Section A: Pure Mathematics [40 marks]

1   The complex numbers z_1 and z_2 are such that $z_1 = 4e^{i\frac{\pi}{3}}$ and $z_2 = 2e^{-i\frac{\pi}{3}}$.

   (i)   On an Argand diagram with origin O, mark out the points which represent
          z_1 and z_2 respectively. Deduce the exact value of $|z_1 - z_2|$. [3]

   (ii)   Write down a complex number a such that $z_1 = az_2$. Hence or otherwise, describe
          geometrically the relation between z_1 and z_2. [2]

2   A sequence u_0, u_1, u_2, ... is such that $u_{n+1} - u_n = -\dfrac{4}{3}\left(\dfrac{1}{3}\right)^n + Q\left(\dfrac{2}{3}\right)^n$, where Q is a
     constant and $n \geq 0$.

   (i)   Given that $u_0 = 3$, $u_1 = \dfrac{4}{3}$, find Q and u_2. [2]

   (ii)   Show algebraically that for any real positive constant ε, there exists an integer n_0
          such that $|u_{n+1} - u_n| \leq \varepsilon$ for $n \geq n_0$. Find an integer n_0 when $\varepsilon = 0.001$. [4]

3   (a)   A curve has equation $y = f(x)$, where

$$f(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq 1, \\ -\dfrac{1}{2}x + \dfrac{3}{2} & \text{for } 1 < x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

   (i)   Sketch the curve for $-1 \leq x \leq 4$. [3]

   (ii)   On a separate diagram, sketch the curve with equation $y = f\left(\dfrac{1}{2}x\right) - 1$, for

          $-1 \leq x \leq 4$. Write down the coordinates of the end points of this curve. [2]

   (b)   State a sequence of transformations that will transform the curve with equation
          $y = \ln x$ onto the curve with equation $y = \ln(2 - x)^3$. [2]

4   The line l_1 has equation $\mathbf{r} = 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ where λ is a parameter. Referred to the origin
     O, the points A and P have position vectors $\mathbf{i} + 4\mathbf{k}$ and $-3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ respectively.

   (i)   Find the cartesian equation of the plane π_1 containing the line l_1 and the point A. [2]

   (ii)   The point Q is the foot of perpendicular of P on l_1. Find the position vector of Q. [3]

   (iii) The line l_2 passes through P and is parallel to the vector $3\mathbf{i} + \mathbf{j} + m\mathbf{k}$ where m is a
          constant. Find the value of m given that l_2 is parallel to π_1.
          h denotes the distance between l_2 and π_1. Find the value of h. [3]

(iv)   The plane π_2 is parallel to π_1. The l_3 passing through P and Q, intersects π_2 at the point R. The points P, Q and R are such that $PQ:PR = 2:5$. Without finding the equation of π_2, find the possible value(s) of the distance between π_1 and π_2 in terms of h.   [2]

5   (i)   Using the substitution $u^2 = x+1$, evaluate $\int_{-1}^{a-1} x\sqrt{x+1}\,dx$, giving your answer in terms of a.   [5]

     (ii)   The curve C has equation $y = x\sqrt{x+1}$. The region R is bounded by C, the lines $y = \sqrt{2}$ and $x = -1$. Find the exact volume of the solid obtained when R is rotated through 2π radians about the line $y = \sqrt{2}$.   [7]

Section B: Statistics [60 marks]

6   Two fair six-sided dice are thrown and the highest score X is recorded.
     (i)   State the probability distribution for X.   [1]
     (ii)   Find the expected value and variance for X.   [3]

7   A group of 8 people consisting of 2 children, 4 women and 2 men goes for lunch, photo-shoot and a game.

     (i)   At lunch, they are seated at a round table such that each child is seated between 2 women and no men are seated together. Find the number of ways they can sit.   [3]
     (ii)   After lunch, they go for a photo-shoot. They stand in a row such that all the women are in descending order of height, from their left to right. Find the number of ways they can stand, assuming that all the women are of distinct height.   [2]
     (iii)   After the photo-shoot, they are split into 2 teams to play a board game. Each team consists of a child, 2 women and a man. Find the number of ways the 2 teams can be formed.   [2]

8   A blind box series named MathBots includes 4 bots and 4 number sets as accessories for the bots. Bots are named α, β and γ with ω as the special character bot. The number sets are $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{N}$. Each box of the series contains one of the bots and one of the number sets.

The data in the table below shows the probabilities of getting boxes with the specified contents.

	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{N}$
α	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{20}$
β	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$
γ	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{1}{20}$
ω	0	0	0	$\frac{1}{10}$

(For example, there is a $\frac{1}{10}$ probability that a randomly chosen box contains ω and $\mathbb{N}$.)

A collector randomly chooses 4 boxes for purchase.

(i)   Show that the probability of the collector getting all the bots and number sets is $\frac{3}{320}$.   [2]

(ii)   Find the probability that the collector gets all the bots given he manages to get all the numbers sets.   [3]

(iii)   Determine if the event that he gets all the bots and the event that he get all the number sets are independent.   [2]

9   Professor Aok conducts a course with grading at the end of each run of the course for certification. Each run of the course takes in exactly 10 students. It is found that on average a student has $p\%$ chance of achieving a distinction. Let X be the number of students who achieve a distinction during a run of the course.

(i)   State an assumption for X to be well modelled by a binomial distribution.   [1]

Assume that X is well modelled by a binomial distribution.

(ii)   Given that the probability 7 students scoring distinction is greater than the probability of any other number of students scoring distinction, find the possible range of values of p.   [2]

(iii)   There is a probability of 0.00228786 that there are only 2 students who achieved distinction in a particular run of the course, find the value of p.   [2]

(iv)   A check revealed that of the 8 recently concluded courses, at least 5 had more than 6 students achieving distinction. Find the probability that out of those recently concluded courses, there are at most 7 runs with more than 6 students achieving distinction.   [3]

10   (a)   With the aid of diagrams, explain the difference between the least square regression lines of y on x and that of x on y. [3]

(b)   To investigate the effect of fertilizers on plant growth, 10 randomly selected plants of the same species are given varying amount of fertilizers and the increase in height of the plants is measured after a week.

Amount of fertilizers, x (ml)	7	5	7	8	6	10	4	9	6	9
Increase in height of plant, y (cm)	9	7	8	10	9	10	3	8	8	10

(i)   Using a scatter diagram, explain if a linear relationship between the amount of fertilizers used and increase in height of plants is appropriate. [2]

(ii)   Another relationship $y = a + b \ln x$ is considered more appropriate. Use this to estimate the amount of fertilizers needed to achieve an increase of 12 cm in the height of the plant. [2]

(iii) Justify if this prediction is reliable. [1]

(iv)   Suppose that the unit of y is in millimetres and that of x remains unchanged. State how the value of the product-moment correlation coefficient will change and find the corresponding equation of the regression line of y' on $\ln x$, where y' is the increase in height in millimetres. [2]

11   Rectangular blocks of wood undergo crafting and lacquering to become ornaments. The masses of the rectangular blocks, X g, are known to have normal distribution with mean 300 g and standard deviation 20 g. Crafting the company logo on a block reduces its mass by 10%. After that, a layer of lacquer is applied to the crafted wooden block that increases its mass by 5%.

(i)   Show that the probability that a piece of completed ornament has mass between 290 g and 350 g is approximately 0.365. [3]

(ii)   During an inspection, completed ornaments are selected randomly and their masses are recorded. Find the probability that the 10th ornament selected is the 3rd piece that has mass between 290 g and 350 g. [2]

The ornament is placed into a gift box before it is sold. The masses of the gift boxes are modelled as 100α % of the masses of the ornaments.

(iii) Given that 40% of the total mass of an ornament and its box exceeds 360 g, determine the value of α. [5]

In another batch of wooden blocks, the manufacturer claims that the masses of the block have normal distribution with mean 130 g and standard deviation 80 g. Justify if this distribution is appropriate. [2]

12   The amount of impurities in filtered water in a town is believed to be 149 units per gallon. A newly developed filter claims to be more effective in filtering out impurities. Fifty-two water samples are collected in the town over a period after the newly developed filter is used. The amount of impurities, x units per gallon, in the sample is summarized by

$$\Sigma x = 7644 \qquad \Sigma(x-20)^2 = 840008$$

(i) Find the unbiased estimates of the mean and variance of the amount of impurities in the water. [2]

(ii) Test, at the 5% level of significance, if the newly developed filter is indeed more effective. [4]

(iii) Briefly explain the meaning of the p-value obtained in the test in part (ii). Sketch a diagram to show this p-value. [2]

The newly developed filter is being used in another town with a different water source. One hundred water samples are collected and it is found that the samples have standard deviation 29.85 units per gallon. The data provides sufficient evidence that the mean amount of impurities differs from 150 units per gallon at the 5% level of significance. Determine the range of values of the sample mean. [4]

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