# TAMPINES MERIDIAN JUNIOR COLLEGE

## JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ______________________________________________

CIVICS GROUP: ________________________________________________

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# H2 MATHEMATICS 9758/01

Paper 1

13 SEPTEMBER 2023

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

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**READ THESE INSTRUCTIONS FIRST**

Write your name and Civics Group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

| For Examiners' Use | |
|---|---|
| **1** | |
| **2** | |
| **3** | |
| **4** | |
| **5** | |
| **6** | |
| **7** | |
| **8** | |
| **9** | |
| **10** | |
| **Total** | |

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

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This document consists of 6 printed pages and 0 blank pages.

**[Turn Over**

1 The hyperbola $C_1$ has equation $\dfrac{(x-2)^2}{3^2}-\dfrac{y^2}{2^2}=1$.

**(a)** Sketch $C_1$, labelling the equations of the asymptotes and coordinates of the axial intercepts. [3]

The curve $C_2$ has equation $(x-2)^2+y^2=k^2$, where $k>0$.

**(b)** State the value of $k$ such that $C_1$ and $C_2$ intersect exactly twice. [1]

2 **(a)** Differentiate $e^{x^2+2x}$ with respect to $x$ and hence find the exact value of $\int_0^1 (x+1)e^{x^2+2x}\ dx$. [3]

**(b)** Using the substitution $x=\sin t$, where $0<t<\dfrac{\pi}{2}$, find $\displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\ dx$ in terms of $x$. [4]

3 In an art lesson, wires are cut and bent to create a series of shapes.

**(a)** Student A cuts a piece of wire of length 100 cm to create a series of squares. The perimeter of the smallest square is 5 cm and the perimeter of each succeeding square is increased by 3 cm. Find the maximum number of squares Student A can form. [3]

**(b)** Student B cuts another piece of wire to create a series of 12 circles, such that the **radius** of the first circle is $x$ cm. The **circumference** of each succeeding circle is $\dfrac{2}{3}$ that of the preceding circle. Given that the total length of wire used by Student B is exactly 100 cm, find the value of $x$. [4]

**4** The curve $C$ has equation $y = 2 + \frac{3a}{x-a}$, where $a$ is a positive constant.

**(a)** Describe fully a sequence of transformations which would transform the curve $y = \frac{1}{x}$ onto the curve $C$. [3]

**(b)** Sketch the graph of $y = \left|2 + \frac{3a}{x-a}\right|$, labelling the equations of the asymptotes and coordinates of the axial intercepts. [3]

**(c)** Use an algebraic method to solve $\left|2 + \frac{3}{x-1}\right| = 3$ and state the solution for $\left|2 + \frac{3}{x-1}\right| < 3$. [4]

**5** **(a)** The polynomial $\mathrm{P}(z)$ is given by $z^3 - 8z + a$, where $a$ is a real constant. Given that $\sqrt{3} + \mathrm{i}$ is a root of the equation $\mathrm{P}(z) = 0$, solve exactly the equation $\mathrm{P}(z) = 0$. [5]

**(b)** It is given that $w = -\sqrt{3} + \mathrm{i}$. Write down the exact value of $\arg(w)$ and hence find the three smallest positive integer values of $n$ such that $w^n$ is purely imaginary. [3]

**6** A curve $C$ has equation $y = ax^2 + bx + c$, where $a, b$ and $c$ are constants.

It is given that $C$ passes through the points $(-2, 17)$, $\left(\frac{1}{2}, \frac{3}{4}\right)$ and $(5, 3)$.

**(a)** Find the equation of $C$. [3]

The function f is defined by

$$\mathrm{f} : x \mapsto ax^2 + bx + c, \quad x \leq 0.$$

**(b)** Using your answer from part **(a)**, find $\mathrm{f}^{-1}(x)$. [3]

The function g is defined by

$$\mathrm{g} : x \mapsto \begin{cases} 3(x-1)^2 & \text{for } 0 \leq x < 2, \\ 5 - |x-4| & \text{for } 2 \leq x < 6. \end{cases}$$

**(c)** It is given that $\mathrm{g}(x) = \mathrm{g}(x+6)$ for all real values of $x$.

Sketch the graph of $y = \mathrm{g}(x)$ for $-4 \leq x < 9$. [4]

**(d)** For the rest of the question, let the domain of g be $[0, 6)$, as originally defined.

The function h is defined by

$$\mathrm{h} : x \mapsto \frac{3}{2} + \frac{3}{2}\cos x, \;\; 0 \leq x \leq \pi.$$

Prove that the composite function $\mathrm{gh}$ exists and find the range of $\mathrm{gh}$. [3]

**7** **(a)** An ellipse has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a$ and $b$ are positive real constants.

Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{b^2x}{a^2y}$ if $y\neq 0$. [1]

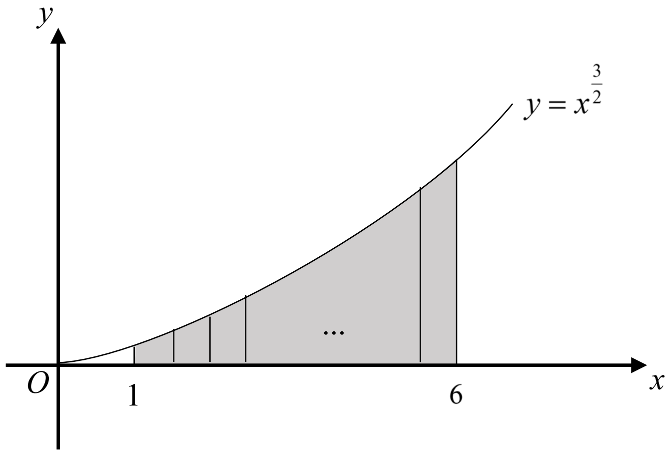
**(b)** The point $P(a\cos\theta, b\sin\theta)$, where $0<\theta<\frac{\pi}{2}$, lies on the ellipse. Show that the equation of the tangent to the ellipse at $P$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$. [4]

**(c)** The tangent found in part **(b)** meets the $x$-axis and $y$-axis at points $R$ and $S$ respectively. Find the area of triangle $ORS$ in terms of $\theta$, $a$ and $b$, where $O$ is the origin. [3]

**(d)** Explain why the minimum area of triangle $ORS$ is $ab$ and state the value of $\theta$ which gives the minimum area of triangle $ORS$. [2]

**8** **(a)** Find the exact value of $\pi\int_1^6 x^3\,\mathrm{d}x$. [1]

**(b)** The diagram shows a sketch of the curve $y = x^{\frac{3}{2}}$, where $x \geq 0$. The region under the curve between $x = 1$ and $x = 6$, shown shaded in the diagram, is $R$. It is required to estimate the volume of the solid formed when $R$ is rotated through $2\pi$ radians about the $x$-axis. The region $R$ is split into $n$ vertical strips of equal width as shown in the diagram below.




Each vertical strip can be approximated by a rectangle. Each rectangle, when rotated through $2\pi$ radians about the $x$-axis, will result in a circular disc. The sum of the volumes of the $n$ circular discs with equal thickness is denoted by $V$. Suppose $V$ gives an overestimation of the volume of revolution of the solid formed when $R$ is rotated through $2\pi$ radians about the $x$-axis.

Show that $V = \dfrac{5\pi}{n}\displaystyle\sum_{r=1}^{n}\left(1 + r\left(\frac{5}{n}\right)\right)^3$. [3]

**(c)** Find an expression for $V$, leaving your answers in the form $\dfrac{5\pi}{4}\left(259 + \dfrac{a}{n} + \dfrac{b}{n^2}\right)$, where $a$ and $b$ are constants to be determined. You may use the results

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \text{ and } \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$ [5]

**(d)** Using your answer in part **(c)**, state the limit of $V$ as $n \to \infty$. Explain how this could be verified by the answer in part **(a)**. [2]

**9** A home interior designer is looking to design a pull-down wardrobe lift. To test his concept, the designer built a small-scale model of it, in which points $(x, y, z)$ are defined relative to a corner of the model at $O(0, 0, 0)$. The rods used in the model can be modelled by equations of straight lines.

The main rod can be modelled by the line $l_1$ with equation $\mathbf{r} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$.

Another rod, line $l_2$, passes through the points $A(5, -5, 8)$ and $B(8, -4, 3)$.

**(a)** A supporting wire connects $B$ to a point $F$ on $l_1$ such that $\overrightarrow{BF}$ is perpendicular to $l_1$. Find the position vector of $F$. [3]

**(b)** A retractable rod, line $l_R$, is placed such that it can be modelled by reflecting $l_2$ about $l_1$. Find a vector equation of $l_R$. [3]

The panels of the small-scale model can be modelled by equations of planes. The main panel, plane $p_1$, contains the main rod $l_1$ and an attachment point $C(0, 0, 6)$. A side panel, plane $p_2$, can be represented by the equation $p_2 : -17x - 37y + 4z = 24$.

**(c)** Show that $p_1$ can be represented by the cartesian equation $-x + y + 5z = 30$. [3]

**(d)** A structural rod, line $l_S$, needs to be placed at the intersection between $p_1$ and $p_2$. Find a vector equation of $l_S$. [2]

**(e)** A decorative plank, plane $p_3$, has equation $p_3 : -3x + \alpha y + 15z = \beta$, where $\alpha, \beta \in \mathbb{R}$. If the distance between $p_1$ and $p_3$ is exactly $3\sqrt{3}$ units, find the possible values of $\alpha$ and $\beta$. [3]

**10** **(a)** Show that the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x} = x^3$ may be reduced by the substitution $y = ux^2$ to $\frac{\mathrm{d}u}{\mathrm{d}x} = x$. Hence find the general solution for $y$ in terms of $x$. [4]

Scientists are investigating the growth of a type of bacteria by monitoring the number of bacteria, $N$, present at time $t$ days after the start of the experiment.

**(b)** In one model, it can be assumed that, at time $t$, the growth rate is proportional to the number of bacteria present. At $t = 0$, the bacteria count is 5000 and the initial growth rate is 200 per day. Write down a differential equation for this model and solve it to obtain a projection for the number of bacteria 50 days after the start of the experiment. [8]

**(c)** After further consideration, the scientists decide to model the growth of the population of bacteria by a type of *Gompertz model* which states that

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN\left(\ln M - \ln N\right),$$

where $k$ and $M$ are positive constants.

Find the general solution of this differential equation and hence give an interpretation of $M$ in the context of the question. [4]

**End of Paper**