National Junior College
2016 – 2017 H2 Further Mathematics
Topic F3: Further Differential Equations (Tutorial)

Basic Mastery Questions

1 Solve the following differential equations, expressing y in terms of t:

(a) $2\sec t\dfrac{\mathrm{d}y}{\mathrm{d}t}=\sqrt{1-y^2}$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{t}{y-t^2y},\quad y(0)=4$

2 Solve the following differential equations, expressing y in terms of x:

(a) $xy'+(2x-3)y=4x^4$

(b) $(1+x)y'+y=\cos x,\quad y(0)=1$

3 Find the general solutions of the following differential equations.

(a) $2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\dfrac{\mathrm{d}y}{\mathrm{d}x}-2y=0$

(b) $16\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-8\dfrac{\mathrm{d}y}{\mathrm{d}x}+y=0$

(c) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-4\dfrac{\mathrm{d}y}{\mathrm{d}x}+21y=0$

4 Find the general solutions of the following differential equations.

(a) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y=10\sin t$

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y=6\mathrm{e}^x-4x$

5 Sketch the phase lines for the following differential equations, and state the equilibrium points.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}t}=10+3y-y^2$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t}=y^2(4-y^2)$

Practice Questions

6 Experiments indicate that rate at which glucose is absorbed by the body is λ times the amount of glucose present in the bloodstream, *G*. Glucose is injected into a patient's bloodstream at a constant rate of *r* units per unit time.

Write a differential equation that models the amount of glucose present in the patient's bloodstream at time *t*.

Solve for *G* in terms of *t*, given that the amount of glucose initially present in the patient's bloodstream is G_0 units.

Find $\lim\limits_{t\to\infty} \mathrm{G}(t)$.

7 The variable *z* satisfies the differential equation $\dfrac{\mathrm{d}z}{\mathrm{d}x} + z(\cot x + \pi - 4x) = 0$.

Using the substitution $z = \dfrac{y}{\sin x}$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (4x - \pi)y$.

Hence find *z* in terms of *x*, given that $z = 2$ when $x = \dfrac{\pi}{2}$.

8 A first order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{p}(x)y = \mathrm{q}(x)y^n \ , \ n \neq 0 \ \text{ and } \ n \neq 1$$

is called a **Bernoulli equation**. Show that the substitution $u = y^{1-n}$ reduces the Bernoulli equation into the linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)\mathrm{p}(x)u = (1-n)\mathrm{q}(x).$$

Hence, solve the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} + xy = xy^2$.

9 Show that the substitution $v = \ln y$ reduces the differential equation

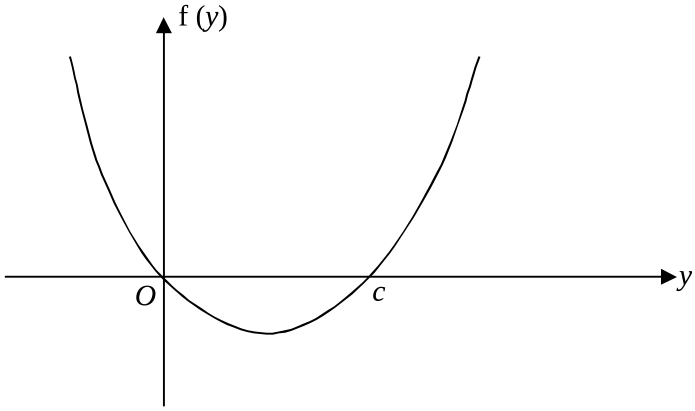
$$\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{P}(x)y = \mathrm{Q}(x)(y \ln y)$$

to the linear equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} + \mathrm{P}(x) = \mathrm{Q}(x)\mathrm{v}(x).$$

Hence solve the equation $x\frac{\mathrm{d}y}{\mathrm{d}x}-4x^2y+2y\ln y=0$.

10 Using the graph of the function $\mathrm{f}(y)$ given below, sketch the phase line for the autonomous equation $\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{f}(y)$. Hence sketch, on a single diagram, the solution curves for **(i)** $y(0)=1.2c$, **(ii)** $y(0)=0.4c$, **(iii)** $y(0)=-0.6c$.



11 When certain kind of chemicals are combined, the rate at which the new compound is form is modelled by the differential equation $\frac{dx}{dt}=k(\alpha-x)(\beta-x)$, where x is the number of grams of the new compound formed and $k>0$ is the constant of proportionality and $0<\alpha<\beta$.

(a) Sketch a phase line diagram. Predict the behaviour of x as $t\to\infty$, given that $\alpha<x<\beta$ when $t=0$.

(b) Consider the case when $\alpha=\beta$. Use a phase line diagram to predict the behaviour of x as $t\to\infty$ when $x(0)<\alpha$.

(c) Show that an explicit solution to the above DE in the case when $k=1$ and $\alpha=\beta$ $=2$ is $x(t)=2-\frac{1}{t+c}$, where c is an arbitrary constant. Find the particular solution satisfying $x(0)=1$. Graph this solution curve. Does the behaviour of the solutions as $t\to\infty$ agree with your answer to part (b)?

12 The population of a new breed of prawns in a farm is being studied. In an initial survey, the number, P (in thousands), of the prawns at time t months is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t}=P(4-P)-n,$$

where n (in thousands) is a positive real constant.

Suppose the farmer intends to harvest 1250 prawns per month. Find the least initial number of prawns that he needs to breed in order for him to ensure a high long-term sustainability of the prawn population.

Solve the differential equation in this case, expressing P in terms of t.

13 For a certain species in a community, there is evidence that there is a minimum population m such that the species will become extinct if the population size of the species falls below m. The condition can be incorporated into the logistic equation by introducing the factor $\left(1-\dfrac{m}{P}\right)$. Thus the population of the species at time t months after an initial survey is modelled by the modified logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{b}\right)\left(1-\frac{m}{P}\right),$$

where k and b are positive constants and b not equal m.

(i) Give a reason why $m < b$ and hence explain why the species will become extinct if the population size of the species falls below m.

(ii) Draw a phase line diagram for the differential equation.

(iii) Comment on the long-term prospects of the population for various values of the initial population.

(iv) Given $\dfrac{\mathrm{d}P}{\mathrm{d}t}$ has a maximum value at $P = q$, sketch the solution curve for which the initial population is between m and q.

14 Find the general solution of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+a)\dfrac{\mathrm{d}y}{\mathrm{d}x}+ay=0$ where the real constant a is such that $a \neq 1$. Find the solution for which $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-1$ when $x = 0$.

15 Show that, if y is a function of x and $x = \mathrm{e}^u$, then $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\mathrm{d}^2y}{\mathrm{d}u^2}-\dfrac{\mathrm{d}y}{\mathrm{d}u}$.

Given that y satisfies the differential equation $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+3x\dfrac{\mathrm{d}y}{\mathrm{d}x}+5y=0$ for $x > 0$.

Use the substitution $x = \mathrm{e}^u$ to show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2}+2\frac{\mathrm{d}y}{\mathrm{d}u}+5y=0.$$

Hence, find the general solution for y in terms of x.

16 Show, by means of the substitution $y = x^{-4}z$, that the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(4x^2+8x\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(3x^2+16x+12\right)y=0,\ \ x>0,$$

can be reduced to the form $\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}+a\frac{\mathrm{d}z}{\mathrm{d}x}+bz=0$, where a and b are to be determined. Hence find the general solution for y in terms of x.

17 Obtain the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+13y=-5\sin 2x+60\cos 2x.$$

Hence, show that if x is large and positive then, whatever the initial conditions,

$$y\approx 5\sin\left(2x+\alpha\right),\ \text{where } \alpha=\tan^{-1}\left(\frac{4}{3}\right).$$

18 Show, by means of the substitution $x=\cos\theta$, that the differential equation

$$\left(1-x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}+n^2 y=2\left(1-x^2\right),$$

where n is a positive integer, may be reduced to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\theta^2}+n^2 y=2\sin^2\theta.$$

Hence obtain the general solution of the original solution for $n\neq 2$ in the form

$$A\cos\left(n\cos^{-1}x\right)+B\sin\left(n\cos^{-1}x\right)+\mathrm{f}\left(x\right),$$

where A and B are arbitraty constants and $\mathrm{f}\left(x\right)$ is a function of x to be determined.

19 Use the substitution $\sin y = x$ to transform the differential equation

(*) $$\cos y\left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right)-\sin y\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+2k\cos y\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)+\sin y=t$$

where k is a real number, into the form

$$a\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}+b\frac{\mathrm{d}x}{\mathrm{d}t}+cx=f\left(t\right)$$

where a, b and c are real numbers and $f(t)$ is a function of t to be determined.

Hence find the general solution to **(*)** in the form $\sin y = g(t)$ where $g(t)$ is some function of t, in each of the following cases:

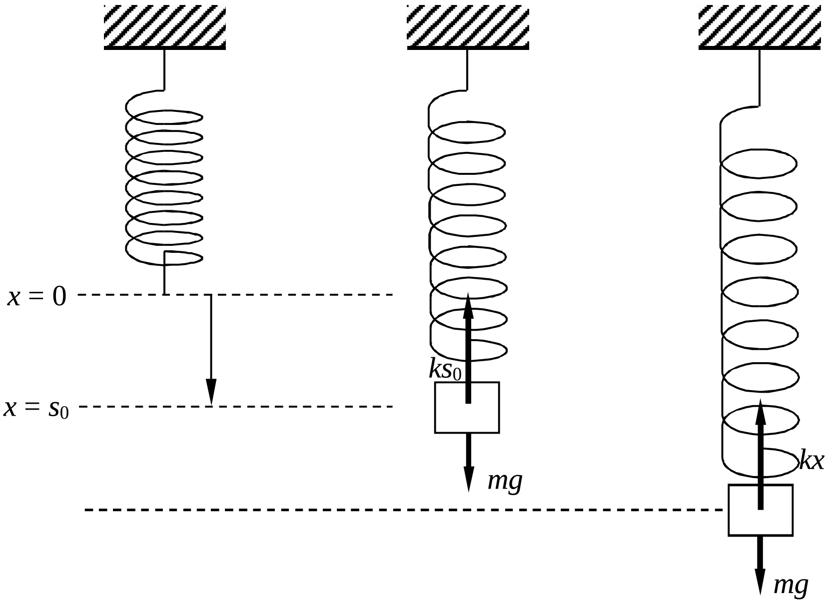
(i) $k > 1$,
(ii) $0 < k < 1$.

Suppose that in case (ii), t is allowed to take large values, show that t must be bounded between two values (in terms of k) which are to be found.

20 The vertical elastic spring system with a mass m attached to the spring can be modelled by the equation

$$mx'' = -kx + mg,$$

where k is the spring constant and g is the acceleration due to gravity (a constant). The length by which the spring is stretched due to the weight of the mass such that the net force acting on the mass is zero is given by s_0, as shown below.



(i) Find, in terms of m, g and k, the value of s_0.

(ii) Verify that $x = s_0$ is a particular solution to the differential equation.

(iii) By using the substitution $x = y + s_0$, show that the differential equation can be reduced to

$$my'' = -ky.$$

Hence find x in terms of t, given that k = 4, m = 1, and the mass is initially momentarily at rest 5 cm from the equilibrium position. [Take g to be 10ms^{-2}.]

21 To avoid a door from being slammed shut (and getting damaged as a result), a door stopper comprising a dashpot is installed in the door. It is desired for the spring-mass-dashpot system to be critically damped.

The spring in the door (that applies a force to shut the door once a user releases the door after opening it) has a spring constant of 250 N/m, while the dashpot has a damping constant of c Ns/m (Newton seconds per metre). The door has a mass of 10 kg. A person opens the door such that the spring is stretched by 1 m from its unstretched length. (*For a simplified model, you may assume that the door is moving in a linear fashion.*)

Taking $t = 0$ to be the time that the person releases the door, find the equation of motion of the door for the following two scenarios:

(a) $c = 60$,
(b) $c = 260$.

For both of the above scenarios, sketch the solution curves.

What is the desired value of the damping constant of the door stopper? Find the equation of motion of the door and sketch the solution curve for this case.

With reference to the solution curves in parts (a) and (b), suggest a possible reason why critical damping is desired.

<u>Numerical Answers</u>

<u>Basic Mastery Questions</u>

1(a) $y = \sin\left(\frac{1}{2}\sin t + C\right)$ **1(b)** $y = \sqrt{16 - \ln\left(1 - t^2\right)}$

2(a) $y = 2x^3 + Cx^3\mathrm{e}^{-2x}$ **2(b)** $y = \dfrac{\sin x + 1}{1 + x}$

3(a) $y = A\mathrm{e}^{-2x} + B\mathrm{e}^{\frac{1}{2}x}$ **3(b)** $y = (A + Bx)\mathrm{e}^{\frac{1}{4}x}$ **3(c)** $y = \mathrm{e}^{2x}(A\cos\sqrt{17}x + B\sin\sqrt{17}x)$

4(a) $y = \mathrm{e}^{-t}\left(A\cos t + B\sin t\right) + 2\sin t - 4\cos t$ **4(b)** $y = A\mathrm{e}^{-x} + B\mathrm{e}^{-2x} + \mathrm{e}^{x} - 2x + 3$

<u>Practice Questions</u>

6 $G = \dfrac{r}{\lambda} + \left(G_0 - \dfrac{r}{\lambda}\right)\mathrm{e}^{-\lambda t}$; $\lim\limits_{t\to\infty} G\left(t\right) = \dfrac{r}{\lambda}$

7 $z\sin x = 2\mathrm{e}^{2x^2 - \pi x}$

8 $y = \dfrac{1}{1 + A\mathrm{e}^{\frac{1}{2}x^2}}$

9 $y = \mathrm{e}^{x^2 + \frac{C}{x^2}}$

11(c) $x(t) = 2 - \dfrac{1}{t+1}$

12 $P = \dfrac{A\mathrm{e}^{\sqrt{11}t}\left(\sqrt{11}+4\right)+4-\sqrt{11}}{2\left(1+A\right)\mathrm{e}^{\sqrt{11}t}}$, where $A = 9.42\times10^{-5}$.

14 $y = A\mathrm{e}^{-ax} + B\mathrm{e}^{-x}$; $y = \mathrm{e}^{-x}$

15 $y = \dfrac{1}{x}\left[A\cos\left(2\ln x\right) + B\sin\left(2\ln x\right)\right]$

16 $y = \dfrac{1}{x^4}(A\mathrm{e}^{-x} + B\mathrm{e}^{-3x})$

17 $y = \mathrm{e}^{-2x}\left(A\cos 3x + B\sin 3x\right) + 3\sin 2x + 4\cos 2x$

18 $y = A\cos\left(n\cos^{-1}x\right) + B\sin\left(n\cos^{-1}x\right) + \dfrac{1}{n^2} + \dfrac{2x^2-1}{4-n^2}$

19(i) $\sin y = A\,\mathrm{e}^{\left(-k+\sqrt{k^2-1}\right)t} + B\,\mathrm{e}^{\left(-k-\sqrt{k^2-1}\right)t} + t - 2k$

19(ii) $\sin y = \mathrm{e}^{-kt}\left[A\cos\left(\sqrt{\left(1-k^2\right)}t\right) + B\sin\left(\sqrt{\left(1-k^2\right)}t\right)\right] + t - 2k$; $2k-1 \le t \le 2k+1$

20(i) $s_0 = \dfrac{mg}{k}$ **(iii)** $x = 2.5\cos\left(2t\right) + 2.5$

21(a) $x(t) = \mathrm{e}^{-3t}\left(\cos 4t + \dfrac{3}{4}\sin 4t\right) = \dfrac{5}{4}\mathrm{e}^{-3t}\cos\left(4t - \tan^{-1}\left(\dfrac{3}{4}\right)\right)$

21(b) $x(t) = \dfrac{25}{24}\mathrm{e}^{-t} - \dfrac{1}{24}\mathrm{e}^{-25t}$; $x(t) = (5t+1)\mathrm{e}^{-5t}$