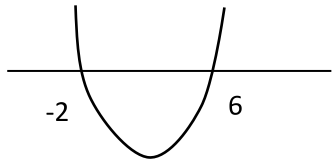
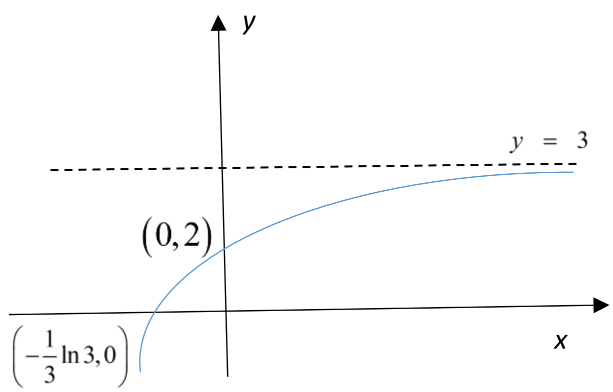
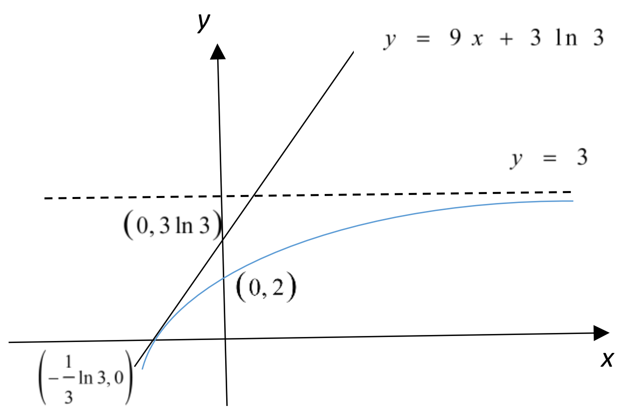
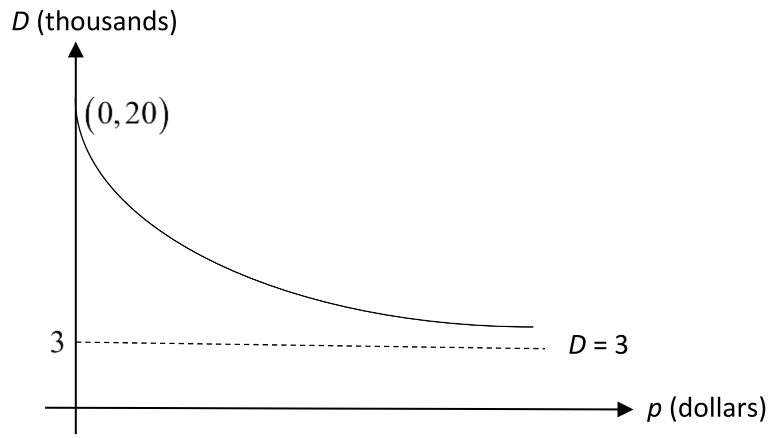
Suggested Solutions for 2022 C2 H1 Preliminary Examinations

No.	Suggested Solutions
1	$x^2 - kx + 3 + k = 0$ has real roots if Discriminant ≥ 0 Discriminant $= k^2 - 4(3+k)$ $= k^2 - 4k - 12$ $= (k+2)(k-6)$  For discriminant ≥ 0, we have $\{k \in \mathbb{R} : k \le -2 \quad \text{or} \quad k \ge 6\}$
2a	$\int \frac{\sqrt[3]{x} - \left(\sqrt{x}\right)^3}{\sqrt{x}}\,\mathrm{d}x = \int x^{-\frac{1}{6}} - x\,\mathrm{d}x = \frac{6}{5}x^{\frac{5}{6}} - \frac{1}{2}x^2 + C$
2(b) **(i)**	$\frac{\mathrm{d}}{\mathrm{d}x}\left(6\ln\sqrt[3]{3x^2+4}\right)$ $= \frac{6}{3}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(3x^2+4\right)\right)$ $= 2\left(\frac{6x}{3x^2+4}\right)$ $= \frac{12x}{3x^2+4}$ <u>Method 2</u> $\frac{\mathrm{d}}{\mathrm{d}x}\left(6\ln\sqrt[3]{3x^2+4}\right)$ $= 6\left(\frac{\frac{1}{3}\left(3x^2+4\right)^{-\frac{2}{3}}6x}{\sqrt[3]{3x^2+4}}\right)$ $= \frac{12x}{3x^2+4}$
(ii)	$\int \frac{x}{3x^2+4}\,\mathrm{d}x = \frac{1}{12}\int \frac{12x}{3x^2+4}\,\mathrm{d}x$ $= \frac{1}{12}\left(6\ln\sqrt[3]{3x^2+4}\right) + c$ $= \frac{1}{2}\ln\sqrt[3]{3x^2+4} + c \quad \text{or} \quad \frac{1}{6}\ln\left(3x^2+4\right) + c$
3(i)	$AD = \sqrt{x^2 + x^2} = \sqrt{2}x$ $\text{PerimeterAEBCGFD} = \frac{1}{2}(2\pi x) + 2AD + 2DF + 4x = 10$ $\pi x + 2\sqrt{2}x + 2DF + 4x = 10$

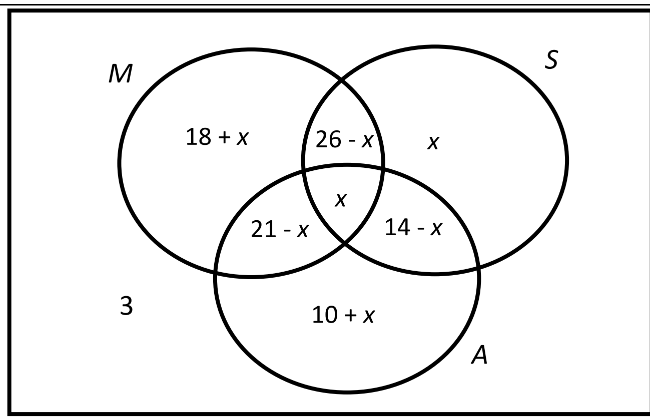
	$DF=\dfrac{10-\left(\pi+2\sqrt{2}+4\right)x}{2}$ $=5-\left(\dfrac{\pi}{2}+\sqrt{2}+2\right)x$
(ii)	Let area of window be A $A=\dfrac{1}{2}\pi x^2+\dfrac{1}{2}(2x+4x)x+(DF)4x$ $=\dfrac{1}{2}\pi x^2+3x^2+\left(5-\left(\dfrac{\pi}{2}+\sqrt{2}+2\right)x\right)4x$ $=\dfrac{1}{2}\pi x^2+3x^2+20x-2\pi x^2-4\sqrt{2}x^2-8x^2$ $=-\dfrac{3}{2}\pi x^2-5x^2-4\sqrt{2}x^2+20x$ $\dfrac{\mathrm{d}A}{\mathrm{d}x}=-3\pi x-10x-8\sqrt{2}x+20$ For max A, $\dfrac{\mathrm{d}A}{\mathrm{d}x}=0\Rightarrow x=\dfrac{20}{3\pi+10+8\sqrt{2}}$ $\dfrac{\mathrm{d}^2A}{\mathrm{d}x^2}=-\left(3\pi+10+8\sqrt{2}\right)<0$ A is maximum when $x=\dfrac{20}{3\pi+10+8\sqrt{2}}$
4(i)	
(ii)	$y=3-\mathrm{e}^{-3x}$ At x-axis, $y=0\Rightarrow \mathrm{e}^{-3x}=3\Rightarrow x=-\dfrac{1}{3}\ln 3$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=3\mathrm{e}^{-3x}$

<table>
<tr>
<td>(iii)</td>
<td>
When $x=-\frac{1}{3}\ln 3$,
$$\frac{dy}{dx}=3e^{-3\left(-\frac{1}{3}\ln 3\right)}=3e^{\ln 3}=9$$
Equation of tangent,
$$y-0=9\left(x-\left(-\frac{1}{3}\ln 3\right)\right)$$
$$y=9x+3\ln 3$$


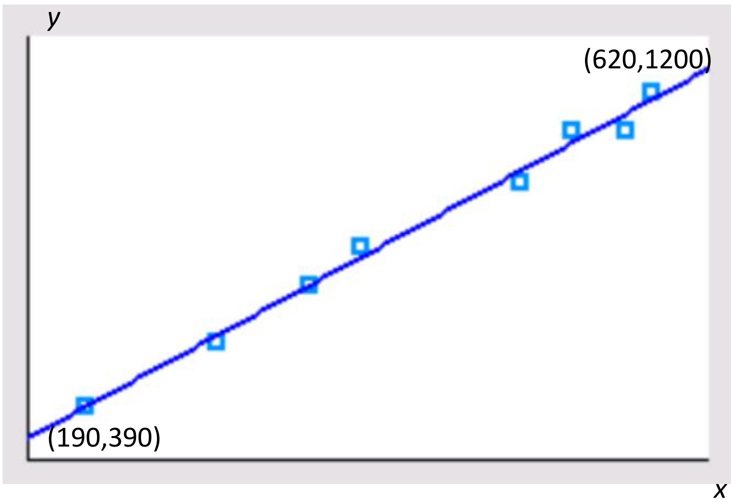
Area
$$=\frac{1}{2}(3\ln 3)\left(\frac{1}{3}\ln 3\right)-\int_{-\frac{\ln 3}{3}}^{0} 3-e^{-3x}\ dx$$
$$=\frac{1}{2}(\ln 3)^2-\left[3x+\frac{e^{-3x}}{3}\right]_{-\frac{\ln 3}{3}}^{0}$$
$$=\frac{1}{2}(\ln 3)^2-\left[\frac{1}{3}-\left(-\ln 3+\frac{e^{\ln 3}}{3}\right)\right]$$
$$=\frac{1}{2}(\ln 3)^2-\frac{1}{3}+\left(-\ln 3+\frac{3}{3}\right)$$
$$=\frac{1}{2}(\ln 3)^2-\ln 3+\frac{2}{3}\ \text{units}^2$$
</td>
</tr>
<tr>
<td>5(i)</td>
<td>
Let x be the selling price of a toy boat.
Let y be the selling price of a toy car.
Let z be the selling price of a toy plane.
</td>
</tr>
</table>

	$x = 2y$ $9x + 5y + 3z = 335$ $3x + 4y = 38 + x + 2z$ Rearranging the equations, we get $x - 2y + 0z = 0$ $9x + 5y + 3z = 335$ $2x + 4y - 2z = 38$ From GC, $x = 22.4$, $y = 11.2$, $z = 25.8$ Hence, the selling price of a toy plane is \$25.80.
(ii)	Substitute D = 20 and p = 0 into $D = 3 + ae^{-0.08p}$, $20 = 3 + ae^{0}$ $a = 17$
(iii)	
(iv)	$D = 3 + 17e^{-0.08p}$ $\frac{dD}{dp} = 17(-0.08)e^{-0.08p}$ $= -1.36e^{-0.08p}$
(v)	$p = \sqrt{0.5(t-9)^2 + 312}$

	$\frac{\mathrm{d}p}{\mathrm{d}t}=\frac{1}{2}\left[0.5\left(t-9\right)^2+312\right]^{-\frac{1}{2}}(0.5)(2)\left(t-9\right)$ $=\frac{1}{2}\left(t-9\right)\left[0.5\left(t-9\right)^2+312\right]^{-\frac{1}{2}}$ $=\frac{t-9}{2\sqrt{0.5\left(t-9\right)^2+312}}$ $\frac{\mathrm{d}D}{\mathrm{d}t}=\frac{\mathrm{d}D}{\mathrm{d}p}\times\frac{\mathrm{d}p}{\mathrm{d}t}$ $=-1.36\mathrm{e}^{-0.08p}\times\frac{t-9}{2\sqrt{0.5\left(t-9\right)^2+312}}$ When $t=20$, $p=\sqrt{0.5\left(20-9\right)^2+312}$ $=19.30025907$ $\frac{\mathrm{d}D}{\mathrm{d}t}=-1.36\mathrm{e}^{-0.08(19.30025907)}\times\frac{20-9}{2\sqrt{0.5\left(20-9\right)^2+312}}$ $=-0.290388\times0.2849703$ $=-0.0828$ (3sf)
(vi)	The monthly demand for toy trains at 20 months from now is decreasing at a rate of 82.8 toys per month.
6(i)	No. of ways = ${}^{17}C_8=24310$
(ii)	**<u>Method 1</u>** <u>Case 1: Husband at either end</u> No. of ways = $7!\times2\times8!$ = 406425600 <u>Case 2: Husband not at either end</u> No. of ways = $14\times8!\times7!$ = 2844979200 Total number of ways = 406425600 + 2844979200 = 3251404800 **<u>Method 2</u>**

	No. of ways = $8! \times 7! \times \left({}^{16}C_1 \times 1\right) = 3251404800$
(iii)	No. of ways = ${}^{8}C_4 \times {}^{9}C_4 = 8820$
(iv)	Case 1: includes wife (but not husband) ${}^{7}C_3 \times {}^{8}C_4 = 2450$ Case 2: includes husband (but not wife) ${}^{7}C_4 \times {}^{8}C_3 = 1960$ Total number of ways = 2450 + 1960 = 4410 Required Probability = $\frac{4410}{8820} = \frac{1}{2}$
7(i)	 $65 = 26 - x + x + 21 - x + \mathrm{n}\left(M \cap S' \cap A'\right)$ $\mathrm{n}\left(M \cap S' \cap A'\right) = 18 + x$ $40 = 26 - x + x + 14 - x + \mathrm{n}\left(S \cap M' \cap A'\right)$ $\mathrm{n}\left(S \cap M' \cap A'\right) = x$ $45 = 21 - x + x + 14 - x + \mathrm{n}\left(A \cap S' \cap M'\right)$ $\mathrm{n}\left(A \cap S' \cap M'\right) = 10 + x$
(ii)	Since M and S are independent, $\mathrm{P}(M \cap S) = \mathrm{P}(M) \times \mathrm{P}(S)$

	$$\frac{26}{92+x}=\frac{65}{92+x}\times\frac{40}{92+x}$$ $$26(92+x)=65\times 40=2600$$ $$2392+26x=2600$$ $$26x=208$$ $$x=8$$
(iii)	$\mathrm{P}(M \mid A)$ is the probability that a customer likes mangoes, given that he or she likes apples. $$\mathrm{P}(M \mid A)=\frac{\mathrm{P}(M\cap A)}{\mathrm{P}(A)}$$ $$=\frac{\left(\frac{21}{100}\right)}{\left(\frac{45}{100}\right)}$$ $$=\frac{7}{15}$$
(iv)	Number of customers who like exactly two different fruits = 18 + 13 + 6 = 37 Number of customers who like only one fruit = 26 + 18 + 8 = 52 Required probability = P(3 like exactly two different fruits and 1 like only one fruit) $=\frac{37}{100}\times\frac{36}{99}\times\frac{35}{98}\times\frac{52}{97}\times 4=0.103$ (3 sf) or $=\frac{{}^{37}C_3\times{}^{52}C_1}{{}^{100}C_4}=0.103$ (3 sf)

8(i)	$\bar{x} = \dfrac{3540}{8} = 442.5, \quad \bar{y} = \dfrac{k+6120}{8}$ Using $\bar{y} = 1.834352\bar{x} + 40.8$, $\dfrac{k+6120}{8} = 1.834352(442.5) + 40.8$ $k = 700.00608$ = 700 (to nearest integer)
(ii)	$r = 0.996$. It indicates a strong positive linear correlation between the number of Nutella brownies sold and the profit.
(iii)	
(iv)	Sub x = 480 into $y = 1.834352x + 40.8$, we obtain $y = 921.28896$ $y \approx \$921.29$ (to 2 d.p. for money) The **product moment correlation** is close to 1 and $x = 480$ is **within the data range** of [190, 620]. Thus, the estimate will be a reliable one.
(v)	As r measures the degree of scatter of the data points, an increase of 80 for all the values of the monthly profit (values of y) will not change the scatter of the data. Hence, there will be no change in the value of r.
9(i)	Let X be the duration of a particular viral infection in a child. Let μ be the mean population duration of a particular viral infection among children and σ^2 be the population variance of a particular viral infection among children. $s^2 = \dfrac{120}{119}(1.215)^2 = 1.48863$, $\bar{x} = 4.07$ $H_0 : \mu = 4.3$ $H_1 : \mu \neq 4.3$ at 5% level of significance

	Under H_0, since $n = 120$ is large, by Central Limit Theorem, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ approximately. $Z = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \sim N(0,1)$ approximately Using a one-tailed test, $\bar{x} = 4.07$ gives *p*-value = 0.038920609 < 0.05 We reject H_0 at 5% level of significance and conclude that there is sufficient evidence there the mean duration of viral infection of the particular viral among children is different from the researchers' claim.
(ii)	Not necessary. Since *n* = 120 is large, by Central Limit Theorem, the sample mean duration of a particular viral infection in children follows a normal distribution approximately.
(iii)	$H_0 : \mu = 4.3$ $H_1 : \mu < 4.3$ Since H_0 is rejected, p-value $\leq \frac{k}{100}$ $\frac{0.038920609}{2} \leq \frac{k}{100}$ $k \geq 1.95$ (3 s.f.)
(iv)	Unbiased estimate of population variance, $s^2 = \frac{\sum (y - \bar{y})^2}{59} = \frac{77.88}{59} = 1.32$
(v)	H_0: $\mu = 4.3$ H_1: $\mu < 4.3$ There is a probability of $\frac{\beta}{100}$ to conclude that the population mean duration of infection being treated with medicine *M* is less than 4.3 days when it is actually 4.3 days.
10 (i)	Let *X* be the mass of a randomly selected Honeycrisp apple. $X \sim N\left(78, 13^2\right)$. $P(70 < X < 90) = 0.5528661 \approx 0.553$
(ii)	Let *Y* be the mass of a randomly selected empty box. $Y \sim N\left(10, 0.8^2\right)$

	Let $T = X_1 + X_2 + X_3 + ... + X_{24} + Y \sim N(1882, 4056.64)$ $P(T < 2000) = 0.9680354 \approx 0.968$
(iii)	The mass of an apples is independent of the mass of another apple. The mass of an apple is independent of the mass of an empty box.
(iv)	k is the largest mass. $P(X \geq k) > 0.55$ From GC, When $k = 75, \quad P(X \geq k) = 0.59125$ $k = 76, \quad P(X \geq k) = 0.56113$ $k = 77, \quad P(X \geq k) = 0.53066$ $k = 76$ Therefore largest $k = 76$ Alternative method $P\left(Z \geq \frac{k-78}{13}\right) > 0.55$ $\frac{k-78}{13} < -0.12566$ $k < 76.366$ largest $k = 76$
(v)	Let A be the cost of a randomly selected Honeycrisp apple. $A \sim N(5.46, 0.8281)$ Let F be the total cost of an empty box and 24 Honeycrisp apples. $F \sim N(131.24, 19.8744)$ $F_1 + F_2 + F_3 \sim N(393.72, 59.6232)$ $P(370 \leq T \leq 420) = 0.998604 \approx 0.999$

11 **(i)(a)**	Let M be the height of a male student and F be the height of a female student. $M \sim N(171,\ 3.58^2)$, $F \sim N(160,\ 5.11^2)$ $\left(\text{P}(F > 158)\right)^2 = 0.6522452^2 = 0.425423 \approx 0.425$
(i)(b)	$\text{P}(F > 158) \times \text{P}(F < 158) \times 2! = 0.4536428 \approx 0.454$
(ii)	Let T be total height of 3 male and 4 female students. $T = F_1 + F_2 + F_3 + F_4 - M_1 - M_2 - M_3$ $T \sim N(127,\ 142.8976)$ $\text{P}(\lvert F_1 + F_2 + F_3 + F_4 - M_1 - M_2 - M_3 \rvert \le 130)$ $= \text{P}\left(-130 \le T \le 130\right)$ $= 0.599078$ ≈ 0.599
(iii)	Let X be the no female students (out of 80) whose height is more than 158 cm. $X \sim B(80,\ 0.6522453)$ $P(X \ge 50) = 1 - P(X \le 49)$ $= 0.73758 \approx 0738$
(iv)	Let Y be number of male students (out of n) who are shorter than 175 cm. $Y \sim B(n,\ \text{P}\left(M < 175\right))$ $\text{P}\left(M < 175\right) = 0.868070857$ $Y \sim B(n,\ 0.868070857)$ $P(Y \le 10) > 0.4$ From GC, $n = 11, \quad \text{P}\left(Y \le 10\right) = 0.78909 > 0.4$ $n = 12, \quad \text{P}\left(Y \le 10\right) = 0.483 > 0.4$ $n = 13, \quad \text{P}\left(Y \le 10\right) = 0.24071 < 0.4$ Largest $n = 12$.