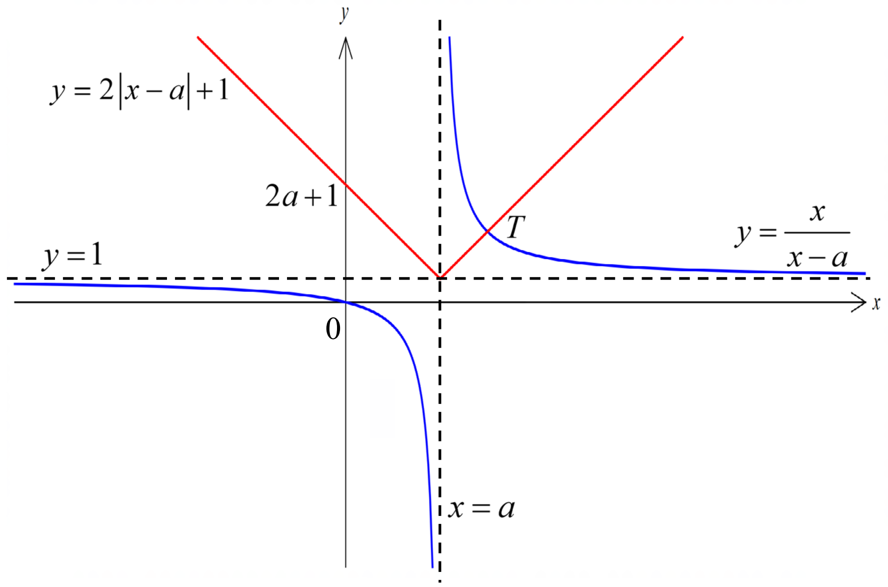
RAFFLES INSTITUTION
2022 Year 6 H2 Mathematics Preliminary Examination Paper 1
Questions and Solutions with comments

1 **(i)** On the same axes, sketch the graphs of $y=\dfrac{x}{x-a}$ and $y=2|x-a|+1$, where a is a positive constant. [3]

(ii) Hence solve the inequality $\dfrac{x}{x-a}>2|x-a|+1$, leaving your answer in terms of a. [3]

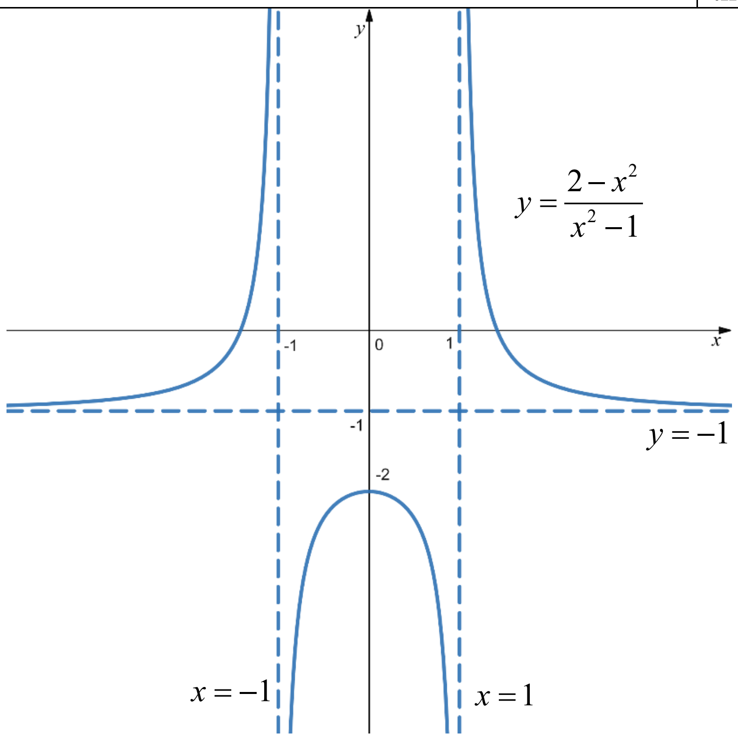
(i)		Note that the asymptotes intersect at $(a, 1)$, and the vertex of the modulus graph is at the same point. Also note the curve passes through the origin.
(ii)	Let the 2 graphs intersect at the point T. To solve for T, consider $\dfrac{x}{x-a}=2(x-a)+1$ $x=2(x-a)^2+x-a$ $(x-a)^2=\dfrac{a}{2}$ $x=a\pm\sqrt{\dfrac{a}{2}}$ Since $x>a$ at the point T, $x=a+\dfrac{\sqrt{2a}}{2}$. Thus the solution is $a<x<a+\dfrac{\sqrt{2a}}{2}$.	The graphs are in **(i)**, so it is easier to solve the equation to find the x-coordinate of the intersection point, then refer to the graphs in **(i)** to write down the solution to the inequality.

2 A function is defined as $\mathrm{f}(x)=\dfrac{1+2x-x^2}{2(x^2-2x)}$, $x\in\mathbb{R}, x\neq 0, 2$.

(i) Show that $\mathrm{f}(x)$ can be written in the form $q\left(\dfrac{2-(x+p)^2}{(x+p)^2-1}\right)$, where p and q are constants to be found. [2]

(ii) Hence describe a sequence of transformations that will transform the graph of $y=\dfrac{2-x^2}{x^2-1}$ onto the graph of $y=\mathrm{f}(x)$. [2]

(iii) Determine, with the help of a sketch or otherwise, the set of values of k for which the equation $\dfrac{2-x^2}{x^2-1}=k$ has no real roots. [2]

(i)	$\dfrac{1+2x-x^2}{2(x^2-2x)}$ $=\dfrac{1}{2}\left[\dfrac{2-(x^2-2x+1)}{(x^2-2x+1)-1}\right]$ $=\dfrac{1}{2}\left[\dfrac{2-(x-1)^2}{(x-1)^2-1}\right]$ Thus $p=-1$ and $q=\dfrac{1}{2}$.	Do show enough working for a 'show' question. Note that a 'show' question is different from a 'verify' question, so here the given answer should not be expanded to compare with the expression for f.
(ii)	The 2 transformations (in either order) are 1. A translation of 1 unit in the positive x-direction 2. A scaling parallel to the y-axis by a factor of $\dfrac{1}{2}$.	Remember to use standard mathematical terms and descriptions. Unacceptable ones include 'along the x-axis', 'in the x-axis', 'on the x-axis'.
(iii)	 The equation $\dfrac{2-x^2}{x^2-1}=k$ has no real roots for values of k in the set $(-2,\ -1]$.	It is usually easier to use the method as suggested in the question. Here the question suggested a sketch, and the correct answer was mainly obtained by those who referred to the graph instead of the discriminant method.

	Alternative Solution: $$\frac{2-x^2}{x^2-1}=k$$ $$2-x^2=kx^2-k$$ $$(k+1)x^2-(k+2)=0$$ If $k=-1$, $(k+1)x^2-(k+2)=0$ becomes $-1=0$, so there is no solution. If $k\neq -1$, the quadratic equation will have no real roots if $$0-4\left[-(k+2)(k+1)\right]<0$$ $$(k+2)(k+1)<0$$ $$-2<k<-1$$ So $-2<k\leq -1$ and the required solution set is $\left(-2,\ -1\right]$.	Note that the discriminant is only used for a quadratic expression. If k = -1, then the equation in the 3rd line is no longer quadratic. So, the case $k=-1$ should be considered separately.

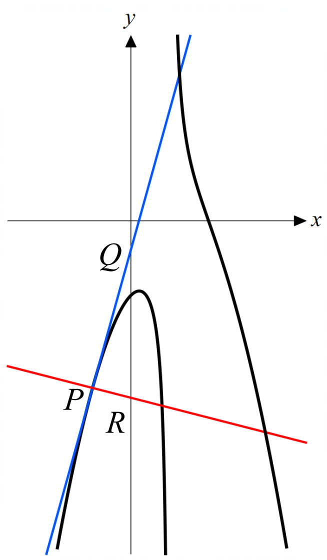
3 A curve has parametric equations

$$x = a\left(1+\frac{1}{t}\right)\ ,\quad y = a\left(t-\frac{1}{t^2}\right),$$

where a is a constant and $t \neq 0$.

(i) Find the equations of the tangent and the normal to the curve at the point P where $t=-\frac{1}{2}$. [5]

(ii) The tangent at P meets the y-axis at Q and the normal at P meets the y-axis at R. Show that the area of triangle PQR is $\frac{241}{120}a^2$. [2]

(i) **[5]**	$\frac{dx}{dt}=-\frac{a}{t^2}$; $\quad \frac{dy}{dt}=a\left(1+\frac{2}{t^3}\right)$ $\frac{dy}{dx}=\frac{a\left(1+\frac{2}{t^3}\right)}{-\frac{a}{t^2}}=-\frac{t^3+2}{t}$ When $t=-\frac{1}{2}$, $\frac{dy}{dx}=\frac{15}{4}$, $x=-a$, $y=-\frac{9}{2}a$ Equation of tangent at P: $y+\frac{9}{2}a=\frac{15}{4}(x+a)$ $y=\frac{15}{4}x-\frac{3}{4}a$ Gradient of normal at P $=-\frac{4}{15}$ Equation of normal at P: $y+\frac{9}{2}a=-\frac{4}{15}(x+a)$ $y=-\frac{4}{15}x-\frac{143}{30}a$	A generally well done question apart from some careless mistakes. Some candidates did not substitute the t-values and had to go through a whole lot of calculation for the next part of the question.
(ii) **[2]**	At point Q, $x=0$, $y=-\frac{3}{4}a$ At point R, $x=0$, $y=-\frac{143}{30}a$ Area of triangle PQR $=\frac{1}{2}\lvert a\rvert\left(\frac{143}{30}\lvert a\rvert-\frac{3}{4}\lvert a\rvert\right)$ $=\frac{1}{2}\left(\frac{241}{60}\right)\lvert a\rvert^2$ $=\frac{241}{120}a^2$ (shown) 	Most mistakes in this part were carried forward from the previous part. Note that the value of a could be positive or negative. Thus, candidates are reminded to be careful with the positive negative signs and ensure that the final expression is obtained correctly.

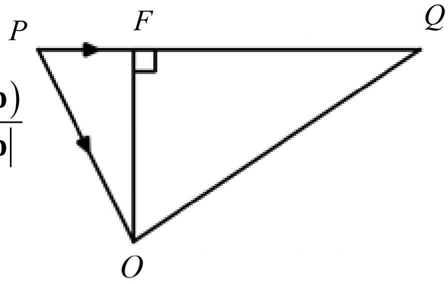
4 **(a)** The point R has position vector $\mathbf{r}$. Given that $\mathbf{r} = \begin{pmatrix} a \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$, where a is a real number, describe geometrically the set of all possible positions of the point R, as a varies. [2]

(b) **(i)** The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively. Show that the point F, the foot of perpendicular from the origin O to the line passing through P and Q, has position vector $(1-\lambda)\mathbf{p}+\lambda\mathbf{q}$, where

$$\lambda = \frac{|\mathbf{p}|^2 - \mathbf{p}.\mathbf{q}}{|\mathbf{q}-\mathbf{p}|^2}.$$ [4]

(ii) Write down an inequality satisfied by λ for F to lie within the line segment PQ. [1]

(a) **[2]**	$\mathbf{r} = \begin{pmatrix} a \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -5a+3 \\ 4a-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} + a\begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix}$, $a \in \mathbb{R}$ R lies on the line passing through the point with coordinates $(-2,3,-2)$, and the line is parallel to the vector $-5\mathbf{j}+4\mathbf{k}$.	Many candidates described the points as "moving 5 units in the negative **j** direction, 4 units in the positive **k** direction" and so on without addressing what would the "set of points" be geometrically. The line should be correct described, or its equation correctly stated.
(b) **[4]**	Since F lies on the line PQ, then $\overrightarrow{OF} = \mathbf{p} + \lambda(\mathbf{q}-\mathbf{p})$, for some $\lambda \in \mathbb{R}$. Since $\overrightarrow{OF}$ is perpendicular to the line PQ, then $(\mathbf{p}+\lambda(\mathbf{q}-\mathbf{p})) \bullet (\mathbf{q}-\mathbf{p}) = 0$ $\mathbf{p}\bullet(\mathbf{q}-\mathbf{p}) + \lambda\|\mathbf{q}-\mathbf{p}\|^2 = 0$ $\mathbf{p}\bullet\mathbf{q} - \|\mathbf{p}\|^2 + \lambda\|\mathbf{q}-\mathbf{p}\|^2 = 0$ $\lambda = \dfrac{\|\mathbf{p}\|^2 - \mathbf{p}\bullet\mathbf{q}}{\|\mathbf{q}-\mathbf{p}\|^2}$ Substitute value of λ into (1) : $\overrightarrow{OF} = \mathbf{p} + \dfrac{\|\mathbf{p}\|^2 - \mathbf{p}\bullet\mathbf{q}}{\|\mathbf{q}-\mathbf{p}\|^2}(\mathbf{q}-\mathbf{p})$ $= \left(1 - \dfrac{\|\mathbf{p}\|^2 - \mathbf{p}\bullet\mathbf{q}}{\|\mathbf{q}-\mathbf{p}\|^2}\right)\mathbf{p} + \dfrac{\|\mathbf{p}\|^2 - \mathbf{p}\bullet\mathbf{q}}{\|\mathbf{q}-\mathbf{p}\|^2}\mathbf{q}$ $= (1-\lambda)\mathbf{p} + \lambda\mathbf{q}$	Candidates who used dot product were generally able to get this part correct. Candidates are reminded that all workings should be showed clearly since this is a "show" question.

	Alternative Method: Consider $\overrightarrow{PF}$ as the projection vector of $\overrightarrow{PO}$ onto $\overrightarrow{PQ}$, $\overrightarrow{PF} = \left(\overrightarrow{PO} \cdot \widehat{\overrightarrow{PQ}}\right)\widehat{\overrightarrow{PQ}}$  $= \dfrac{-\mathbf{p} \cdot (\mathbf{q} - \mathbf{p})}{\lvert \mathbf{q} - \mathbf{p} \rvert} \times \dfrac{(\mathbf{q} - \mathbf{p})}{\lvert \mathbf{q} - \mathbf{p} \rvert}$ $= \dfrac{\lvert \mathbf{p} \rvert^2 - \mathbf{p} \cdot \mathbf{q}}{\lvert \mathbf{q} - \mathbf{p} \rvert^2}(\mathbf{q} - \mathbf{p})$ Hence we have $\overrightarrow{OF} = \overrightarrow{OP} + \overrightarrow{PF}$ $= \mathbf{p} + \dfrac{\lvert \mathbf{p} \rvert^2 - \mathbf{p} \cdot \mathbf{q}}{\lvert \mathbf{q} - \mathbf{p} \rvert^2}(\mathbf{q} - \mathbf{p})$ $= \left(1 - \dfrac{\lvert \mathbf{p} \rvert^2 - \mathbf{p} \cdot \mathbf{q}}{\lvert \mathbf{q} - \mathbf{p} \rvert^2}\right)\mathbf{p} + \left(\dfrac{\lvert \mathbf{p} \rvert^2 - \mathbf{p} \cdot \mathbf{q}}{\lvert \mathbf{q} - \mathbf{p} \rvert^2}\right)\mathbf{q}$ $= (1 - \lambda)\mathbf{p} + \lambda \mathbf{q}$ where $\lambda = \dfrac{\lvert \mathbf{p} \rvert^2 - \mathbf{p} \cdot \mathbf{q}}{\lvert \mathbf{q} - \mathbf{p} \rvert^2}$ (shown)	Candidates who used the projection vector method, with careful consideration of the directions of the vectors, were also able to conclude correctly. It is important to consider the direction of the vectors when using this method that involves projection vector. Thus, Both $\overrightarrow{FP}$ and $\overrightarrow{PF}$ are parallel to $\overrightarrow{PQ}$, but the correct direction is determined by $\overrightarrow{PO}$. Thus, $\overrightarrow{FP} \neq \left(\overrightarrow{PO} \cdot \widehat{\overrightarrow{PQ}}\right)\widehat{\overrightarrow{PQ}}$. Candidates who used the length of projection method are reminded to be careful when dealing with the modulus sign. Using merely length of projection would have omitted the direction.
(b)(ii) **[1]**	For F to lie within the line segment PQ, $0 \le \lambda \le 1$	Generally well done for those who attempted this part.

5 **Do not use a calculator in answering this question.**
The complex numbers z and w are given by

$$z=\sin\left(\frac{\pi}{6}\right)+\mathrm{i}\cos\left(\frac{\pi}{6}\right) \text{ and } w=\sqrt{2}\left[\sin\left(\frac{\pi}{6}\right)+\mathrm{i}\cos\left(\frac{\pi}{3}\right)\right].$$

(i) Find $|z|$ and $\arg(z)$. Hence find the value of z^3. [3]

(ii) By considering some suitable form of w or otherwise, find w^4. [3]

(iii) Hence find the value of $z^{2022}-w^{2020}$. [2]

(i) **[3]**	$z=\sin\left(\frac{\pi}{6}\right)+\mathrm{i}\cos\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{2}-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$ $=\cos\left(\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}\right)=\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$. $\therefore |z|=1$ and $\arg(z)=\frac{\pi}{3}$. Hence $z^3=\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)^3=\mathrm{e}^{\mathrm{i}\pi}=-1$. **Alternatively**: $z=\sin\left(\frac{\pi}{6}\right)+\mathrm{i}\cos\left(\frac{\pi}{6}\right)=\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$. $\lvert z\rvert=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$, $\arg(z)=\tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)=\frac{\pi}{3}$	$\mathrm{e}^{\mathrm{i}\theta}=\cos\theta+\mathrm{i}\sin\theta$ ***not*** $\sin\theta+\mathrm{i}\cos\theta$. "Hence", so need to use $\lvert z\rvert$ and $\arg(z)$ to find z^3. You are expected to simplify $\mathrm{e}^{\mathrm{i}\pi}$ to give -1.
(ii) **[3]**	$w=\sqrt{2}\left[\sin\left(\frac{\pi}{6}\right)+\mathrm{i}\cos\left(\frac{\pi}{3}\right)\right]=\sqrt{2}\left(\frac{1}{2}+\mathrm{i}\frac{1}{2}\right)=\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}$. Thus $w^4=\left(\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2}\right)^4=\left(\frac{1}{2}+\mathrm{i}-\frac{1}{2}\right)^2=\mathrm{i}^2=-1$ **Alternatively:** $\lvert w\rvert=\sqrt{\left(\sqrt{2}\sin\frac{\pi}{6}\right)^2+\left(\sqrt{2}\cos\frac{\pi}{3}\right)^2}=\sqrt{2\left(\frac{1}{2}\right)^2+2\left(\frac{1}{2}\right)^2}=1$. $\arg(w)=\tan^{-1}\left(\frac{\sqrt{2}\cos\frac{\pi}{3}}{\sqrt{2}\sin\frac{\pi}{6}}\right)=\tan^{-1}(1)=\frac{\pi}{4}$. Thus $w^4=\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^4=\mathrm{e}^{\mathrm{i}\pi}=-1$.	Show working clearly as calculator is not allowed in this question. Expected to simplify $\mathrm{e}^{\mathrm{i}\pi}$ to -1 *unless* the question asked for w^4 in the form $r\mathrm{e}^{\mathrm{i}\theta}$.
(iii) **[2]**	Hence $z^{2022}-w^{2020}=\left(z^3\right)^{674}-\left(w^4\right)^{505}$ $=(-1)^{674}-(-1)^{505}$ $=1-(-1)$ $=2$ OR $\mathrm{e}^{674\pi\mathrm{i}}-\mathrm{e}^{505\pi\mathrm{i}}$ $=\mathrm{e}^{0\pi\mathrm{i}}-\mathrm{e}^{\pi\mathrm{i}}$ $=1-(-1)$ $=2$	Use $z^3=w^4=-1$ or $\mathrm{e}^{\mathrm{i}\pi}$ from **(i)** and **(ii)**.

6 **(a)** A sequence is such that $u_1 = p$, where p is a constant, and $u_{n+1} = \frac{4}{u_n}$, for $n > 0$.

Describe how the sequence behaves when

(i) $p = 2$, [1]

(ii) $p = 3$. [1]

(b) Another sequence $v_1, v_2, v_3, \ldots$ is such that $v_n = v_{n-1} + n$, where $n \geq 2$, and $v_1 = A$.

Find in terms of A and n,

(i) v_n, [4]

(ii) $\sum_{r=1}^{n} v_r$. [3]

(You need not simplify your answer. You may use the result $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.)

(a)(i) **[1]**	$u_1 = 2$ $u_2 = \frac{4}{u_1} = \frac{4}{2} = 2$ $u_3 = \frac{4}{u_2} = \frac{4}{2} = 2$ The sequence is a constant sequence of 2.	Question asked for a description of the behaviour of the sequence, so stating $u_n = 2 \ \forall n$ is technically not sufficient. Note the difference between the terms 'sequence' and 'series'. Series refer to the sum of the terms in a sequence.
(a)(ii) **[1]**	$u_1 = 3$ $u_2 = \frac{4}{u_1} = \frac{4}{3}$ $u_3 = \frac{4}{u_2} = \frac{4}{\frac{4}{3}} = 3$ The sequence alternates between 3 and $\frac{4}{3}$.	Again, the description should be precise, such that the terms alternate between 3 and $\frac{4}{3}$, and not that it 'repeats' after every 2 terms. Also, stating the sequence $3, \frac{4}{3}, 3, \frac{4}{3}, \ldots$ is technically insufficient.

(b)(i) [4]	$v_2 = v_1 + 2$ $v_3 = v_2 + 3$ $= v_1 + 2 + 3$ $v_4 = v_3 + 4$ $= v_1 + 2 + 3 + 4$ $v_n = v_1 + 2 + 3 + 4 + ... + n$ $= v_1 + \frac{n-1}{2}(2+n)$ $= v_1 + \frac{(n-1)(n+2)}{2}$ $\therefore v_n = A + \frac{(n-1)(n+2)}{2}$	It is a good habit to write down the first few terms so that one can see the pattern. Many students fail to identify that *n* is <u>NOT</u> a constant. When we sum an AP, it is important to identify the number of terms, first and last terms correctly.
(b)(ii) [2]	$\sum_{r=1}^{n} v_r = \sum_{r=1}^{n}\left(A + \frac{(r-1)(r+2)}{2}\right)$ $= \sum_{r=1}^{n}\left(A + \frac{r^2+r-2}{2}\right)$ $= \sum_{r=1}^{n} A + \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$ $= \sum_{r=1}^{n}(A-1) + \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{r=1}^{n} r$ $= n(A-1) + \frac{1}{12}n(n+1)(2n+1) + \frac{1}{4}(n)(n+1)$	Poor notation was observed. Here we are summing v_r and so the expression to sum should be $v_r = A + \frac{(r-1)(r+2)}{2}$ instead of $v_n = A + \frac{(n-1)(n+2)}{2}$. For the latter case, it should be noted that $\sum_{r=1}^{n} v_n = nv_n$ instead and hence incorrect.

7 It is given that $y=\sqrt{2+\cos^2 x}$. Show that

(i) $$2y\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin 2x,$$ [1]

(ii) $$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-\cos 2x.$$ [1]

Hence find the Maclaurin series of y, up to and including the term in x^4. [5]

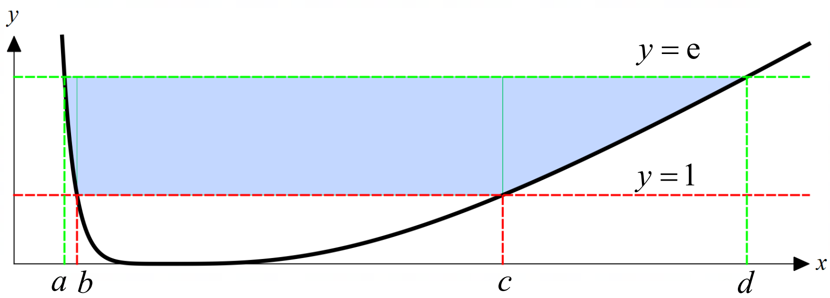
By substituting $x=\frac{\pi}{6}$, show that $\sqrt{11}\approx 2\sqrt{3}\left(1-\frac{\pi^2}{216}+\frac{\pi^4}{31104}\right)$. [2]

(i) **[1]**	$y=\sqrt{2+\cos^2 x} \quad\Rightarrow\quad y^2=2+\cos^2 x$ Differentiating w.r.t x, $2y\frac{\mathrm{d}y}{\mathrm{d}x}=2\cos x(-\sin x)=-\sin 2x$ (shown) ------ (1)	Most of the students make use of implicit differentiation after squaring the both sides. Some had attempted to differentiate directly.
(ii) **[1]**	Differentiating (1) w.r.t x, $2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-2\cos 2x$ $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-\cos 2x$ (shown) ------ (2)	Most of the students make use of implicit differentiation. For those who make use of direct differentiation are not able to complete it correctly.
[5]	Differentiating (2) w.r.t x, $y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\sin 2x$ $y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=2\sin 2x$ ------- (3) Differentiating (3) w.r.t x, $y\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)+3\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)+3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=4\cos 2x$ $y\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+4\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)+3\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2=4\cos 2x$ ------ (4) When $x=0, y=\sqrt{3}, \frac{\mathrm{d}y}{\mathrm{d}x}=0, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{\sqrt{3}}{3}, \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0, \frac{\mathrm{d}^4y}{\mathrm{d}x^4}=\sqrt{3}$ By Maclaurin's Theorem, $y\approx\sqrt{3}-\frac{\sqrt{3}}{6}x^2+\frac{\sqrt{3}}{24}x^4$	Generally well done. Most of the students make use of implicit differentiation to find $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$. However common mistakes like $\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]=2\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]=2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ are observed.
[2]	$\sqrt{2+\cos^2\left(\frac{\pi}{6}\right)}\approx\sqrt{3}-\frac{\sqrt{3}}{6}\left(\frac{\pi}{6}\right)^2+\frac{\sqrt{3}}{24}\left(\frac{\pi}{6}\right)^4$ $\sqrt{2+\left(\frac{\sqrt{3}}{2}\right)^2}\approx\sqrt{3}-\frac{\sqrt{3}}{216}\pi^2+\frac{\sqrt{3}}{31104}\pi^4$ $\frac{\sqrt{11}}{2}\approx\sqrt{3}\left(1-\frac{\pi^2}{216}+\frac{\pi^4}{31104}\right)$ $\sqrt{11}\approx 2\sqrt{3}\left(1-\frac{\pi^2}{216}+\frac{\pi^4}{31104}\right)$ (shown)	Majority are able to make use of the above Maclaurin series to show the required results. However, appropriate intermediate should be shown.

8 A curve C is defined by $y=\dfrac{(\ln x)^4}{\sqrt{x}}$ where $0<x<10$.

(i) Find the exact volume generated when the area bounded by C, the x-axis and the lines $x=1$ and $x=\mathrm{e}$ is rotated about the x-axis through 360°. [4]

(ii) Find the area enclosed by C and the lines $y=1$ and $y=\mathrm{e}$. [6]

(i) **[4]**	Required volume $=\pi\int_1^{\mathrm{e}}\left(\dfrac{(\ln x)^4}{\sqrt{x}}\right)^2\,\mathrm{d}x$ $=\pi\int_1^{\mathrm{e}}\dfrac{1}{x}(\ln x)^8\,\mathrm{d}x$ $=\pi\left[\dfrac{(\ln x)^9}{9}\right]_1^{\mathrm{e}}$ $=\dfrac{\pi}{9}$	Generally well done although quite a number of candidates solved the integral using the method of integration by parts. Note that the use of GC is not allowed as question states 'exact volume'.
(ii) **[6]**	 $\dfrac{(\ln x)^4}{\sqrt{x}}=\mathrm{e}\quad\Rightarrow\quad x=0.32735,\ 4.76172$ $\dfrac{(\ln x)^4}{\sqrt{x}}=1\quad\Rightarrow\quad x=0.40892, 3.17520$ Let $a=0.32735$, $b=0.40892$, $c=3.1752$ and $d=4.7617$ (5 sf) <u>Method 1</u> Required area $=(d-a)\mathrm{e}-\int_a^b\dfrac{(\ln x)^4}{\sqrt{x}}\,\mathrm{d}x-(c-b)-\int_c^d\dfrac{(\ln x)^4}{\sqrt{x}}\,\mathrm{d}x$ = 6.2482 (5 sf) = 6.25 (3 sf). <u>Method 2</u> Required area $=\int_a^b\left(\mathrm{e}-\dfrac{(\ln x)^4}{\sqrt{x}}\right)\mathrm{d}x+(c-b)(\mathrm{e}-1)+\int_c^d\left(\mathrm{e}-\dfrac{(\ln x)^4}{\sqrt{x}}\right)\mathrm{d}x.$ = 6.2482 (5 sf) = 6.25 (3 sf). <u>Method 3</u> Required area $=\int_a^d\left(\mathrm{e}-\dfrac{(\ln x)^4}{\sqrt{x}}\right)\mathrm{d}x-\int_b^c\left(1-\dfrac{(\ln x)^4}{\sqrt{x}}\right)\mathrm{d}x.$ = 6.2482 (5 sf) = 6.25 (3 sf).	Some identified the required region wrongly. Do check that your identified region is indeed defined only by C and the lines $y=1$ and $y=\mathrm{e}$. Give the x-coordinates of the points of intersection to at least 5 sig. fig. Attempts to express x in terms of y were futile. This part is best done by using any of the 3 methods stated here. Ensure sufficient accuracies used in intermediate workings to ensure a correct final answer.

9 The functions f, g and h are defined as follows:

$$\mathrm{f}: x \mapsto x^3 - 1, \quad x \in \mathbb{R},$$
$$\mathrm{g}: x \mapsto \mathrm{e}^{-3x}, \quad x \in \mathbb{R}, x < 0,$$
$$\mathrm{h}: x \mapsto \frac{x+1}{x-1}, \quad x \in \mathbb{R}, x \neq 1.$$

(i) Define in a similar form, the inverse functions g^{-1} and h^{-1}. [4]

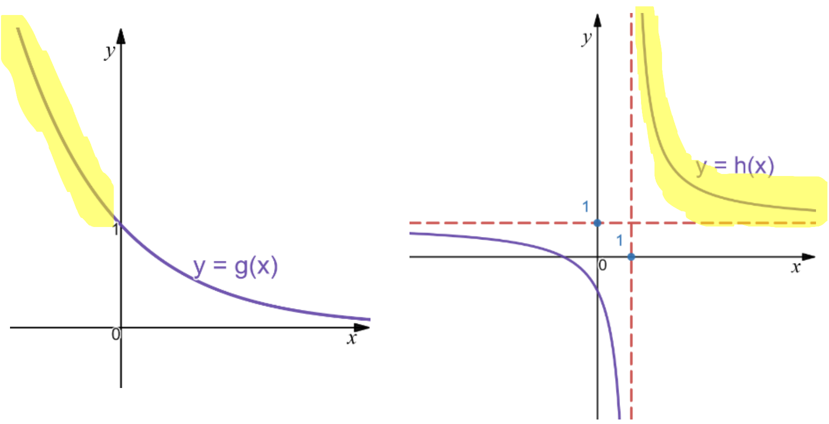
(ii) Write down the rule of the composite function ff . [1]

(iii) It is known that the equation $\mathrm{ff}(x) = 0$ has only one real root α . Find the exact value of α and hence show that $\mathrm{g}^{-1}(\alpha) = k \ln 2,$ where *k* is a constant to be found. [2]

(iv) Show that the composite function hg exists and write down the range of this function. [2]

(v) Denoting composite functions hh as h^2, hhh as h^3 and so on, find the value of *x* for which $\mathrm{h}^m(x) = \mathrm{h}^{-1}(-1),$ where *m* is a positive even integer. [2]

(i)	Let $y = \mathrm{e}^{-3x}$. Then $-3x = \ln y \Rightarrow x = -\frac{1}{3}\ln y$. So $\mathrm{g}^{-1}: x \mapsto -\frac{1}{3}\ln x, \quad x \in \mathbb{R}, \quad x > 1.$ Let $y = \frac{x+1}{x-1}$. Then $yx - y = x + 1 \Rightarrow yx - x = y + 1$ $\Rightarrow x(y-1) = y + 1$ $\Rightarrow x = \frac{y+1}{y-1}$. So $\mathrm{h}^{-1}: x \mapsto \frac{x+1}{x-1}, \quad x \in \mathbb{R}, x \neq 1.$	Generally well done except expressing final answer in a similar form as given in the questions or careless mistake in computing the domains of the inverse functions.
(ii)	$\mathrm{ff}(x) = (x^3 - 1)^3 - 1$.	Some students misinterpreted the question as stating condition of composite function which is not the case.
(iii)	Given that $\mathrm{ff}(\alpha) = 0$ means $\left(\alpha^3 - 1\right)^3 - 1 = 0 \Rightarrow \left(\alpha^3 - 1\right)^3 = 1$ $\Rightarrow \alpha^3 - 1 = 1$ $\Rightarrow \alpha^3 = 2$ $\Rightarrow \alpha = \sqrt[3]{2}.$ So $\mathrm{g}^{-1}(\alpha) = -\frac{1}{3}\ln\alpha = -\frac{1}{3}\ln 2^{\frac{1}{3}} = -\frac{1}{9}\ln 2$ (Shown). Thus $k = -\frac{1}{9}$.	Avoid careless mistakes such as writing $\sqrt[3]{2}$ as $3\sqrt{2}$ or writing index form of $\sqrt[3]{2}$ as $2^{\frac{3}{2}}$ which eventually cause the mistake of finding the value of *k*.

(iv)	Since $R_g = (1, \infty) \subseteq D_h = (-\infty, 1) \cup (1, \infty)$, hg exists (Shown). $(-\infty, 0) \xrightarrow{g} (1, \infty) \xrightarrow{h} (1, \infty)$ The range of the function is $(1, \infty)$.  	Generally well done, note about the notion of range and domain of function. To find the range of composite function, it is useful to visualize from the graphs of 2 functions based on their domains given.
(v)	Since $h^2(x) = x$ (i.e. h is a self-inverse function), and $h^m(x) = x$ if m is an even integer. Thus $h^m(x) = h^{-1}(-1)$ becomes $x = h^{-1}(-1) = 0$.	Quite a number of students did not show $h^2(x) = x$ clearly which is key step in finding the value of x.

10 Glucose in the blood stream is reduced at a rate proportional to the amount of glucose present in the blood stream.
Let G milligrams (mg) be the amount of glucose in 1 decilitre (dL) of blood stream at time t minutes and let λ denote the positive constant of proportionality.

(i) Write down a differential equation relating G, λ and t. Solve this differential equation to find an expression of G in terms of λ and t. [3]

Through extensive research, doctors recommended that a healthy glucose level in the blood stream of an adult should be between 70 mg/dL to 100 mg/dL.
An adult patient, Neo, has 80 mg/dL of glucose in his blood stream at time $t = 0$ minutes. Due to a medical condition, he is not able to extract glucose from the food he eats. As he is not able to replenish the glucose in his blood stream, he is thus losing his glucose in the blood stream at a rate (in mg/dL per minute) corresponding to $\lambda = 0.005$ per minute.

(ii) Find the approximate time it would take for Neo's glucose level in the blood stream to fall below the healthy range if there is no intake of glucose. [2]

In order to maintain Neo's glucose level in the blood stream in the healthy range, glucose is injected into his blood stream intravenously at a constant rate of μ mg/dL per minute.

(iii) Write down a differential equation relating G, μ and t to model Neo's situation. Solve this differential equation to find an expression of G in terms of μ and t. [5]

(iv) Explain whether it is recommended to keep injecting glucose at a rate of 0.7 mg/dL per minute into Neo's blood stream. [2]

(v) Find the maximum range of values of μ so that Neo's glucose level in the blood stream will always be in the healthy range. [1]

(i)	$\frac{\mathrm{d}G}{\mathrm{d}t} = -\lambda G$ $\int \frac{1}{G}\,\mathrm{d}G = \int -\lambda\ \mathrm{d}t$ $\ln G = -\lambda t + C$, where $C \in \mathbb{R}$ $G = A\mathrm{e}^{-\lambda t}$, where $A = \mathrm{e}^{C}$.	Generally well done. There were a handful of students who missed out the minus sign when forming the DE.
(ii)	When $t = 0$, $G = 80 \Rightarrow A = 80$ $G = 80\mathrm{e}^{-0.005t}$ For $G \le 70$, $\mathrm{e}^{-0.005t} \le 0.875$ $-0.005t \le \ln 0.875$ $t \ge 26.706$ The approximate time is 26.7 minutes.	Most students solve equality instead of inequality. The common mistake here was giving the answer as 27minutes. This were mostly from students who chose to use the GC to get the answer.
(iii)	$\frac{\mathrm{d}G}{\mathrm{d}t} = \mu - 0.005G = -0.005(G - 200\mu)$ $\int \frac{1}{G - 200\mu}\,\mathrm{d}G = \int -0.005\ \mathrm{d}t$ $\ln\lvert G - 200\mu\rvert = -0.005t + D$, where $D \in \mathbb{R}$ $G - 200\mu = B\mathrm{e}^{-0.005t}$, where $B = \pm\mathrm{e}^{D}$. When $t = 0$, $G = 80 \Rightarrow B = 80 - 200\mu$ $\therefore G = 200\mu + (80 - 200\mu)\mathrm{e}^{-0.005t}$	Most students were able to form the DE and perform the integration. However, quite many did not solve for the final answer by putting in the initial condition

(iv)	When $\mu = 0.7$, $G = 140 - 60\mathrm{e}^{-0.005t}$ Observe that when $t \to \infty$, $G \to 140$. This means it is **not recommended** as after a long period of time, the glucose level in the blood stream will exceed 100 mg/dL. Alternatively: For $G \geq 100$, we have $140 - 60\mathrm{e}^{-0.005t} \geq 100$ $\Rightarrow 60\mathrm{e}^{-0.005t} \leq 40$ $\Rightarrow t \geq 81.093$ This means it is **not recommended** as after approximately 81 minutes the glucose level in the blood stream will exceed 100 mg/dL.	Quite many students gave a sweeping statement that the glucose level will exceed 100mg/dL without justification.
(v)	$70 < 200\mu < 100 \Rightarrow 0.35 < \mu < 0.5$	

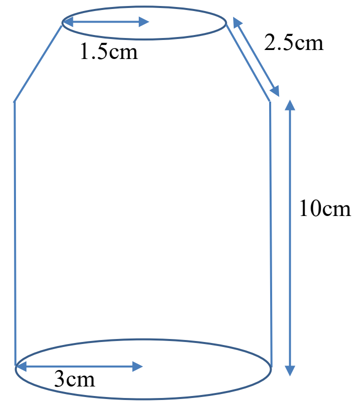
11 [It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]

Globally, 2 trillion drink cans are produced every year. Drink cans constitute part of the packaging cost for beverage companies and using the appropriate material in appropriate amounts will enable them to save costs. A new beverage company wants to produce drink cans using a particular material for the top and bottom of a cylindrical shaped drink can, and another material for the curved body. For a fixed thickness, the material for the top and bottom costs $1.20/m^2 and the material for the curved side costs $0.90/m^2.

(i) For a fixed volume of 100π ml per can, show, using differentiation, that the radius r of the most economical can is approximately 3.35cm and evaluate the corresponding height h. (1ml = 1cm^3) [7]

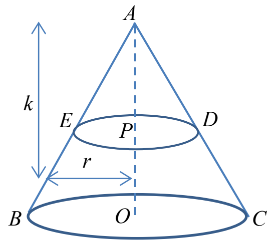
(ii) Hence, find the cost of the most economical can, giving your answer correct to the nearest 0.1 cent. [1]

In order to make the can more attractive, the company redesigns the can such that the base radius of the can is 3cm and the height of the cylindrical part is 10cm. At the top of each can, there is a tapering in before being covered by a circular lid of radius 1.5cm, with the dimensions of the can as shown below.



(iii) The beverage is pumped into the can at a rate of 90π ml/s.
Find the rate at which the liquid level in the can is increasing when it is 1cm from the lid of the can. [6]

(i) **[8]**	Let C be the material cost of a can $\pi r^2 h = 100\pi$ $h = \dfrac{100}{r^2}$ ------ (1) $C = 2\pi rh(0.9\times10^{-4}) + 2\pi r^2(1.2\times10^{-4})$ ------ (2) Sub (1) into (2), $C = 2\pi r\dfrac{100}{r^2}(0.9\times10^{-4}) + 2\pi r^2(1.2\times10^{-4})$ $= \dfrac{0.018\pi}{r} + 0.00024\pi r^2$ $\dfrac{dC}{dr} = -\dfrac{0.018\pi}{r^2} + 0.00048\pi r$ when $\dfrac{dC}{dr} = 0,\ r^3 = \dfrac{0.018}{0.00048}$ $r = 3.3472 = 3.35$ (3sf) (shown) $h = 8.9258$ or 8.9107 (if use r = 3.35) $\dfrac{d^2C}{dr^2} = \dfrac{0.036\pi}{r^3} + 0.00048\pi$, for $r > 0$, $\dfrac{d^2C}{dr^2} > 0$, so C is minimum Thus, the most economical can has a radius of 3.35 cm (3sf) and height 8.93 cm (3sf)	Most students are able to get the relationship between h and r. Quite a number of students went to find the surface area instead of cost and therefore unable to find the correct r.
(ii) **[1]**	The cost of the can, $C = \dfrac{0.018\pi}{3.35} + 0.00024\pi(3.35)^2 = 0.0253$ (3sf) The most economical can costs 2.5 cents each. (1dp)	Most students are able to get this part correct.

(iii) **[6]**	 Refer to the diagram above. Let *BC* be the diameter of the top of the cylindrical part of can with *O* as the centre and let *ED* be the diameter of the lid with *P* as the centre. The lines *BE* and *CD* are extended to meet at *A* as shown in the diagram. The points *A*, *B*, *C*, *D* and *E* lie in the same plane and *ABC* and *AED* form two right cones. Let $OB = 3$ cm, $PE = 1.5$ cm, then $BE = EA = 2.5$ cm. Hence $BA = 5$ cm, $OA = 4$ cm. Let *r* be the radius of the liquid surface, and *k* be the vertical distance from the liquid surface to *A*. Using similar triangles, $\frac{r}{k} = \frac{3}{4}$ Volume of liquid above level *BC*, $V = \frac{1}{3}\pi(3^2)(4) - \frac{1}{3}\pi r^2 k$ $= \frac{1}{3}\pi(3^2)(4) - \frac{3}{16}\pi k^3$ $\Rightarrow \frac{\mathrm{d}V}{\mathrm{d}k} = -\frac{9}{16}\pi k^2$ When the liquid level is 1cm from the lid of the can, $k = 3$ $\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}k}{\mathrm{d}V}$ $= 90\pi \times \left(-\frac{16}{9\pi(3)^2}\right)$ $= -\frac{160}{9}$ Since *k* is decreasing at $\frac{160}{9}$ cm/s, it follows that the liquid level in the can is increasing at $\frac{160}{9}$ cm/s when it is 1cm from the top of the can.	Most students are able to get the relationship between *r* and *k*. However, most students have trouble trying to set up the volume equation. A small handful of students fail to recognise that 1cm from the lid means *k*=3