}(\sin\theta)^3+\ldots$$

Note that the dots at the end of the above 2 expressions indicate that they continue for as long as the powers of $\cos\theta$ are non-negative and the powers of $\sin\theta$ is smaller than or equal to n. The precise expression for the last term will depend on whether n is even or odd.

Example 5

By considering $(\cos\theta+\mathrm{i}\sin\theta)^2$, prove the double angle formulae for cosine and sine functions.

Solution

By Binomial Theorem, we have

$$(\cos\theta+\mathrm{i}\sin\theta)^2=(\cos^2\theta-\sin^2\theta)+\mathrm{i}(2\sin\theta\cos\theta)$$

By De Moivre's Theorem, we have

$$(\cos\theta+i\sin\theta)^2=\cos 2\theta+i\sin 2\theta$$

$=$

Equating real and imaginary parts, we get

Comparing real part: $\cos 2\theta=\cos^2\theta-\sin^2\theta$

" im part: $\sin 2\theta=2\sin\theta\cos\theta$

which are the double angle formulae for cosine and sine functions!

Example 6

Use De Moivre's Theorem to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$. Hence express $\tan 3\theta$ in terms of $\tan\theta$. By using a suitable value for θ, find the exact value of $\tan\dfrac{\pi}{12}$.

$z=(\cos\theta+i\sin\theta)^3$

$z^3=\cos 3\theta+i\sin 3\theta$

$\cos 3\theta=\mathrm{Re}(z)=$

Solution

By Binomial Theorem, we have

$\cos^3\theta+\cos^2\theta\, i\sin\theta+\cos\theta\, i^2\sin^2\theta+i^3\sin^3\theta$

$=\cos^3\theta+i\cos^2\theta\sin\theta-\cos\theta\sin^2\theta-i\sin^3\theta$

$$(\cos\theta+\mathrm{i}\sin\theta)^3=\cos^3\theta+3\mathrm{i}\cos^2\theta\sin\theta+3\mathrm{i}^2\cos\theta\sin^2\theta+(\mathrm{i}\sin\theta)^3$$
$$=\cos^3\theta+3\mathrm{i}\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-\mathrm{i}\sin^3\theta$$

$\cos^3\theta-\cos\theta\sin^2\theta$

By De Moivre's Theorem, we have

$$\cos 3\theta+i\sin 3\theta=(\cos\theta+i\sin\theta)^3$$

$\sin 3\theta$

$=\cos^2\theta\sin\theta-\sin^3\theta$

Equating real and imaginary parts, we get

$\tan 3\theta=\dfrac{\cos^2\theta\sin\theta-\sin^3\theta}{\cos^3\theta-\cos\theta\sin^2\theta}$

show

$$\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$$

$\frac{\pi}{12}=3\theta$

$\theta=\frac{\pi}{36}$

~~LHS = RHS~~

$\tan\frac{\pi}{12}=0.0874$

~~real parts are equate~~

$$\sin 3\theta=$$

$$=3\cos^2\sin\theta-\sin^3\theta$$

Given that $z=\cos\theta+i\sin\theta$, the following 2 results can be used to express $\cos^n\theta$, $\sin^n\theta$ in terms of $\cos n\theta$ and $\sin n\theta$.

$z+z^*$ $\quad$ $z-z^*$

Result 1 : $z+\frac{1}{z}=2\cos\theta$ and $z-\frac{1}{z}=2i\sin\theta$ ← for proving trigo identities

Proof : Let $z=\cos\theta+i\sin\theta$, then $\frac{1}{z}=\cos\theta-i\sin\theta$

Hence $z+\frac{1}{z}=2\cos\theta$ and $z-\frac{1}{z}=2i\sin\theta$

Result 2: $z^n+\frac{1}{z^n}=2\cos n\theta$ and $z^n-\frac{1}{z^n}=2i\sin n\theta$

Proof : Let $z=\cos\theta+i\sin\theta$, then $\frac{1}{z}=\cos\theta-i\sin\theta$

By De Moivre's Theorem,

$z^n=\cos n\theta+i\sin n\theta$, $\frac{1}{z^n}=\cos n\theta-i\sin n\theta$

Hence $z^n+\frac{1}{z^n}=2\cos n\theta$ and $z^n-\frac{1}{z^n}=2i\sin n\theta$

$z=\cos\theta+i\sin\theta$

Example 7

Show that $\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$.

Solution

Let $z=\cos\theta+i\sin\theta$, then $\frac{1}{z}=\cos\theta-i\sin\theta$,

Hence $z+\frac{1}{z}=2\cos\theta$ [From Result 1]

$$(2\cos\theta)^4=\left(z+\frac{1}{z}\right)^4$$

$$16\cos^4\theta=z^4+4z^3\left(\frac{1}{z}\right)+6z^2\left(\frac{1}{z}\right)^2+4z\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^4$$

$$=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$$

$$=2\cos 4\theta+4(2\cos 2\theta)+6 \text{ [from Result 2]}$$

$$\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$$

Example 8

Express $\sin^5\theta$ in terms of multiple angles of θ.

Solution

Let $z=\cos\theta+\mathrm{i}\sin\theta$, then $\frac{1}{z}=\cos\theta-\mathrm{i}\sin\theta$.

Hence $z-\frac{1}{z}=2i\sin\theta$ [from result 1]

$(2i\sin\theta)^5=\left(z-\frac{1}{z}\right)^5$

$32i\sin^5\theta=z^5-5z^4\left(\frac{1}{z}\right)+10z^3\left(\frac{1}{z}\right)^2-10z^2\left(\frac{1}{z}\right)^3+5z\left(\frac{1}{z}\right)^4-\left(\frac{1}{z}\right)^5$

$=\left(z^5-\frac{1}{z^5}\right)-5\left(z^3-\frac{1}{z^3}\right)+10\left(z-\frac{1}{z}\right)$

$=2i\sin5\theta-5(2i\sin3\theta)+10(2i\sin\theta)$ [from result 2]

$\sin^5\theta=\frac{1}{16}(\sin5\theta-5\sin3\theta+10\sin\theta)$

De Moivre's Theorem is useful in finding the sum of series that involves the sine and cosine function.

Let $z=\cos\theta+\mathrm{i}\sin\theta$.

For $n\in\mathbb{Z}^+$, $z+z^2+\ldots+z^n=\frac{z(1-z^n)}{1-z}$ [Sum of the first n terms of a G.P.]

Since $z+z^2+\ldots+z^n=(\cos\theta+\mathrm{i}\sin\theta)+(\cos2\theta+\mathrm{i}\sin2\theta)+\ldots+(\cos n\theta+\mathrm{i}\sin n\theta)$,

then $\cos\theta+\cos2\theta+\ldots+\cos n\theta=\mathrm{Re}(z+z^2+\ldots+z^n)=\mathrm{Re}\left(\frac{z(1-z^n)}{1-z}\right)$

and $\sin\theta+\sin2\theta+\ldots+\sin n\theta=\mathrm{Im}(z+z^2+\ldots+z^n)=\mathrm{Im}\left(\frac{z(1-z^n)}{1-z}\right)$.

let $z=\cos\theta+i\sin\theta$, $z^*=\cos\theta-i\sin\theta$, $z-\frac{1}{z}=2i\sin\theta$

~~$z^5=\cos5\theta+i\sin5\theta=(\cos\theta+i\sin\theta)^5$~~ $z^5=\cos5\theta+i\sin5\theta \Rightarrow z^5-\frac{1}{z^5}=2i\sin5\theta$

$(2i\sin\theta)^5=\left(z-\frac{1}{z}\right)^5$

$32i\sin^5\theta=z^5+5z^4\left(-\frac{1}{z}\right)+10z^3\left(-\frac{1}{z}\right)^2+10z^2\left(-\frac{1}{z}\right)^3+5z\left(-\frac{1}{z}\right)^4+\left(-\frac{1}{z}\right)^5$

$=z^5-5z^3+10z-\frac{10}{z}+\frac{5}{z^3}-\frac{1}{z^5}$

$=z^5-\frac{1}{z^5}-5z^3+\frac{5}{z^3}+10z-\frac{10}{z}$

$=2i\sin5\theta-10i\sin3\theta+20i\sin\theta$

$\sin^5\theta=\frac{1}{16}(\sin5\theta-5\sin3\theta+10\sin\theta)$ ✓

Example 9

Prove that $\sin x+\sin 2x+\ldots+\sin nx=\dfrac{\sin\frac{nx}{2}\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}, x\neq 2n\pi, n\in\mathbb{Z}^+$.

Solution

Let $z=\cos x+i\sin x$

$\sin x+\sin 2x+\ldots+\sin nx=\operatorname{Im}(z+z^2\ldots+z^n)$

$(z+z^2+\ldots+z^n)=z\left(\frac{1-z^n}{1-z}\right)$ $\because$ GP, first term $=z$, ratio $=z$, no of terms $=n$

$=e^{ix}\left(\frac{1-e^{inx}}{1-e^{ix}}\right)$ make conjugate for +/−

$=e^{ix}\frac{e^{i\frac{nx}{2}}}{e^{i\frac{x}{2}}}\left(\frac{\overset{z^*}{e^{-i\frac{nx}{2}}}-\overset{z}{e^{i\frac{nx}{2}}}}{e^{-i\frac{x}{2}}-e^{i\frac{x}{2}}}\right)=$

$=e^{i\frac{n+1}{2}x}\left(\frac{-2i\sin\frac{nx}{2}}{-2i\sin\frac{x}{2}}\right)$

$=\left(\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right)e^{i\frac{n+1}{2}x}$

$\sin x+\sin 2x+\ldots+\sin nx=\left(\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right)\operatorname{Im}\left(\cos\frac{n+1}{2}x+i\sin\frac{n+1}{2}x\right)$

$=\dfrac{\sin\frac{nx}{2}\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}$ (proved)

APPENDIX

PROOF OF DE MOIVRE'S THEOREM FOR INTEGER INDEX

Proof of De Moivre's Theorem for positive integers

Let P_n be the statement $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta),\ n \in \mathbb{Z}^+$.

When $n = 1$, LHS $= (\cos\theta + \mathrm{i}\sin\theta)^1 = \cos\theta + \mathrm{i}\sin\theta$ and

RHS $= \cos(1\theta) + \mathrm{i}\sin(1\theta) = \cos(\theta) + \mathrm{i}\sin(\theta)$

Hence LHS = RHS and P_1 is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos(k\theta) + \mathrm{i}\sin(k\theta)$

To prove P_{k+1} is true, i.e. $(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = \cos\left((k+1)\theta\right) + \mathrm{i}\sin\left((k+1)\theta\right)$

$$
\begin{aligned}
\text{LHS} &= (\cos\theta + \mathrm{i}\sin\theta)^{k+1} \\
&= (\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta) \\
&= (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta) \\
&= \left[\cos k\theta\cos\theta - \sin k\theta\sin\theta + \mathrm{i}(\cos k\theta\sin\theta + \sin k\theta\cos\theta)\right] \\
&= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta \qquad \text{using the addition formulae for sine and cosine.} \\
&= \text{RHS}
\end{aligned}
$$

Therefore P_{k+1} is true if P_k is true.

Since P_1 is true, by mathematical induction, P_n is true for all positive integers.

Proof of De Moivre's Theorem for negative integers

Let $n = -m, m \in \mathbb{Z}^+$. Then

$$
\begin{aligned}
(\cos\theta + \mathrm{i}\sin\theta)^n &= (\cos\theta + \mathrm{i}\sin\theta)^{-m} \\
&= \frac{1}{(\cos\theta + \mathrm{i}\sin\theta)^m} \\
&= \frac{1}{(\cos m\theta + \mathrm{i}\sin m\theta)} \\
&= \frac{(\cos m\theta - \mathrm{i}\sin m\theta)}{(\cos^2 m\theta + \sin^2 m\theta)} \\
&= (\cos(-m\theta) + \mathrm{i}\sin(-m\theta)) \\
&= \cos n\theta + \mathrm{i}\sin n\theta
\end{aligned}
$$

Hence the result is true for negative integers as well.

SUMMARY