RIVER VALLEY HIGH SCHOOL
2023 JC2 Preliminary Examination
Higher 2

NAME	

CLASS							INDEX NUMBER	

MATHEMATICS **9758/01**

Paper 1 **15 Sep 2023**

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your class, index number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
Total	

Calculator Model:

This document consists of 25 printed pages and 3 blank pages.

1 (i) Without using a calculator, solve $\dfrac{4x^2-4x+1}{1+x-2x^2}<0$. [3]

(ii) Hence solve $\dfrac{\left(2^{x+1}-1\right)^2}{1+2^x-2^{2x+1}}\le 0$. [3]

2 (i) The complex number $z=a+\mathrm{i}$, where a is a real constant, is such that the modulus of $\dfrac{z-1}{z^*(1+\mathrm{i})}$ is $\dfrac{1}{2}$. Find the value of a where $a<2$. [3]

(ii) It is given that a is the value found in part (i). The complex number w has argument $\dfrac{\pi}{4}$. Find the 3 smallest positive integer of n such that the complex number $\left(\dfrac{z-1}{z^*w}\right)^n$ is purely imaginary with the imaginary part negative. [3]

3 A sequence $u_1,\ u_2, u_3,\ ...$ is such that $u_n=\dfrac{1}{n}$ for $n\ge 1$.

(i) Show that for $n\ge 2$, $u_{n-1}-2u_n+u_{n+1}=\dfrac{A}{n^3-n}$, where A is a constant to be found. [2]

(ii) Hence find $\displaystyle\sum_{n=2}^{N}\frac{1}{n^3-n}$. (You need not express the answer as a single fraction.) [2]

(iii) Explain why $\displaystyle\sum_{n=2}^{N}\frac{1}{n^3-n}$ converges. [1]

(iv) Use your result in part (ii) to find $\displaystyle\sum_{n=3}^{N}\frac{1}{(n-2)(n-1)(n)}$. [2]

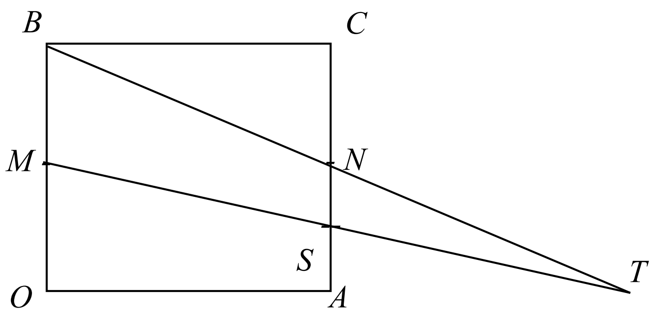
4 It is known that x_0 is one of the roots of the equation $x^4-4x^3+6x^2-ax+b=0$, where a and b are real.

(i) Show algebraically that $x_0{}^*$ is also a root. [1]

(ii) Given that $x_0=2-\mathrm{i}$, find the values of a and b and the other roots. [4]

(iii) Hence, solve $by^4-ay^3+6y^2-4y+1=0$. [2]

5



OACB is a square with origin *O*. Points *A* and *B* are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. *M* and *N* are midpoints of *OB* and *AC* respectively. *S* is on *AC* such that $AS : AC = 1 : 4$. *MS* produced and *BN* produced meet at point *T*.

(i) Find the position vector $\overrightarrow{OS}$ in terms of **a** and **b**. [2]

(ii) Find vector equations of the line *MS* and the line *BN*. [3]

(iii) By finding the position vector of T, deduce that *T*, *A* and *O* are collinear points. [2]

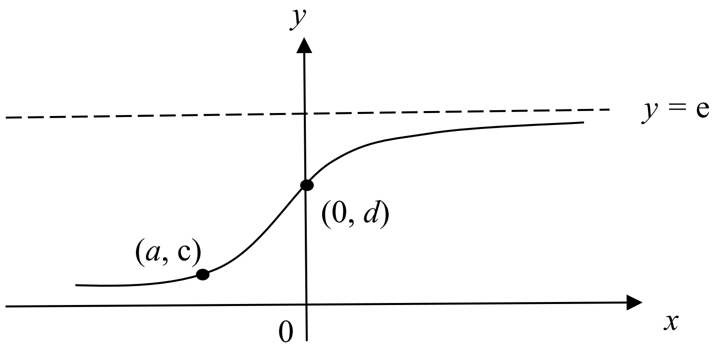
6 A function f is said to be self-inverse if $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$.
Show that f given by

$$\mathrm{f}: x \mapsto \frac{ax+b}{x-a}, \text{ where } x \in \mathbb{R}, x \neq a,$$

for some constants $a, b \in \mathbb{R}$, is a self-inverse function. [2]

State a geometrical relationship between $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$. [1]

The graph below shows the function g defined for $x \in \mathbb{R}$. The equation of its asymptotes are $y = 0$ and $y = \mathrm{e}$ and it passes through the points (a, c) and $(0, d)$.



Determine if gf exists. If it exists, give its domain and range. [4]

7 (a) An arithmetic series has non-zero first term a and exactly n terms, where n is odd. Given that the sum of the first and middle terms equals the last term, find the middle term in terms of a. [3]

(b) A convergent geometric series with positive terms has first term $3p^3$ and common ratio $\frac{p}{q^2}$, where $q \neq 0$.

(i) If the nth term is m, show that

$$\ln m = \ln 3 + (n+A)\ln p + (B-2n)\ln q,$$

where A and B are constants to be determined. [2]

(ii) Suppose $q = 2p$, find the range of values of p. [3]

8 The graph of a function, $y=\mathrm{f}(x)$, cuts the y-axis at 1 and has a gradient of -1 at the same point.

(i) Given that $(1-x^2)\mathrm{f}''(x) - x\mathrm{f}'(x) = 0$, find the first three non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. [4]

(ii) Hence, find the series expansion of $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$ up to and including the term in x^2. [2]

(iii) Deduce the series expansion of $\ln|\mathrm{f}(x)|$ up to and including the term in x^2. [2]

9 The curve C has equation $y=\dfrac{ax^2+bx-14}{x+c}$. It is given that the equations of the asymptotes of C are $x=3$ and $y=x+5$.

(i) Determine the values of a, b and c. [3]

(ii) Sketch the graph of C indicating clearly the equations of the asymptotes, coordinates of axial intercepts and stationary points. [3]

(iii) On the same axes, sketch the graph of $\dfrac{(x-3)^2}{m^2}-(y-8)^2=1$, where m is a positive real constant, indicating clearly the equations of the asymptotes [2]

(iv) Hence state the range of values for m for which the equation $\dfrac{(x-3)^2}{m^2}-\left(\dfrac{x^2-6x+10}{x-3}\right)^2=1$ has roots. [1]

10 The curve C has parametric equations $x=\sqrt{4+t^2}$, $y=t^2$, $t\ge 0$.

(i) Find the equation of the tangent to curve C at point $P\left(\sqrt{4+p^2}, p^2\right)$. [3]

(ii) The tangent at P meets the x-axis at point Q. Point M is the midpoint of the line segment PQ. Find the cartesian equation of the curve traced out by M as p varies. [3]

(iii) Show that the area of the region bounded by C, the x-axis and the line $x=3$ is given by $\displaystyle\int_a^b \frac{t^3}{\sqrt{4+t^2}}\,\mathrm{d}t$, where a and b are constants to be determined. Hence evaluate the integral to find the exact area of the region. [4]

11 The *Law of Universal Gravitation* states that the force F causing any two bodies A and B with constant masses to be attracted toward each other, can be expressed as $F=\dfrac{k}{r^2}$, where r is the distance between them and k is a positive constant.

Body A is situated at the point $(1, 0)$ while body B is in motion along the path described by the equation $\dfrac{x^2}{4}+y^2=1$.

The bodies A and B have constant masses, and F denotes the gravitational force between them.

(i) Find an expression for F in terms of k and x. [2]

(ii) Using differentiation, find the maximum value of F as x varies, leaving your answer in terms of k. [5]

(iii) Sketch the graph of F as x varies, stating clearly the coordinates of the turning point and end points. [2]

(iv) Write down the minimum value of F in terms of k. [1]

An ellipse with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ where $a>b$, has two foci at the points with coordinates $(c, 0)$ and $(-c, 0)$ where c is a constant such that $0<c<a$.

(v) Write down the coordinates of the points when B is equidistant from the two foci. Hence, determine the force F between A and B at these points. [2]

12 In a tank at an aquarium, fish pollute the water at a constant rate. A filter is installed into a new clean tank to remove the pollutant. The rate at which it removes the pollutant from the tank is proportional to the square root of the amount of pollutant present in the tank at that instant.

Immediately after being cleaned of all pollutant, the amount of pollutant in the tank is increasing at a rate of 5 units per day. It is known that eventually, with the filter installed in a new clean tank, the amount of pollutant in the tank would stabilize at 75 units.

(i) Show that $\frac{\mathrm{d}Q}{\mathrm{d}t} = 5 - \sqrt{\frac{Q}{3}}$, where Q is the amount of pollutant present at time t. [3]

(ii) Using the substitution $x = \sqrt{\frac{Q}{3}}$, show that the differential equation in part (i) can be written as $6x\frac{\mathrm{d}x}{\mathrm{d}t} = 5 - x$. [2]

(iii) Hence or otherwise, show that the particular solution of the differential equation in part **(i)** is

$$\left(a - \sqrt{\frac{Q}{3}}\right)^{b} \mathrm{e}^{\sqrt{\frac{Q}{3}}} = m\,\mathrm{e}^{pt}$$

where a, b, p and m are constants to be determined. [6]

The water in the tank becomes toxic if the pollutant level gets too high. To avoid this, the tank is cleaned when the pollutant level reaches 48 units.

(iv) By comparing the time taken for the pollutant level to reach 48 units with and without the filter, comment on the effectiveness of the filter. [2]

END OF PAPER

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