2023 Promotional Examination Revision Package

1

Sequences and Series

1. **(A Level N84/P1/Q1)**

(a) The sum, S_n, of the first n terms of an arithmetic progression is given by $S_n = pn + qn^2$. Given also that $S_3 = 6$ and $S_5 = 11$,

(i) Find the values of p and q, [3]

(ii) Deduce, or find otherwise, an expression for the nth term and the value of the common difference. [3]

(b) Find the set of values of θ lying in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ such that the sum to infinity of the geometric series $1 + \sin\theta + \sin^2\theta + ...$ is greater than 2. [4]

2. **(NJC14/Promo/Q11)**

(a) The sum of the first n terms of a sequence $u_1, u_2, u_3, ...$ is given by

$$S_n = \frac{c}{k-1}\left[1 - \left(\frac{1}{k}\right)^n\right],$$

where c and k are non-zero real constants, with $k > 1$.

(i) Show that $u_n = ck^{-n}$ for all positive integers n. [2]

(ii) Prove that the sequence $\{u_n\}$ is a geometric progression.

Hence explain why the sum to infinity of the geometric progression exists, and evaluate it in terms of c and k. [4]

(b) A patient is administered a 500 mg dose of Drug A on the first day. Each subsequent day, the dosage of Drug A is reduced by 20 mg.

(i) Find the total amount of Drug A (in mg) administered at the end of 2 weeks. [2]

Another patient is administered a 500 mg dose of Drug B every 6 hours. At the end of each 6-hour period, 32% of the amount of drug present at the start of the 6-hour period remains.

(ii) Find the amount of Drug B (in mg) present immediately after the patient takes the 3rd dose. [2]

(iii) An overdose occurs when the amount of Drug B exceeds 735 mg. Calculate the maximum number of doses the patient can take to avoid overdosing. [3]

3. **(RI14/Promo/Q6)**

The annual wage of a certain occupation offered by Company P begins with an initial amount of \$9500 and increases by \$400 every year for 14 years till it reaches \$15100 and remains constant at \$15100 thereafter.

Let $p(n) = a_1 + a_2 + a_3 + \ldots + a_n$, where a_i denotes the annual wage in the i^{th} year offered by Company P for i = 1, 2, 3, …, n.

(i) Show that $p(n) = \begin{cases} 200n^2 + 9300n & \text{for } n \leq 15, \\ 15100n + A & \text{for } n > 15, \end{cases}$

where A is a constant to be determined. [4]

(ii) The annual wage of the same occupation offered by Company Q begins with the same initial amount of \$9500 but it increases at a fixed rate of r % every year.

Sam has been offered a job for the same occupation at both Company P and Q. Find the value of r, correct to 2 decimal places, such that the total annual wages earned by Sam if he works for Company P for 15 years is equivalent to the total annual wages earned if he works for Company Q for 12 years. [4]

4. **(AJC16/Promo/Q11)**

A publisher tracks the sales of a new book. It is found that 3^n copies of the book are sold in the first week, where n is a positive integer ($n \geq 3$). In week 2 he sells 3^{n-1} copies more than in week 1, and in week 3 he sells $\left(3^{n-1} + 3^{n-2}\right)$ copies more than in week 2.

(i) By finding and simplifying the number of copies sold in each week from week 1 to week 3 in the form $a\left(3^n\right)$, show that the numbers of copies sold in each of the first three weeks form a geometric progression. [3]

In fact, starting from week 3, the number of copies sold in each week forms an arithmetic progression with common difference -48.

(ii) Given that the number of copies sold falls to zero in week K, show that

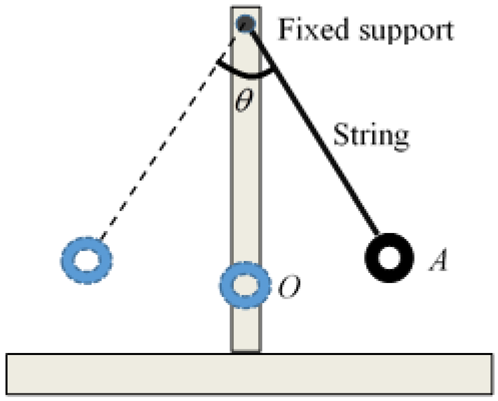
$$K = 3 + 3^{n-3} .$$ [3]

(iii) Hence find the total number of copies sold from week 1 up to and including week K, expressing your answer in terms of n. [3]

(iv) The publisher printed 20,000 copies of the book. What is the greatest value of n such that the books will not be out of stock? [3]

5. **(SRJC16/Promo/Q10)**

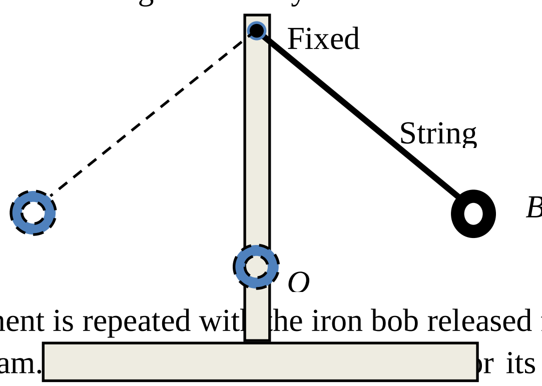
In a simple pendulum experiment, an iron bob is suspended by a string from a fixed support. The iron bob is then released from the position A as seen in the diagram, and the angle θ made from its swings are recorded until the iron bob comes to a rest at position O. A swing is defined as the complete movement of the iron bob from one side of O to the other. For instance, the first swing would be from right to left, and the second swing from left to right and so on.



When the iron bob swings for the first time, the angle θ recorded is 80°. In each of the subsequent swings, the angle θ recorded is $\frac{3}{4}$ times that of the previous swing.

(i) Find the angle recorded at the 7th swing. [2]

(ii) At the n^{th} swing, the angle recorded first falls below 3% that of the first swing. Find the sum of the angles made by the iron bob after n swings. [4]



The experiment is repeated with the iron bob released from the position B instead, as seen in the diagram. [illegible]r its first swing is 120°. In each of the subsequent swings, the angle recorded is 3° less than that of the previous swing and the experiment is stopped after 40 swings.

(iii) Find the angle recorded at the 19th swing. [2]

(iv) Given that the sum of the angles made by the iron bob in the first n swings is less than 1550°, find the greatest possible value of n. [3]

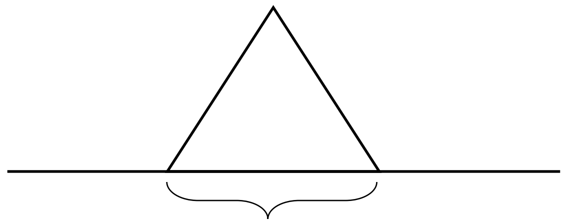
6. **(ACJC18/Promo/Q13)**

The Koch snowflake is one of the earliest example of a fractal, a pattern that produces a picture which contains an infinite number of copies of itself.

The snowflake is constructed by starting with an initial equilateral triangle, then recursively adding layers of equilateral triangles as follows:

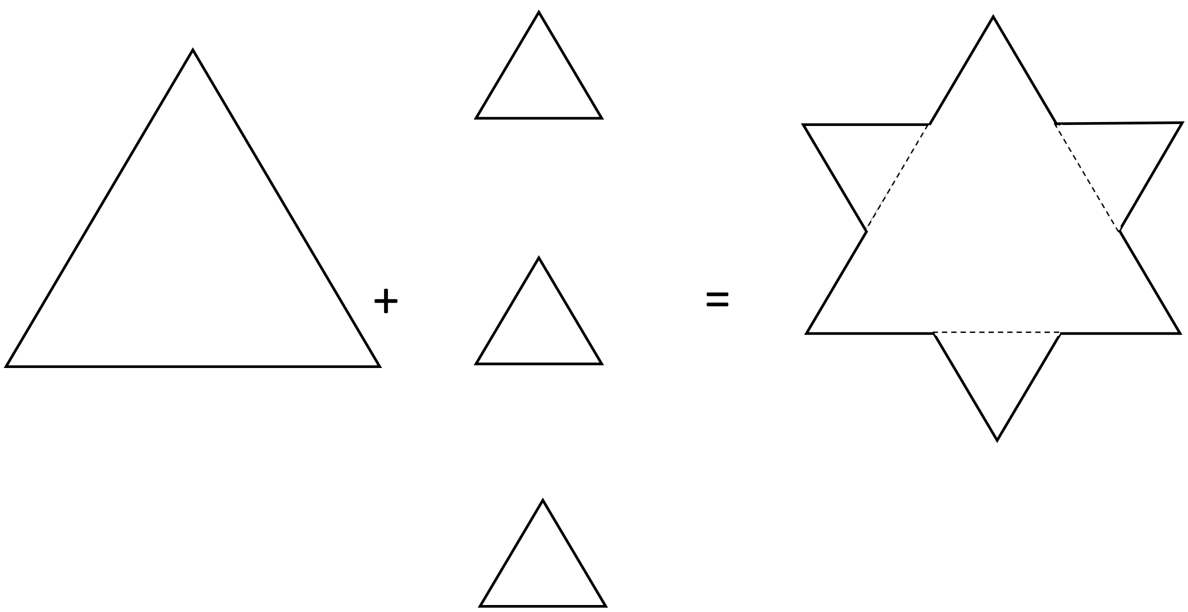
(I) divide each line segment into three segments of equal length,

(II) an equilateral triangle is drawn with the middle segment from step (I) as its base and pointing outward, as shown below.

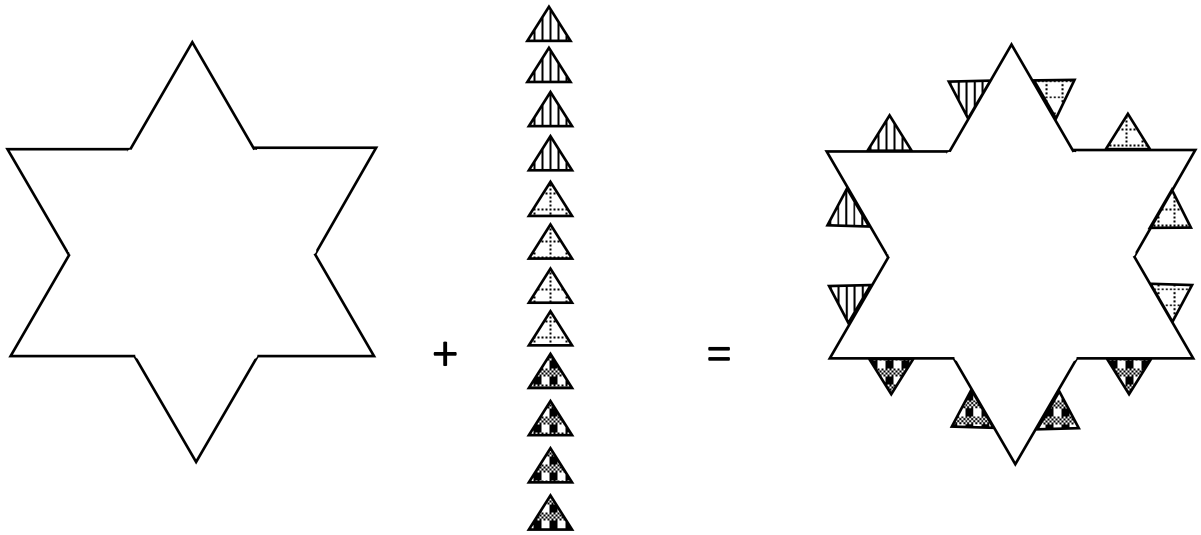


middle third segment of line

The diagrams below show how the first, second and third layers of triangles are added to the initial triangle.

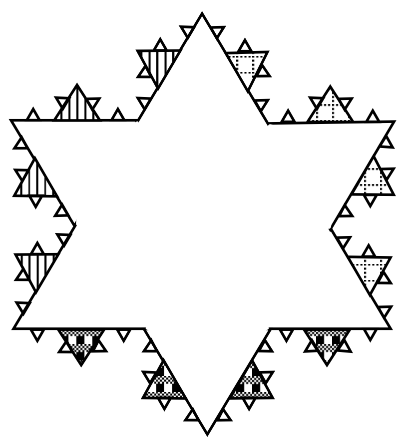


first layer of triangles



second layer of triangles

After a third layer of triangles is added, the snowflake is shown in the following diagram.



The Koch snowflake is formed when this process of adding layers of triangles repeats over and over again infinitely.

(i) Given that the initial triangle has side of length x cm, by first writing down the area of each triangle in the first and second layers, deduce the area of each triangle in the r^{th} layer. [3]

(ii) By first writing down the number of triangles in the first, second and third layers, deduce the number of triangles in the r^{th} layer. [2]

(iii) Show that the total area of triangles in the r^{th} layer is $\left(\frac{3\sqrt{3}}{16}\right)\left(\frac{4}{9}\right)^r x^2$ [2]

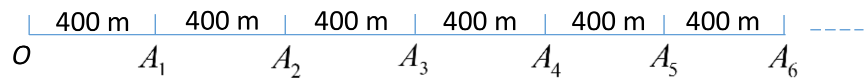
(iv) Hence find the total area of the Koch snowflake, explaining why it is a finite value. [4]

7. **(DHS18/Promo/Q10)**

To celebrate SG55 in the Year 2020, an athlete has two proposals for a training programme to complete a 55-km run around Singapore.

Proposal (1): To run 800 m on day 1, 960 m on day 2, 1152 m on day 3, and on each successive day, to run a distance that is a constant multiple of the distance run in the previous day.

Proposal (2): To run from a starting point O to and from a series of points A_1, A_2, A_3, …. , increasingly far away in a straight line. The distances between the adjacent points are all 400 m as shown in the figure below.



On day 1, the athlete is to run from O to A_1 and back to O.

On day 2, the athlete is to run from O to A_2 and back to O, in addition to what he has done on the previous day, i.e. O to A_1 to O, and then O to A_2 to O.

This pattern continues for the subsequent days.

(i) For Proposal (1), find the least value of n for which the distance run on day n exceeds 55 km. [4]

(ii) For Proposal (2), show that the distance run on day N is $400N^2 + 400N$ and hence find the least value of N for which the distance run on day N exceeds 55 km. Determine the distance from O and the direction of travel, of the athlete after he has run exactly 55 km. [5]

In order to support his training programme, the athlete needs to ensure a sufficient daily calorie intake based on the sum of two components:

- the athlete's Basal Metabolic Rate (BMR),
- the number of calories burnt in a day due to the athlete's running.

A person's BMR can be calculated using this equation:

BMR = $[A\times(\text{Weight in kg})]+[B\times(\text{Height in cm})]+[C\times(\text{Age in years})]+5$

where A, B and C are constants.

Based on the above equation, the table below shows the respective biodata and BMR of three persons.

	Weight (kg)	Height (cm)	Age (years)	BMR
Person 1	60	165	25	1511.25
Person 2	78	175	30	1728.75
Person 3	80	178	27	1782.5

Given that the athlete's biodata is 65 kg, 170 cm, 30 years old, and assuming that he burns 500 calories in a day due to his running, calculate the daily calorie intake he needs. [4]

8. **(NYJC17/Promo/Q12)**
Two athletes signed up for the 13 km Nanyang Marathon. In preparation for this marathon, the athletes planned a 16-day personalised training programme as follows:

Athlete *A* will run 0.4 km on the first day of the training programme and for each subsequent day, the distance he will run is increased by *x* km.

Athlete *B* will run 0.6 km on the first day of the training programme and for each subsequent day, the distance he will run is 10% more than the distance ran for the previous day.

(i) Find the minimum value of *x* such that Athlete *A* is able to run at least 13 km on the last day of the training programme. [2]

(ii) Athlete *B* targets to run a total distance of 26 km in these 16 days. Can he achieve this target? You must show sufficient workings to justify your answer. [2]

After 2 days of training, Athlete *B* modified his training programme such that on Day *n*, $3 \le n \le 16$, he will run $\left[\frac{10}{21}(1.05)^n + n\right]$ km. There are no further changes to his training programme.

(iii) Show that the total distance Athlete *B* will run by the end of Day *m*, where $3 \le m \le 16$, can be expressed as

$$P\left(Q^m\right)+\frac{(m-2)(m+3)}{2}+R,$$

where *P*, *Q* and *R* are exact constants to be determined. [5]

(iv) Athlete *B* feels that running more than 13 km per training session may be too strenuous. Assuming he follows the new training programme but limit his running distance to 13 km per training session, find the total distance he will run in the 16 days of training. [4]

9. **(RI17/Promo/Q3)**

(a) A piece of paper in the form of a semi-circle of radius *r* is cut into twelve sectors such that the areas are in arithmetic progression, and the area of the biggest sector is three times that of the smallest sector.

Find the exact area of the smallest sector in terms of *r*. [2]

(b) An arithmetic series *A* has first term *a* and common difference *d*, where *a* and *d* are non-zero. A convergent geometric series *G* has common ratio *r*.

The first three terms of *G* are equal to the first, eleventh and seventeenth terms of *A*, respectively.

(i) Find *r*. [4]

Using your answer in part **(i)**, find the exact ratio of the sum to infinity of *G* to the sum of the first four terms of *G*. [2]

10. **(RI18/Promo/Q5)**

A geometric progression $u_1,\ u_2,\ u_3,\ \ldots$ has first term 2 and common ratio r, where r is a positive constant. These terms are grouped into sets containing 1, 4, 7, … terms so that the number of terms in each set after the first is three more than the number of terms in the previous set.

$$\{u_1\},\ \{u_2,\ u_3,\ u_4,\ u_5\},\ \{u_6,\ u_7,\ u_8,\ u_9,\ u_{10},\ u_{11},\ u_{12}\},\ \ldots\ .$$

(i) Write down expressions in terms of n, for

(a) the number of terms in the nth set, [1]

(b) the total number of terms in the first n sets. [1]

(ii) Deduce, in terms of n and r, the first term in the $(n+1)$th set. [2]

(iii) Given that the nth set contains the term u_{2019}, using an algebraic approach, show that $3n^2 - n - 4036 > 0$ for $r > 1$. Hence find the value of n. [3]

11. **(VJC Promo 9758/2018/Q9)**

Alan manufactures and sells 3D printers while Bob is a smartphone App developer selling Apps. They started their businesses at the same time. Alan's monthly sales (in number of printers sold) could be modelled by an arithmetic progression with common difference d, where $d \neq 0$ while Bob's monthly sales (in number of downloads) could be modelled by a geometric progression with common ratio r. It is also known that Alan's monthly sales in the 1^{st}, 3^{rd} and 13^{th} month were equal to Bob's in the 1^{st}, 2^{nd}, and 3^{rd} month respectively.

(i) Show that $r^2 - 6r + 5 = 0$. [2]

(ii) Hence, find the exact value of r, justifying that this is the only answer. [2]

(iii) Explain clearly why every term of this geometric progression is also a term in the arithmetic progression. [2]

It is further given that Alan makes a profit of \$2000 per 3D printer sold while Bob makes a profit of \$0.01 each time the App is downloaded.

(iv) Bob's App was downloaded once in the 1^{st} month and his monthly profit first exceeds Alan's in the x^{th} month, find the value of x. [3]

12. **(YIJC Promo 9758/2020/Q13)**

A mask production factory produces 5000 reusable masks in the first week. In each subsequent week, the number of reusable masks produced is 200 more than the previous week.

(i) Find the number of reusable masks produced in the 10th week. [2]

(ii) Find the minimum number of weeks required by the factory to produce a total number of at least 100 000 reusable masks. [3]

Due to a certain flu pandemic, the demand for reusable masks has increased drastically. If the demand for reusable masks in the preceding week is *M*, it would become $100+aM$ this week.

(iii) Given that the demand for reusable masks in the first week is 500, show that the demand for reusable masks in the *N*th week can be written as $\dfrac{100}{a-1}\left(5a^{N}-4a^{N-1}-1\right)$. [3]

It is now given that $a = 1.15$.

To meet the growing demand of reusable masks, the factory adjusts its production capacity such that the number of reusable masks produced from the 10th week onwards is *k* more than the previous week.

(iv) Find the least value of *k* required by the factory in order to meet the total demand for the first 25 weeks. [4]

13. **(A Level N09/P1/Q3 modified)**

(i) Show that $\dfrac{1}{n-1}-\dfrac{2}{n}+\dfrac{1}{n+1}=\dfrac{A}{n^3-n}$ where *A* is a constant to be determined. [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{n}\frac{1}{r^3-r}$.

[There is no need to express your answer as a single algebraic fraction.]

(iii) Give a reason why the series $\displaystyle\sum_{r=2}^{\infty}\frac{1}{r^3-r}$ is convergent and write down its value.[2]

(iv) Find $\displaystyle\sum_{r=n+1}^{2n}\frac{1}{r^3-r}$.

[There is no need to express your answer as a single algebraic fraction.]

14. **(NYJC19/Promo/Q5)**

(i) It is given that $\dfrac{4r}{r^4+4}=\dfrac{1}{(r-1)^2+1}-\dfrac{1}{(r+1)^2+1}$. Show that

$$\sum_{r=1}^{n}\frac{r}{r^4+4}=\frac{3}{8}-\frac{1}{4n^2+4}-\frac{1}{4(n+1)^2+4}.$$ [3]

Using your answer from part **(i)**,

(ii) deduce an expression in terms of k for the infinite sum

$$\frac{2k+1}{(2k+1)^4+4}+\frac{2k+2}{(2k+2)^4+4}+\frac{2k+3}{(2k+3)^4+4}+\dots,$$

(There is no need to express your answer as a single algebraic fraction.) [3]

(iii) show that $\displaystyle\sum_{r=2}^{n+1}\frac{r-1}{(r-1)^4+4}<\frac{3}{8}$. [2]

15. **(SRJC14/Promo/Q9)**

Given that $\mathrm{f}(r)=\dfrac{4^{r+1}(r-1)}{3(r+2)}$,

(i) Show that $\mathrm{f}(r)-\mathrm{f}(r-1)=\dfrac{4^r r^2}{(r+1)(r+2)}$.

[2]

(ii) Hence, find $\displaystyle\sum_{r=1}^{n}\frac{4^r r^2}{(r+1)(r+2)}$ in terms of n. [3]

(iii) Using the result in **(ii)**, find $\displaystyle\sum_{r=3}^{n}\frac{4^{r-2}(r-1)^2}{r(r+1)}$. [3]

16. **(ACJC10/Promo/Q8)**

Given that $u_r=\dfrac{1}{r!}$. Show that $u_r-u_{r+1}=\dfrac{1}{r!+(r-1)!}$. [1]

Hence evaluate the series $\dfrac{1}{2!+3!}+\dfrac{1}{3!+4!}+\dfrac{1}{4!+5!}+\dots+\dfrac{1}{(N-1)!+N!}$ in terms of N. [3]

Deduce that $\displaystyle\sum_{r=3}^{N}\frac{1}{r!+(r+2)!}<\frac{1}{6}$. [2]

17. **(HCI12/Promo/Q4)**

By considering an arithmetic series, show that

$$\frac{1}{1+2+3+...+n}=2\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

[3]

Hence find $\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+...+\frac{1}{1+2+3+...+n}$. [3]

18. **(TJC14/Promo/Q5)**

Express $\frac{2r+3}{r(r+1)}$ in partial fractions. [2]

Denoting $S_n=\sum_{r=1}^{n}\left[\frac{2r+3}{r(r+1)}\left(\frac{1}{3^r}\right)\right]$, find S_n in terms of n. [3]

Hence determine whether the series converges. [1]

19. **(AJC14/Promo/Q3)**

Show that $\frac{1}{r-1}-\frac{1}{r+1}=\frac{2}{(r-1)(r+1)}$. Hence find $\sum_{r=2}^{n}\frac{1}{(r-1)(r+1)}$. [4]

Deduce the exact value for $\sum_{r=2}^{\infty}\left(\mathrm{e}^{-2r}+\frac{1}{r(r+2)}\right)$. [4]

20. **(RVHS14/Promo/Q4)**

Prove that $\ln\left(r+\frac{r}{r+1}\right)=\ln r-\ln(r+1)+\ln(r+2)$. [2]

Hence, find in terms of n, $\ln\left(2+\frac{2}{3}\right)+\ln\left(3+\frac{3}{4}\right)+\ln\left(4+\frac{4}{5}\right)+...+\ln\left(n-1+\frac{n-1}{n}\right)$,

giving your answer in the form $\ln\left(\frac{(n+1)!}{k}\right)$, where k is a constant to be found. [4]

21. **(ACJC Promo 9758/2020/Q10)**

By considering partial fractions or otherwise, show that

$$\sum_{r=1}^{n} \frac{2r+3}{r(r+1)(r+2)} = A - \frac{3}{2(n+1)} - \frac{1}{2(n+2)},$$

where A is a constant to be determined. [4]

(i) Explain why $\sum_{r=1}^{\infty} \frac{2r+3}{r(r+1)(r+2)}$ converges, and state the convergence limit. [2]

(ii) Find the least value of n such that $\sum_{r=1}^{n} \frac{2r+3}{r(r+1)(r+2)} > \frac{8}{5}$. [2]

(iii) Evaluate

$$\frac{9}{3\times4\times5} + \frac{11}{4\times5\times6} + \frac{13}{5\times6\times7} + \ldots\ldots + \frac{2N+1}{N\left(N^2-1\right)},$$

leaving your answer in terms of N. [3]

Answers

1. **(a)** **(i)** $p = \frac{17}{10}, q = \frac{1}{10}$ **(ii)** $\frac{1}{5}n + \frac{8}{5}$; $\frac{1}{5}$ **(b)** $\left\{\theta \in \mathbb{R} \,\middle|\, \frac{\pi}{6} < \theta < \frac{\pi}{2}\right\}$

2. **(a)** **(ii)** $S_\infty = \frac{c}{k-1}$ **(b)(i)** 5180 **(b)(ii)** 711 **(b)(iii)** 6

3. **(i)** $A = -42000$ **(ii)** $r = 8.39$

4. **(i)** $T_1 = 3^n$, $T_2 = 3^n + 3^{n-1} = 3^n\left(1+\frac{1}{3}\right) = \frac{4}{3}\left(3^n\right)$,

$$T_3 = 3^n + 3^{n-1} + 3^{n-1} + 3^{n-2} = 3^n\left(1+\frac{1}{3}+\frac{1}{3}+\frac{1}{9}\right) = \frac{16}{9}\left(3^n\right)$$

T_1, T_2, T_3 form a geometric progression with common ration 4/3.

[Alternatively, show that $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{4}{3}$]

(ii) $k = 3 + 3^{n-3}$ **(iii)** $\left[29 + 8(3^{n-3})\right]\left(3^{n-2}\right)$ **(iv)** greatest n = 6

5. **(i)** 14.2° (1 d.p) **(ii)** 314.3° **(iii)** 66° (1 d.p) **(iv)** greatest $n = 15$

6. **(i)** $\frac{\sqrt{3}}{4}\left(\frac{x}{3^r}\right)^2$, **(ii)** $3\times4^{r-1}$, **(iii)** $\frac{3\sqrt{3}}{16}\left(\frac{4}{9}\right)^r x^2$, **(iv)** $\frac{2\sqrt{3}}{5}x^2$

7. **(i)** 25 **(ii)** 12; 2200m from O and running away from O, 2067.5 calories

8. **(i)** 0.84, Athlete B is unable to achieve his target, $P = 10$, $Q = 1.05$, $R = -9.765$

(iv) 135 km

9. **(a)** $\frac{\pi}{48}r^2$ **(b)(i)** $r = \frac{3}{5}$ **(b)(ii)** $625:544$

10. **(i)(a)** $3n-2$ **(i)(b)** $\frac{n}{2}[3n-1]$ **(ii)** $2r^{\frac{n}{2}[3n-1]}$ (iii) 37

11. **(ii)** $r=5$ **(iv)** $x = 11$ **12.** (i) 6800, (ii) 16 weeks, (iv) $k = 543$

13. **(ii)** $\frac{1}{2}\left[\frac{1}{2}-\frac{1}{n}+\frac{1}{n+1}\right]$ **(iii)** $\frac{1}{4}$ **(iv)** $\frac{1}{4n}+\frac{1}{4n+2}-\frac{1}{2n+2}$

14. **(ii)** $\frac{1}{16k^2+4}+\frac{1}{4(2k+1)^2+4}$

15. **(ii)** $\frac{4^{n+1}(n-1)}{3(n+2)}+\frac{2}{3}$ **(iii)** $\frac{4^{n-1}(n-2)}{3(n+1)}$

16. $\frac{1}{3!}-\frac{1}{(N+1)!}$ **17.** $2\left(\frac{1}{3}-\frac{1}{n+1}\right)$ **18.** $\frac{3}{r}-\frac{1}{r+1}$; $1-\frac{1}{3^n(n+1)}$

19. $\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}$; $\frac{1}{e^4-e^2}+\frac{5}{12}$

20. $\ln\left(\frac{(n+1)!}{3n}\right)$

21. $A=\frac{7}{4}$ (i) $\frac{7}{4}$ (ii) $n = 13$ (iii) $\frac{5}{8}-\frac{3}{2N}-\frac{1}{2(N+1)}$

2

Graphs and Transformations

1. **(A Level N01/FMP1/Q1)**

The curve C has equation $y=\dfrac{x^2-2x-3}{x+2}$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Draw a sketch of C, which should include the asymptotes, and state the coordinates of the points of intersection of C with the x-axis. [3]

(iii) On the same diagram draw a sketch of the curve $y=\dfrac{4}{(x+4)^2}$, showing clearly the asymptotes. [2]

(iv) Hence show that the equation $x^4+6x^3-3x^2-60x-56=0$ has exactly 2 real roots. [2]

2. **(NYJC14/Promo/Q10)**

The curve C has equation $y=\dfrac{x^2-x+7}{x-2}$.

(i) Find the equations of the asymptotes of C. [2]

(ii) Using an algebraic method, prove that C cannot exist for $a<y<b$, where a and b are real constants to be determined.[3]

(iii) Sketch C, labelling clearly the asymptotes, turning point(s) and intersection(s) with the axes, where applicable. [3]

(iv) State the range of values of x for which C is increasing and concave downwards. [1]

(v) Using part **(iii)** and by considering another appropriate graph, determine the number of real solutions to the equation $\left|x^2-x+7\right|=\left|\dfrac{2-x}{x^2}\right|$. [2]

3. **(A Level N08/P1/Q9modified)**

It is given that $\mathrm{f}(x)=\dfrac{ax+b}{cx+d}$, for non-zero constants a, b, c and d.

(i) Given that $ad-bc\neq 0$, show by differentiation that the graph of $y=\mathrm{f}(x)$ has no turning points. [3]

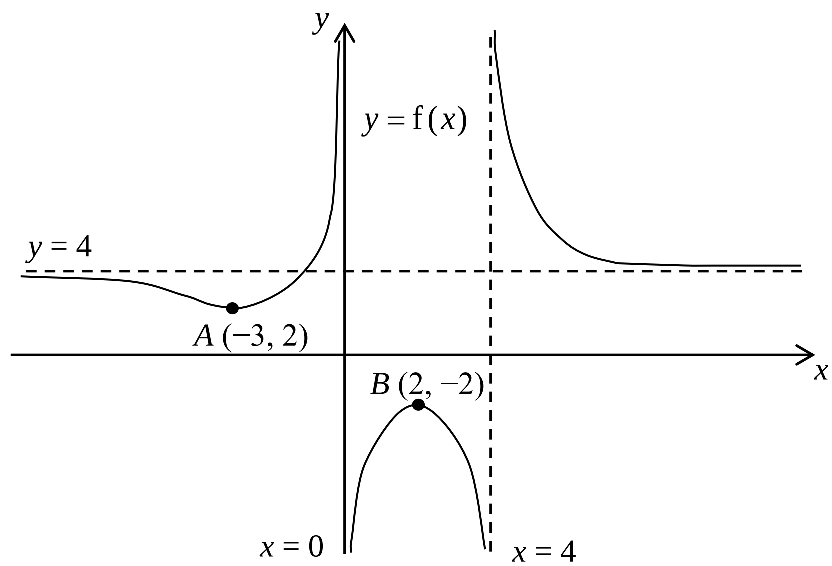
(ii) What can be said about the graph of $y=\mathrm{f}(x)$ when $ad-bc=0$? [2]

(iii) Deduce from part **(i)** that the graph of $y=\dfrac{3x-7}{2x+1}$ has a positive gradient at all points of the graph. [1]

(iv) Sketch the graph of $y=\dfrac{3x-7}{2x+1}$ including the coordinates of the points where the graphs cross the axes and the equations of any asymptotes. [2]

4. **(TPJC13/Promo/Q9)**

The graph of $y=\mathrm{f}(x)$ shown below has asymptotes $x=0$, $x=4$ and $y=4$. The curve has a minimum point at A (−3, 2) and a maximum point at B (2, −2).



On separate diagrams, sketch the graphs of

(i) $y=\mathrm{f}(|x|)$, [2]

(ii) $y=\mathrm{f}(2x-1)$, [2]

(iii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

(iv) $y=\mathrm{f}'(x)$. [3]

State clearly the equations of the asymptotes, the coordinates of any intersections with the axes and the coordinates of the points corresponding to A and B.

5. **(JJC10/Promo/Q8)**

The curve C has an equation $y=\dfrac{ax+3}{x+b}$ where a, b are positive integers, and $ab<3$.

(i) State the equations of the asymptotes of C, in terms of a and b. [2]

(ii) Sketch the graph of C, labeling all asymptotes and axial intercepts. [3]

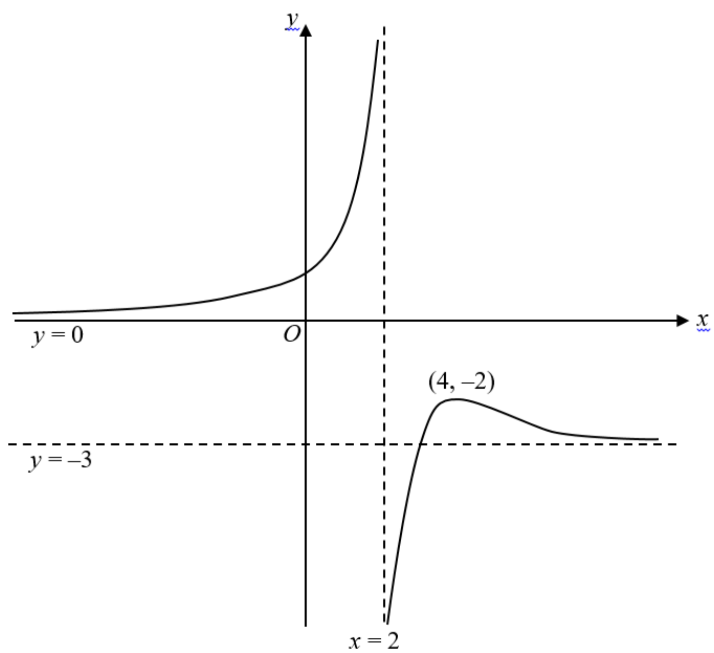
Given that the curve C intersects the straight line, $y=x+3$, at $(-2,1)$ and $(0,3)$.

(iii) Show that $a=2$ and $b=1$. [3]

(iv) State a sequence of geometrical transformations that will transform the graph of $y=\dfrac{1}{x}$ to the graph of C. [2]

6. **(NYJC19/Promo/Q7)**

(a) The diagram below shows the graph with equation $y=\mathrm{f}(x)$. It has a maximum point at $(4,-2)$ and asymptotes $x=2$, $y=-3$ and $y=0$.



Sketch on separate diagrams, the graphs of

(i) $y=3\mathrm{f}(2x+1)$, [3]

(ii) $y=\dfrac{1}{\mathrm{f}(x)}$. [3]

(b) A curve $y = \mathrm{g}(x)$ undergoes, in succession, the following transformations to get $y = \frac{1}{x}$:

A: A scaling with scale factor $\frac{1}{3}$ parallel to the *y*-axis

B: A reflection in the *y*-axis

C: A translation of 1 unit in the direction of the *x*-axis.

Find $\mathrm{g}(x)$, showing your workings clearly. [3]

7. (ACJC18/Promo/Q6)

The curve C_1 is defined by the equation

$$y = \frac{1}{x(x-2a)},\ x \neq 0,\ x \neq 2a\ ,$$

where *a* is a positive real number.

(i) By first sketching the graph of $y = x(x-2a)$, sketch on a separate diagram the graph of C_1, indicating clearly the equations of all asymptotes, and the coordinates of the turning points and the points where the curve crosses the axes, if any. [3]

Another curve C_2 is defined by the equation $b(x-a)^2 + a^2y^2 = a^2b$, where *b* is a positive real number.

(ii) Sketch C_2 on the same diagram as C_1, indicating the coordinates of the point on C_2 with the smallest *y*-value. [2]

(iii) Deduce the range of values of *b*, in terms of *a*, such that C_1 and C_2 intersect twice. [1]

8. **(DHS19/Promo/Q10)**

The curve C_1 is defined by $y = 2x + 1 + \dfrac{b}{x+a}, x \neq -a$, where a and b are constants. Given that
C_1 has two stationary points, what can be said about the values of a and b? [3]

It is given that $a = b = 1$.
Sketch C_1, indicating in your graph any points where C_1 crosses the axes and the equations of any asymptotes. [2]

(a) The curve C_2 is defined by $\left(x+p\right)^2 + y^2 = R^2$, where p is a real constant and $R > 0$. Sketch the graph of C_2 on the same diagram as C_1. [1]

It is given that C_1 and C_2 intersect. By considering the stationary points of C_1, find the minimum value of R as p varies and state the corresponding value of p. Express both your answers in exact form. [3]

(b) Sketch the graph of $y = \mathrm{f}\,'(x)$, where $\mathrm{f}(x) = \left|2x + 1 + \dfrac{1}{x+1}\right|$. [2]

9. **(DHS20/Promo/Q6)**

A hyperbola C, with asymptotes of equation $y = 2x + 10$ and $y = -2x - 6$, has turning points $(-4,\ 5)$ and $(-4,\ -1)$. Find the centre of C and determine the equation of C. [3]

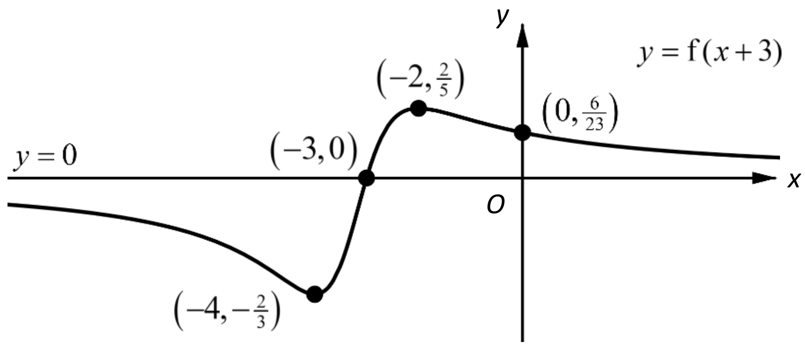
Consider the graph of $y = \mathrm{f}(x)$ where $\mathrm{f}(x)$ is the function describing the hyperbola above its centre.

(i) By considering C, or otherwise, sketch the curve $y = \mathrm{f}(|x|)$, indicating the equations of any asymptote(s) and the coordinates of any point(s) where the curve crosses the axes. [4]

(ii) The line $y = k$ cuts the curve $y = \mathrm{f}(|x|)$ at two points $(\alpha,\ k)$ and $(\beta,\ k)$, where $\alpha < \beta$. Explain why α satisfies the equation $4\alpha^2 - 32\alpha + 73 - (k-2)^2 = 0$. [2]

10. **(ACJC16/Promo/Q4)**

The diagram below shows the graph of $y=\mathrm{f}(x+3)$. The graph has axial intercepts at $(-3,0)$ and $\left(0,\dfrac{6}{23}\right)$, turning points at $\left(-4,-\dfrac{2}{3}\right)$ and $\left(-2,\dfrac{2}{5}\right)$, and a horizontal asymptote $y=0$.



(i) Given that the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ has an oblique asymptote $y=x+\dfrac{1}{2}$, sketch the graph of $y=\dfrac{1}{\mathrm{f}(x)}$, showing clearly the coordinates of any axial intercepts and turning points, and equations of asymptotes. [3]

(ii) Sketch the graph of $y=-\dfrac{\mathrm{f}'(x)}{\left[\mathrm{f}(x)\right]^2}$, indicating the coordinates of any intersections with the x-axis and the equations of any asymptotes. [2]

11. **(MJC10/Promo/Q4)**

The curve C_1 has equation $k^2x^2+y^2=k^2$, where $k>1$.

(i) Sketch C_1, indicating the axial intercepts, asymptotes and stationary points, if any.

[2]

(ii) Hence, or otherwise, sketch $k^2x^2+\dfrac{1}{4}y^2=k^2$, where $k>1$, indicating the axial intercepts, asymptotes and stationary points, if any. [1]

C_1 undergoes a single transformation to become C_2.

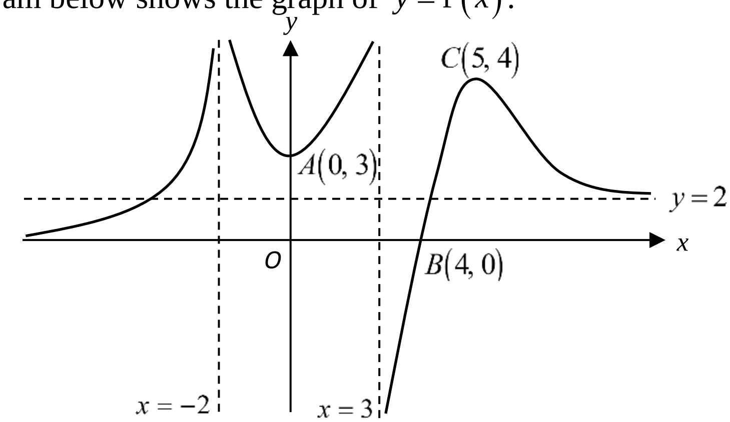
(iii) Given that C_2 has a line of symmetry $y = 2$ and passes through the origin, state the value of k. [1]

(iv) Describe a geometrical transformation by which C_2 may be obtained from C_1. [1]

(v) Write down the equation of C_2. [1]

12. **(CJC14/Promo/Q10modified)**

(a) The diagram below shows the graph of $y = \mathrm{f}(x)$.



The points A, B and C have coordinates $(0, 3)$, $(4, 0)$ and $(5, 4)$ respectively.
The equations of the asymptotes are $x = -2$, $x = 3$, $y = 0$ and $y = 2$. Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(-x-1)$ [2]

(ii) $y = \mathrm{f}'(x)$ [3]

stating clearly in each case the axial intercepts, the asymptotes and the points corresponding to A, B and C where appropriate.

(b) A curve C undergoes in succession, the following transformations:

I: A translation of 2 units in the negative y–direction

II: A translation of 3 units in the positive x–direction

III: Scaling parallel to x–axis by a scale factor of $\frac{1}{2}$

The equation of the resulting curve is $x^2 + y^2 = 4$. Determine the equation of the curve. [3]

13. **(DHS18/Promo/Q4)**
The curve C has equation

$$y=\frac{x^2+2x+q}{x-p},\ \ x\neq p,$$

where p and q are constants. It is given that the line $y=x+3$ is an asymptote of C. Show that $p=1$. [2]

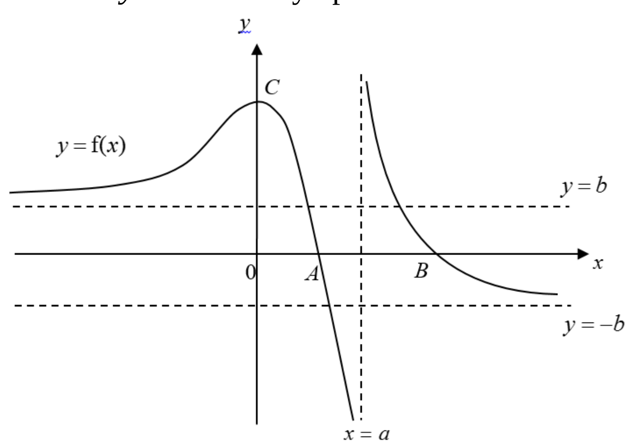
(i) If C has 2 stationary points, show that $q>-3$. [3]

(ii) Sketch C for $q=1$. By adding another graph, deduce that for all positive β, the equation

$$(\beta+1)x^2+2x+1=\beta x^3$$

has exactly one real root. [5]

14. **(NJC10/Promo/Q4)**
The diagram shows a sketch of the graph of $y=\mathrm{f}(x)$. The curve cuts the x-axis at the point A (2, 0) and point B (5, 0), and has a turning point at C (0, $b+2$).
The lines $x=a$, $y=b$ and $y=-b$ are asymptotes to the curve.



On separate diagrams, sketch the graphs of

(a) $y=\mathrm{f}'(x)$, where $\mathrm{f}'(x)$ is the derivative function of $\mathrm{f}(x)$, [2]

(b) $y=\mathrm{f}\left(-|x|\right)-b$ [3]

In each case, show clearly the axial intercepts, the asymptotes and the coordinates of turning points where applicable.
State the range of values of x such that $y=\mathrm{f}(x)$ is increasing. [1]

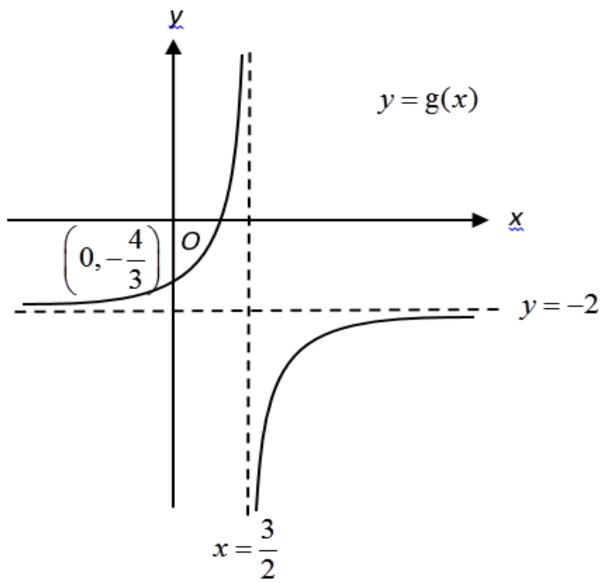
15. **(TJC19/Promo/Q1)**

(i) Sketch the curve with equation $y = \left|\frac{\alpha x}{x+1}\right|$, where α is a positive constant, stating the equations of the asymptotes. On the same diagram, sketch the line with equation $y = \alpha x - 2$. [3]

(ii) Solve the inequality $\left|\frac{\alpha x}{x+1}\right| \geq \alpha x - 2$, giving your answers in term of α. [3]

16. **(VJC19/Promo/Q7)**

The diagram below shows the graph of $y = \mathrm{g}(x)$, where $\mathrm{g}(x) = \frac{ax+b}{2x+c}$.



Determine the values of a, b and c. [3]

It is also given that $\mathrm{g}(x) = \mathrm{f}\left(\frac{1}{2}x - 1\right)$. State a sequence of 2 transformations that will map the graph of $y = \mathrm{g}(x)$ to the graph of $y = \mathrm{f}(x)$.

Find $\mathrm{f}(x)$. [5]

17. (VJC16/Promo/Q7)

(i) A circle with equation $x^2+(y-k)^2=25$, where k is a constant, cuts the parabola with equation $4y=x^2$ at four real and distinct points. Find the set of values of k. [3]

(ii) A curve with equation $4x^2+y^2-2y-24=0$ is transformed by a translation of 6 units in the positive y-direction, followed by a stretch parallel to the x-axis with scale factor 2. Find the equation of the new curve. [3]

State, with a reason, the number of times the parabola $4y=x^2$ will cut this new curve. [1]

18. (NYJC10/Promo/Q10)

The curve C has equation $y=\dfrac{x^2+a}{x-a}$, $x \neq a$ and a is a non-zero constant.

(i) Given that the oblique asymptote of C is $y = x + 2$, find the value of a. [3]

(ii) Find the range of values of a if C has two turning points. [3]

(iii) Using the value of a found in part **(i)** and using an algebraic method, show that C cannot exist for $y_1 < y < y_2$ where values of y_1 and y_2 are to be determined in exact form. [3]

(iv) Hence sketch the graph of C, showing clearly the equations of the asymptotes and coordinates of the stationary points. [3]

19. (RI14/Promo/Q3)

The curve C has equation

$$\frac{(x-1)^2}{9}-y^2=1.$$

(i) Sketch the curve C, showing clearly the equations of asymptotes, axial intercepts and coordinates of turning points, if any. [3]

(ii) Given that k is a positive constant and C intersects the curve with equation

$$y^2+\frac{x^2}{k^2}=1$$

at exactly two distinct points, state the range of values of k. [2]

20. (NYJC18/Promo/Q4)

Sketch the graph of $y=\dfrac{4\lambda-x^2}{x^2+\lambda}$ in each of the following two cases:

(i) $\lambda>0$, [2]

(ii) $\lambda<0$. [3]

Your sketches should include clear labelling of asymptotes, the exact coordinates of any points where the curve crosses the *x*- and *y*- axes and stationary point(s).

By using your graph in **(ii)** and considering a suitable graph whose cartesian equation is to be stated, find the positive value of *h* such that the equation

$$\left(\frac{4\lambda-x^2}{hx^2+h\lambda}\right)^2=1+\frac{x^2}{\lambda}$$

where $\lambda<0$, has only one real root.

State the value of the real root for this value of *h*. [3]

Answers

1. **(i)** $x=-2,\ y=x-4$ **(ii)** $(-1, 0), (3, 0)$

2. **(i)** $x=2,\ y=x+1$ **(ii)** $a=-3$ and $b=9$ **(iv)** $x<-1$
(v) 2 solutions

3. **(ii)** $y=\mathrm{f}(x)$ is a horizontal line $y=\dfrac{a}{c}$

5. **(i)** $y=a,\ \ x=-b$

6. **(b)** $\mathrm{g}(x)=\dfrac{3}{1-x}$

7. **(iii)** $b>\dfrac{1}{a^4}$

8. $a\in\mathbb{R}, b>0$ **(a)** $\min R=2\sqrt{2}-1,\ p=1-\dfrac{1}{\sqrt{2}}$

9. $(-4,\ 2)$; $(y-2)^2-4(x+4)^2=9$

11. **(iii)** $k=2$ **(v)** $4x^2+(y-2)^2=4$

12. **(b)** $\left(\dfrac{x+3}{2}\right)^2+(y-2)^2=4$

14. $x<0$

15. **(ii)** $x\le\dfrac{1+\sqrt{2\alpha+1}}{\alpha},\ x\ne-1$

16. a = -4, b = 4, c = -3; $\dfrac{-8x-4}{4x+1}$

17. **(i)** $\left\{k\in\mathbb{R}: 5<k<\dfrac{29}{4}\right\}$ **(ii)** $(x)^2+(y-7)^2=5^2$; 4 times

18. **(i)** $a=2$ **(ii)** $a<-1$ or $a>0$

(iii) $y_1=4-2\sqrt{6},\quad y_2=4+2\sqrt{6}$

19. **(ii)** $2<k<4$

20. $h=4, x=0$

3

Inequalities and Systems of Equations

1. (ACJC14/C1MidYear/Q1)

Without using a calculator, solve the inequality $\dfrac{(x+1)(x^2-3x+3)}{(3-x)} \geq 0$. [3]

Hence find the range of values of x for which $\dfrac{(x^2+1)(x^4-3x^2+3)}{(3-x^2)} \geq 0$. [2]

2. (TJC14/C1MidYear/Q5)

Without using a graphic calculator, solve the inequality $\dfrac{x-7}{x^2-7} \leq 1$. [4]

Hence solve the inequality $\dfrac{|x|+7}{x^2-7} \geq -1$. [3]

3. (CJC13/C1BT/Q6)

(a) **(i)** Show algebraically that $x^2+x+4>0$ for all real values of x. [2]

(ii) Hence, without using a calculator, solve the inequality $\dfrac{1-x}{2+x} \leq \dfrac{2}{x}$. [3]

(b) **(i)** Sketch in a single diagram, the graphs of $y=\left|\dfrac{x+1}{x-1}\right|$ and $y=-(x+3)^3$. [2]

(ii) Hence solve the inequality $\left|\dfrac{x+1}{x-1}\right| > -(x+3)^3$. [2]

(iii) Using the answer obtained in (ii), deduce the range of values of x that satisfies $\left|\dfrac{e^x+1}{e^x-1}\right| > -(e^x+3)^3$. [3]

4. **(IJC12/PrelimP1/Q5)**

Sketch, on the same diagram, the graphs of $y=\ln(2x+9)$ and $y=\sqrt{10-x^2}$, including the coordinates of the points where the graphs cross the x-axis and the equations of any asymptotes. [3]

Hence solve the inequality $\ln(2x+9)\geq\sqrt{10-x^2}$. [5]

Deduce the solution to the inequality $\ln(2|x|+9)\geq\sqrt{10-x^2}$. [2]

5. **(ACJC14/Promo/Q2)**

(i) Without using a calculator, solve the inequality $\dfrac{4}{x+3}\leq x$. [3]

(ii) Hence solve the inequality $\dfrac{4x}{1+3x}\leq\dfrac{1}{x}$. [3]

6. **(NYJC10/Promo/Q4)**

Using an algebraic method, solve $\dfrac{1}{x}\geq\dfrac{1-x}{x^2+4x}$. [3]

Hence, deduce the range of values of x which satisfy

(i) $\dfrac{1}{|x|}\geq\dfrac{1-|x|}{x^2+4|x|}$; [2]

(ii) $\dfrac{1}{x-2}\leq\dfrac{3-x}{x^2-4}$. [3]

7. **(HCI13/PrelimP1/Q1)**

Given that a is a positive real number, solve the inequality $\dfrac{x^2-x+2}{x^2+(1-a)x-a}\leq 0$, leaving your answer in terms of a. [3]

8. **(AJC12/PrelimP1/Q3)**

Without using a calculator, solve the inequality $\dfrac{6x-4}{x-3}\leq 1-2x,\ x\neq 3$. [3]

Hence find the exact range of values of θ for which $\dfrac{6-4\operatorname{cosec}\theta}{1-3\operatorname{cosec}\theta}\leq 1-2\sin\theta$, where $0<\theta<2\pi$. [4]

9. (RI2016/Promo/3)

Do not use a calculator in answering this question.

Solve the inequality $\dfrac{x+4}{-x^2+2x+3}<1$. [4]

Deduce the solution of

(i) $\dfrac{x^2+4}{-x^4+2x^2+3}<1$, [2]

(ii) $\dfrac{x-4}{x^2+2x-3}<1$. [2]

10. (A-Level 2017/Paper I/Q2)

(i) On the same axes, sketch the graphs of $y=\dfrac{1}{x-a}$ and $y=b|x-a|$, where a and b are positive constants. [2]

(ii) Hence, or otherwise, solve the inequality $\dfrac{1}{x-a}<b|x-a|$. [4]

11. (AJC13/Promo/Q6)

Without using a calculator, solve $\dfrac{x(4x-1)}{2x-1} < 3x+1$. [2]

Hence, find the solutions of the inequalities

(a) $x-5 < 3x+1 < \dfrac{x(4x-1)}{2x-1}$,

(b) $\dfrac{\cos x\,(4\cos x+1)}{2\cos x+1} > 3\cos x-1$ for $0 \le x \le \pi$,

leaving your answers in exact form. [6]

12. (JPJC19/Promo/Q3)

A curve C has equation $y=\dfrac{2}{x}-\dfrac{3}{x^2}$. Using an algebraic method, find the set of values of x for which C is

(a) decreasing, [3]

(b) concave upwards. [3]

13. **(NYJC16/Promo/Q1)**

Let $\mathrm{f}(x)$ be a cubic polynomial. It is given that the graph of $y=\mathrm{f}(x)$ passes through the origin and has a stationary point at the point $(-6,-3)$. The tangent at the point where $x=-4$ is parallel to the line $6y=x+1$. Find $\mathrm{f}(x)$. [4]

14. **(RI(JC)13/Promo/Q2)**

A shop selling biscuits gave the following offers. A customer who bought a total of 5 or more packets of biscuits was given a discount, regardless of the types of biscuits bought. A 10 % discount was given if 5 to 8 packets were bought and a 20 % discount if more than 8 packets were bought.

The following table shows the number of packets bought and the total amounts that three customers paid.

Type of Biscuits	Mrs Chan	Mr Lim	Mr Tan
Cream	3	5	2
Marie	1	2	1
Almond	3	5	1
Total amount paid in $	10.35	15.72	6.40

Calculate the price per packet for each type of biscuit before the discount. [4]

15. **(MJC13/C1MidYear/Q3)**

(i) In the first round of a game show, Mary has to decipher a 3-digit code. These three digits are the first three terms of a sequence u_n respectively.
Given that u_n is a quadratic polynomial in *n*, $n \in \mathbb{Z}^+$ and $u_4 = 14$, $u_6 = 40$ and $u_{10} = 140$, find u_n in terms of *n* and deduce the 3-digit code. [4]

(ii) In the next round, Mary is given another new sequence v_n and v_n is a quadratic polynomial in *n*. It is given that $v_1 = 1$, $v_2 = 2$ and the constant term in the quadratic polynomial is negative.
Mary claimed that "Every term in the sequence v_n is positive".
Determine the sign of the coefficient of n^2 in the quadratic polynomial v_n and hence, show that Mary is wrong. [3]

16. (ACJC13/C1MidYear/Q3)

An investment firm bought a total of 20,000 shares belonging to three local companies on the Stock Exchange Market in January 2013 and then sold them all in February of the same year. The brokerage charges transaction fees of 3 cents per share per transaction (buying and selling of shares are separate transactions).

The following table shows the prices at which this firm bought and sold the stocks.

	Buying Price per share $	Selling Price per share $	Net Earnings per share $
Bao Steel	8.10	6.90	–1.26
Ocean Shipping	2.21	2.40	
Ping An Insurance	4.04	4.50	

The portfolio indicates that the firm held twice as many Ocean Shipping shares as Bao Steel. In the sale of all 20,000 stocks, this firm made an overall net profit of $1400.

(i) Find the net earnings per share in dollars (after subtracting transaction fees) for Ocean Shipping and Ping An Insurance, and write clearly a system of linear equations with three unknowns using the information provided. [3]

(ii) Solve the system of linear equations, and determine the number of shares of each company held by the firm during the period from January to February 2013. [1]

17. (NYJC12/PrelimP1/Q1)

It is given that $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants. The curve C with equation $y = \mathrm{f}(x)$ passes through $(0, -1)$ and has a maximum point at $(-1, 1)$. The area bounded by C, the x-axis and the lines $x = 2$ and $x = 3$ is $\frac{31}{4}$ units^2. Given that $\mathrm{f}(x) > 0$ for $2 < x < 3$, find the values of a, b, c and d. [4]

18. (NJC Promo 9758/2020/Q1)

A curve has equation $y = ax^3 + bx^2 + cx$ where a, b and c are constants. The curve passes through the point with coordinates $(1, 12)$ and the gradient of the curve is –112 at the point where $x = 3$. When the curve is reflected in the y-axis, the resulting curve passes through the point $(-2, 3)$. Find the values of a, b and c. [4]

Answers

1. $-1 \le x < 3$; $-\sqrt{3} < x < \sqrt{3}$

2. $x < -\sqrt{7}$ or $x > \sqrt{7}$ or $0 \le x \le 1$; $x > \sqrt{7}$ or $x < -\sqrt{7}$ or $x = 0$

3. **(a)(ii)** $x < -2$ or $x > 0$

(b)(ii) $x > -3.84, x \ne 1$ **(iii)** $x \in \mathbb{R}, x \ne 0.$

4. $-3.16 \le x \le -2.95$ or $1.88 \le x \le 3.16$; $-3.16 \le x \le -1.88$ or $1.88 \le x \le 3.16$

5. **(i)** $-4 \le x < -3$ or $x \ge 1$ **(ii)** $-\frac{1}{3} < x \le -\frac{1}{4}$ or $0 < x \le 1$

6. $-4 < x \le -\frac{3}{2}$ or $x > 0$

(i) $x < 0$ or $x > 0$ **(ii)** $x < -2$ or $\frac{1}{2} \le x < 2$

7. $-1 < x < a$

8. $x \le -\frac{1}{2}$ or $1 \le x < 3$; $\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$ or $\theta = \frac{\pi}{2}$

9. $x < -1$ or $x > 3$; (i) $x < -\sqrt{3}$ or $x > \sqrt{3}$, (ii) $x > 1$ or $x < -3$

10. **(ii)** $x < a$ or $x > a + \frac{1}{\sqrt{b}}$

11. $-\frac{1}{\sqrt{2}} < x < \frac{1}{2}$ or $x > \frac{1}{\sqrt{2}}$ **(a)** $-3 < x < -\frac{1}{\sqrt{2}}$ or $\frac{1}{2} < x < \frac{1}{\sqrt{2}}$

(b) $\frac{\pi}{4} < x < \frac{2\pi}{3}$ or $\frac{3\pi}{4} < x \le \pi$

12. **(a)** $x < 0$ or $x > 3$ **(b)** $x > \frac{9}{2}$

13. $\mathrm{f}(x) = \frac{1}{72}x^3 + \frac{1}{4}x^2 + \frac{3}{2}x$

14. $a = 1.60, b = 1.45, c = 1.75$

15. $u_n = 2n^2 - 7n + 10$; 547

16. **(i)** \$0.13; \$0.40 **(ii)** 3000, 6000, 11000

17. $a = 1, b = 0, c = -3, d = -1$ **18.** $a = -6.5, b = 9, c = 9.5$

4

Functions

1. **(AJC14/Promo/Q9)**

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{3-x^2}{x^2+1}, \quad x \in \mathbb{R}, x > a.$$

Given that f^{-1} exists, state the least value of a. [1]

Use this value of a for the rest of this question.

(i) Find f^{-1}, expressing your answer in similar form. [2]

(ii) Sketch the graphs of f and f^{-1} on the same diagram, giving the equations of any asymptotes and the exact coordinates of any points of intersection with the axes. [3]

Given that the graphs of f and f^{-1} intersect at only one point, find the coordinates of this point of intersection. [1]

(iii) The region R is bounded by the graph of y = f(x), y = x and the x-axis. Find the exact volume of the solid generated when R is rotated through 2π radians about the y-axis. [3]

2. **(CJC14/Promo/Q9)**

The functions f and g are defined as follows.

$$\mathrm{f} : x \mapsto \frac{x^2-1}{x^2+1}, \quad \text{for } x \in \mathbb{R}.$$

$$\mathrm{g} : x \mapsto \ln(x+2), \text{ for } x \in \mathbb{R}, x > -2.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$, and write down the range of f.

Explain why the composite function gf exists. [3]

(ii) Solve the inequality $\mathrm{gf}(x) > 1$. [3]

(iii) Give a reason why function f^{-1} does not exist. [2]

If the domain of f is restricted to the interval $(-\infty, k]$, where k is a constant, state the greatest value of k for which the function f^{-1} exists.

Thus, find $\mathrm{f}^{-1}(x)$, and write down the domain and range of f^{-1}. [6]

(iv) Let the domain of f be $(-\infty, k]$ as found in **(iii)**.

If the graphs of $y = \mathrm{f}(x)$ and $y = mx - 1$, where m is a constant, have three points of intersection, find the range of possible values of m. [3]

3. **(AJC16/Promo/Q9)**

The function f is defined by $\mathrm{f}: x \mapsto x|x-2|$, $x \in \mathbb{R}$.

(i) Sketch the graph of $y=\mathrm{f}(x)$. If the domain of f is restricted to $x \leq m$ such that f^{-1} exists, state the maximum value of m . [2]

Use the value of *m* in **(i)** for the rest of this question.

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

(iii) It is given that $\mathrm{fg}(x)=-\tan^2 x$, where $|x|<\frac{\pi}{2}$. Find $\mathrm{g}(x)$. [2]

4. **(DHS14/Promo/Q3)**

The function f is defined by

$$\mathrm{f}(x)=\begin{cases}2x+3 & \text{for } 0<x\leq 4,\\ -4x+27 & \text{for } 4<x\leq 6,\end{cases}$$

and that $\mathrm{f}(x)=\mathrm{f}(x+6)$ for all real values of x.

(i) Find the value of $\mathrm{f}(-17)+\mathrm{f}(17)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-8\leq x\leq 13$. [3]

(iii) Hence find the exact value of $\int_{-2}^{6}\mathrm{f}(x)\,\mathrm{d}x$. [2]

5. **(EJC18/Promo/Q4)**

The function g is defined by

$$\mathrm{g}: x \mapsto \sqrt{2-\frac{2}{9}x^2},\ x\in\mathbb{R},\ 0\leq x<3.$$

(i) Show that g^2 exists. [1]

(ii) Find an expression for $\mathrm{g}^2(x)$ and find the exact range of g^2. [3]

(iii) Another function, h, is such that $\mathrm{hg}^2(x)=\mathrm{g}(x)$. State the relationship between h and g, and find the exact value of $\mathrm{hg}\left(\sqrt{2}\right)$. [2]

6. **(PJC14/Promo/Q7)**

The functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2-2x-3 \quad , \quad x\in\mathbb{R},$$

$$\mathrm{g}: x \mapsto \frac{2x+1}{x+5} \quad , \quad x\in\mathbb{R},\ x>-5.$$

(i) Explain why the composite function gf exists and define gf, including its domain. Find the range of gf. [4]

(ii) Explain with the aid of a diagram why the inverse function f^{-1} does not exist. f has an inverse if its domain is restricted to $x \le \lambda$. Write down the largest value of λ.

Hence find f^{-1}, stating its domain. [5]

7. **(RI16/Promo/Q11)**

Do not use a calculator in answering this question.

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{1}{x-1}, \quad x \in \mathbb{R}, \quad x > 1.$$

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain and range of f^{-1}. [4]

(ii) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, giving the equations of any asymptotes. [3]

The function g with domain $(0, \infty)$ is defined such that the composite function gf exists and it is defined by

$$\mathrm{gf} : x \mapsto \ln\left(x^2 - x\right), \quad x \in \mathbb{R}, \quad x > 1.$$

(iii) Find an expression for g(x). [2]

(iv) Use the fact that g is decreasing to solve the inequality $\mathrm{gf}(x) > \mathrm{g}(x)$. [3]

8. **(MJC14/Promo/Q9)**

Two functions f and g are defined by

$$\mathrm{f} : x \mapsto x^2 + 2x + 1, \qquad x \in \mathbb{R};$$

$$\mathrm{g} : x \mapsto \frac{1}{x+1}, \qquad x > -1.$$

(i) Give a reason why the inverse of f does not exist. [1]

The domain of f is now restricted to $x \le k$, where k is a real constant.

(ii) Given that f^{-1} exists, state the largest value of k. [1]

(iii) Using the value of k found in part **(ii)**, find $\mathrm{f}^{-1}(x)$. [3]

In the rest of the question, the domain of f is $x \in \mathbb{R}$ as originally defined.

(iv) Show that gf exists. Find gf and its range. [5]

9. **(JJC16/Promo/Q7)**

The functions f and g are defined by

$$\mathrm{f}: x \mapsto |x-2|, \qquad x \in \mathbb{R},$$
$$\mathrm{g}: x \mapsto x^2 - x, \qquad x \in \mathbb{R},\ x \le 0.$$

(i) By considering the graphs of f and g, solve exactly the inequality $\mathrm{f}(x) > \mathrm{g}(x)$. [4]

(ii) Explain why the composite function fg exists. [2]

(iii) Find fg(x), giving the domain of fg. [2]

(iv) Show by means of a graphical argument that $(\mathrm{f\,g})^{-1}$ does not exist. [2]

(v) If the domain of fg is restricted to $x \le a$, state the greatest value of a for $(\mathrm{f\,g})^{-1}$ to exist. With this domain, find $(\mathrm{f\,g})^{-1}(x)$. [4]

10. **(NJC14/Promo/Q9 modified)**

Functions f and g are defined such that

$$\mathrm{f}: x \mapsto \cos^{-1}\left(x^2\right), \qquad -1 \le x \le 1,$$
$$\mathrm{g}: x \mapsto x^3 + 1, \qquad x \in \mathbb{R}.$$

(i) Explain why the composite function fg does not exist. [1]

The function h is defined such that $\mathrm{h}(x) = \mathrm{g}(x)$ and the domain of h is $-\dfrac{5}{4} \le x \le 0$.

(ii) Find the range of fh in exact form. [2]

(iii) Determine all the possible value(s) of x that satisfies $\mathrm{g}^{-1}\left(x^2\right) = 2$.

Hence explain why $\mathrm{h}^{-1}\left(x^2\right) = 2$ has no solution. [4]

11. **(VJC14/Promo/Q8)**

The function f is defined by

$$\mathrm{f}: x \mapsto \frac{-x-2}{5x+1}, \qquad x \in \mathbb{R},\ x \ne -\frac{1}{5}.$$

(i) Explain why both the function f^{-1} and composite function f^2 exist. [4]

(ii) Find $\mathrm{f}^2(x)$ and state the range of f^2. [3]

(iii) Determine the solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{2x-2}{5x+4}, \qquad x \in \mathbb{R},\ x \ne -\frac{4}{5}, x \ne -\frac{1}{5}, x \ne \frac{2}{5}.$$

(iv) Verify that $g^2(x) = f(x)$. [1]

(v) Given that k is an integer, find $g^{49}(k)$, giving your answer in terms of k. [3]

12. **(RI14/Promo/Q9)**
The functions f and g are defined by

$$f : x \mapsto \frac{ax+3}{x+b}, \qquad x \in \mathbb{R}, \quad x \neq -b,$$
$$g : x \mapsto 2x+1, \qquad x \in \mathbb{R}.$$

(i) Given that the graph of f passes through the point with coordinates $\left(-\frac{3}{2}, 0\right)$ and has a vertical asymptote with equation $x = -1$, state the values of a and b. [2]

(ii) Sketch the graph of f and find its range. Hence show that f is one-one and find f^{-1}. [7]

(iii) Show that the composite function gf^{-1} exists and find $gf^{-1}(x)$. [2]

13. **(RI14/C1MYE/Q9)**
The function f is defined by

$$f : x \mapsto (x-2)(4-x), \; x \in \mathbb{R}, x \geq 2.$$

(i) Sketch the graph of $y = f(x)$, stating the coordinates of any turning points and points of intersection with the axes. [2]

(ii) If the domain of f is further restricted to $2 \leq x \leq m$, state with a reason the set of values of m for which the function f^{-1} exists. [2]

In the rest of this question, the domain of f is $2 \leq x \leq m_0$, where m_0 is the greatest value of m found in **(ii)**.

(iii) Find $f^{-1}(x)$ and write down the domain of f^{-1}. [3]

The function g has domain $\mathbb{R}$.

(iv) Explain why gf exists. [1]

(v) Given that

$$gf : x \mapsto 2 + 2(x-3)^2 - (x-3)^4, \; x \in \mathbb{R}, \; 2 \leq x \leq m_0,$$

find $g(x)$. [3]

14. **(SAJC14/Promo/Q9)**

The function f is defined by

$$\mathrm{f}: x \mapsto \frac{1}{2}x^2 - x + \beta,\ x \in \mathbb{R}.$$

It is given that $\beta > \frac{3}{2}$.

(i) Sketch the graph of $y = \mathrm{f}(x)$, indicating any axial intercept(s) and stationary point(s) and explain graphically why the inverse function does not exist. [2]

(ii) If the domain of f is now restricted to $x \geq k$, state the least value of k for which the function f^{-1} exists. Hence, sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$ on the same diagram, showing clearly the relationship between the three graphs. [4]

In the rest of the question, the domain of f is $\mathbb{R}$ as originally defined.

(iii) Another function g is defined by

$$\mathrm{g}: x \mapsto \sin x \text{ for } 0 \leq x \leq 2\pi.$$

Given that fg exists, find the range of fg. [2]

15. **(TJC14/Promo/Q11)**

The functions f and g are defined by

$$\mathrm{f}(x) = \begin{cases} \sqrt{\left(3 - \frac{x}{4}\right)} & \text{for } -4 \leq x \leq 4 \\ -x - 2 & \text{for } x < -4 \end{cases}$$

$$\mathrm{g}(x) = \mathrm{e}^x,\ x \in \mathbb{R},\ x < 0.$$

(i) Sketch the graph of f. [2]

(ii) Define $\mathrm{f}^{-1}(x)$ in a similar form. [4]

(iii) Find the set of values of x for which $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. [2]

(iv) Explain why the composite function gf does not exist. [2]

16. **(VJC20/Promo/Q6)**

The functions f and g are defined by

$$\mathrm{f}: x \mapsto \left|3\mathrm{e}^{-x} - 8\right| - 2, \qquad x \in \mathbb{R}, x \geqslant \lambda,$$
$$\mathrm{g}: x \mapsto a\sin x, \qquad x \in \mathbb{R}, 0 < x < \pi,$$

where a is a positive constant.

(i) Find the smallest value of λ, in exact form, for f^{-1} to exist. [2]

For the rest of the question, λ takes the value found in part **(i)**.

(ii) Find $f^{-1}(x)$. [3]

(iii) Find, in terms of *a*, the range of fg. [2]

Answers

1. Least $a = 0$

(i) $f^{-1} : x \mapsto \sqrt{\dfrac{3-x}{x+1}}, \quad x \in \mathbb{R}, \quad -1 < x < 3$

(ii) Point of intersection is (1,1)

(iii) $4\pi\left(\ln 2 - \dfrac{1}{3}\right)$ units³

2. **(i)** $R_f = [-1,1)$ **(ii)** $x > 2.47$ or $x < -2.47$

(iii) Greatest value of $k = 0$; $f^{-1}(x) = -\sqrt{\dfrac{1+x}{1-x}}$; $D_{f^{-1}} = [-1,1)$, $R_{f^{-1}} = (-\infty, 0]$

(iv) $-1 < m < 0$

3. **(i)** maximum $m = 1$ **(ii)** $f^{-1}(x) = 1 - \sqrt{1-x}$, $D_{f^{-1}} = (-\infty, 1]$

(iii) $g(x) = 1 - \sec x$

4. **(i)** 12 **(iii)** 56

5. **(ii)** $R_{g^2} = \left[\dfrac{\sqrt{14}}{3}, \sqrt{2}\right)$ **(iii)** $hg(\sqrt{2}) = \sqrt{2}$

6. **(i)** $gf : x \mapsto \dfrac{2x^2 - 4x - 5}{x^2 - 2x + 2}, \quad x \in \mathbb{R}$; $R_{gf} = [-7, 2)$ **(ii)** Largest $\lambda = 1$;

$f^{-1} : x \mapsto 1 - \sqrt{x+4}, \quad x \geq -4$

7. **(i)** $f^{-1}(x) = \dfrac{1}{x} + 1$, $D_{f^{-1}} = (0, \infty)$, $R_{f^{-1}} = (1, \infty)$ **(iii)** $g(x) = \ln\left(\dfrac{x+1}{x^2}\right)$

(iv) $x > \frac{1}{2} + \frac{\sqrt{5}}{2}$

8. **(ii)** Largest value of $k = -1$ **(iii)** $\mathrm{f}^{-1}(x) = -1 - \sqrt{x},\ x \geq 0$

(iv) $\mathrm{gf}(x) = \frac{1}{x^2 + 2x + 2}$, $D_{\mathrm{gf}} = \mathbb{R}$, $R_{\mathrm{gf}} = (0,1]$

9. **(i)** $-\sqrt{2} < x \leq 0$ **(iii)** $\mathrm{fg}(x) = \mathrm{f}(x^2 - x) = |x^2 - x - 2|$, $x \in \mathbb{R}$, $x \leq 0$

(v) Greatest value of $a = -1$, $(\mathrm{fg})^{-1} = \frac{1 - \sqrt{4x+9}}{2}$, $D_{(\mathrm{fg})^{-1}} = [0, \infty)$

10. **(ii)** $\mathrm{R}_{\mathrm{fh}} = \left[0, \frac{\pi}{2}\right]$ **(iii)** $x = \pm 3$

11. **(ii)** $\mathrm{f}^2(x) = x$, $\mathrm{R}_{\mathrm{f}^2} = \left(-\infty, -\frac{1}{5}\right) \cup \left(-\frac{1}{5}, \infty\right)$

(iii) $x \in \mathbb{R}, x \neq -\frac{1}{5}$ **(v)** $\frac{2k-2}{5k+4}$

12. **(i)** $a = 2, b = 1$ **(ii)** $\mathrm{R}_{\mathrm{f}} = \mathbb{R} \setminus \{2\}$; $\mathrm{f}^{-1} : x \mapsto \frac{3-x}{x-2}$, $x \in \mathbb{R}$, $x \neq 2$

(iii) $\mathrm{gf}^{-1}(x) = \frac{4-x}{x-2}$, $x \in \mathbb{R}$, $x \neq 2$

13. **(ii)** $[2,3]$ **(iii)** $\mathrm{f}^{-1}(x) = 3 - \sqrt{1-x}$ with domain $[0,1]$

(v) $\mathrm{g}(x) = 3 - x^2$

14. **(ii)** Least value of $k = 1$ **(iii)** $\left[\beta - \frac{1}{2}, \frac{3}{2} + \beta\right]$

15. **(ii)** $\mathrm{f}^{-1}(x) = \begin{cases} 4(3 - x^2) & \text{for } \sqrt{2} \leq x \leq 2 \\ -x - 2 & \text{for } x > 2 \end{cases}$ **(iii)** $[\sqrt{2}, 4]$

16. **(i)** $\lambda = -\ln \frac{8}{3}$ **(ii)** $\mathrm{f}^{-1}(x) = -\ln\left(2 - \frac{x}{3}\right)$

(iii) $(3, 6 - 3\mathrm{e}^{-a}]$

5

Differentiation and its Applications

1. **[MJC14/Promo/Q5, CJC14/Promo/Q2, MJC13/Promo/Q2, CJC17Promo/Q5(parts), YJC18/Promo/Q5)]**

(a) Differentiate $\frac{x-2x^3}{\ln x}$ with respect to *x*. [2]

(b) Given that $0 < x < \frac{\pi}{2}$, show that $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^{-1}(\cos x)\right] = k$, where *k* is a real constant to be determined. [3]

(c) Given that $\mathrm{e}^{xy} = (1+y^2)^2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *x* and *y*, simplifying your answer. [4]

(d) The variables *x* and *y* are related by $3^y = xy - \sec x$.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *x* and *y*. [4]

(e) The parametric equations of a curve C are $x = \sin^{-1}(1-t),\ \ y = \mathrm{e}^{\sqrt{2t-t^2}}$.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *t*. [4]

(f) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\log_x 2)$. [2]

(g) Let $y = x^{2x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *x*. [3]

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following, simplifying your answers:

(h) Find $y = \ln(x\cos^2 x), 0 < x < \frac{\pi}{2}$. [2]

(i) $\frac{1}{x} - \frac{1}{y} = \frac{1}{a}$, where a is an arbitrary constant. [2]

(j) $y = x\tan^{-1}\sqrt{x}$. [2]

2. **(RI13/Promo/Q1)**

A curve C is given by the equation $\sqrt{x} + \sqrt{y} = \sqrt{a}$, for $x > 0,\ y > 0$, where a is a positive constant.

(i) Show that C has no stationary point. [3]

(ii) What can be said about the tangents to C as $x \to 0$? [1]

3. **(TJC10/Promo/Q6)**

A curve is defined by the parametric equations $x = 4t^3$ and $y = \frac{1}{t^2}$, where *t* is a real non-zero parameter.

(i) Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. Hence, find the equation of the tangent to the curve at the point $(4t^3, \frac{1}{t^2})$. [4]

(ii) Find the values of t at the points where the tangents to the curve pass through the point (1, 2). [3]

4. **(SRJC13/Promo/Q3)**

A curve is defined by the equation

$$2x^3 + \mathrm{e}^y + \frac{1}{y} = 0 \quad \text{where } y \neq 0 .$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

Given that the normal to the curve at point P is parallel to the x-axis, find the y-coordinate of P, leaving the answers to 3 decimal places. [4]

5. **(MJC17/Promo/Q5)**

Suppose the following facts are known about the function g and its derivatives:

$$\mathrm{g}\left(\frac{\pi}{2}\right) = 1, \quad \mathrm{g}'\left(\frac{\pi}{2}\right) = 0, \quad \mathrm{g}''\left(\frac{\pi}{2}\right) = -1$$

Consider the function $\mathrm{f}(x) = \mathrm{e}^{\mathrm{g}(x)}$.

(i) Using differentiation, show that $\mathrm{f}(x)$ has a stationary point at $x = \frac{\pi}{2}$ and determine the nature of this stationary point. [5]

(ii) Given that $\mathrm{g}(x)$ is $\sin x$, show that $\mathrm{f}(x)$ is increasing on the interval where $-\frac{\pi}{2} < x < \frac{\pi}{2}$. [2]

6. **(TJC13/Promo/Q10)**

A curve C is given parametrically by the equations

$$x = 2\cos^3\theta, \qquad y = 2\sin^3\theta \qquad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Show that the normal at the point with parameter θ has equation

$$y\sin\theta = x\cos\theta + 2(\sin^4\theta - \cos^4\theta) .$$ [4]

The normal at the point Q where $\theta = \frac{\pi}{6}$, cuts C again at the point P, where $\theta = p$.

Show that $\sin^3 p - \sqrt{3}\cos^3 p + 1 = 0$ and hence find the coordinates of P. [5]

7. **(ACJC20/Promo/Q11)**

A curve C has parametric equations $x = \frac{t}{\sqrt{1-4t^2}}$ and $y = \sin^{-1} 2t$, for $-\frac{1}{2} < t < \frac{1}{2}$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2(1-4t^2)$ and explain why the gradient is always positive for all points on the curve. [4]

(ii) Describe the behavior of C as $x \to \pm\infty$. Sketch C, stating clearly the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

(iii) Given that A is the point on C with parameter $\frac{1}{2\sqrt{2}}$, find the equation of l, the tangent to C at the point A, leaving your answer in exact form. [3]

Show that l meets the curve C again at another point B and find the coordinates of B. [2]

8. **(NYJC13/Promo/Q8 modified)**

The equation of a curve is $y(x+2)^2+2y^2(x+2)-12x=0$, where $x, y \in \mathbb{R}^+$.

(i) Show that the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x=2$ is $\frac{1}{16}$. [5]

(ii) Find the equation of the normal to the curve at the point where $x=2$. [2]

(iii) Given that the normal in **(ii)** meets the line $x=2$ at the point P and the line $x=0$ at the point S. Find the exact area of triangle OSP, where O is the origin. [2]

9. **(JJC13/Promo/Q10)**

(a) The curve C is defined by

$$x=\mathrm{e}^{3t}, \qquad y=t^2, \qquad \text{where } t \geq 0.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t and determine the value of t for which $\frac{\mathrm{d}y}{\mathrm{d}x}$ is zero. [3]

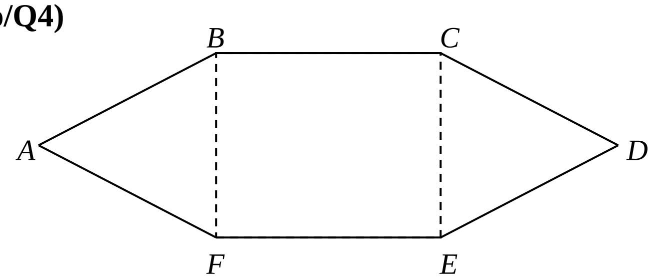
(ii) Sketch the graph of C. [2]

(b) The equation of a curve C is $x^2-2xy+2y^2=k$, where k is a constant. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [3]

Given that C has two points for which the tangents are parallel to the line $y=x$, find the range of values of k. [3]

Given that $k=4$, find the exact coordinates of each point on the curve C at which the tangent is parallel to the y-axis. [4]

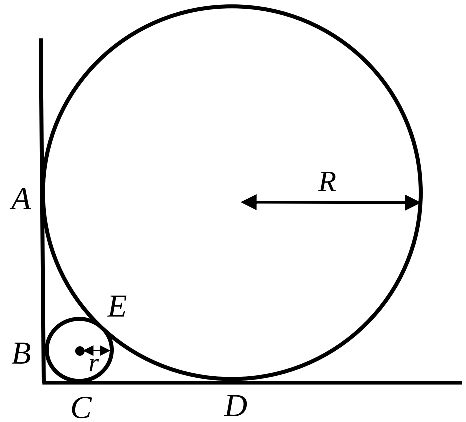
10. **(PJC13/Promo/Q4)**



$ABCDEF$ is a hexagon with a fixed perimeter where ABF and CDE are equilateral triangles and $BCEF$ is a rectangle as shown in the diagram above. If the area of the hexagon is a maximum, show that the ratio of AB to the perimeter of the hexagon is $(4+\sqrt{3}):26$. [5]

11. **(AJC13/Promo/Q11)**

Two solid cylinders of the same height are placed at a corner of the wall such that the vertices A, B, C and D touch the wall. At point E, the two cylinders touch each other. The diagram below shows a cross section of the cylinders.



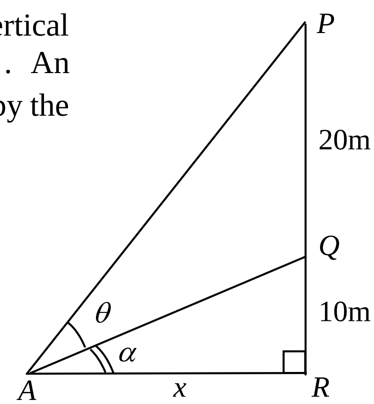
Let r be the radius of the small cylinder and R be the radius of the big cylinder.

(i) Show that $R=(\sqrt{2}+1)^2 r$. [2]

(ii) Given that the volume of the small cylinder is $\frac{16\pi}{\sqrt{2}+1}$ cm^3, find the exact value of the radius r such that the surface area of the big cylinder is a minimum. [5]

12. (JJC10/Promo/Q7)

In the figure shown, PQ is an advertisement painted on a vertical wall PQR of a building with $PQ=20\text{m}$ and $QR=10\text{m}$. An observer at A is x metres from the wall. The angle subtended by the advertisement at his eye is θ and angle QAR is α.



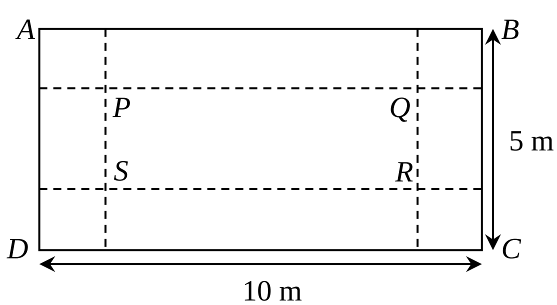
(i) By considering $\tan(\theta+\alpha)$, or otherwise, show that

$$\tan\theta=\frac{20x}{x^2+300}.$$ [3]

(ii) By differentiation, show that $\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{20(300-x^2)}{x^4+1000x^2+90000}$.

Hence, find the exact value of x for which θ is a maximum. [6]

13. (YJC13/Promo/Q13)



The diagram shows a rectangular piece of metal sheet $ABCD$ of sides 5 metres and 10 metres. A square of side x metres is removed from each corner of $ABCD$. The remaining shape is now folded along PQ, QR, RS and SP to form an open rectangular box of height x metres.

(i) Show that the volume V cubic metres of the box is given by

$$V = 4x^3 - 30x^2 + 50x.$$ [1]

(ii) Use differentiation to find the exact value of x which gives a stationary value of V and explain why there is only one answer. [4]

14. **(MJC13/Promo/Q12)**

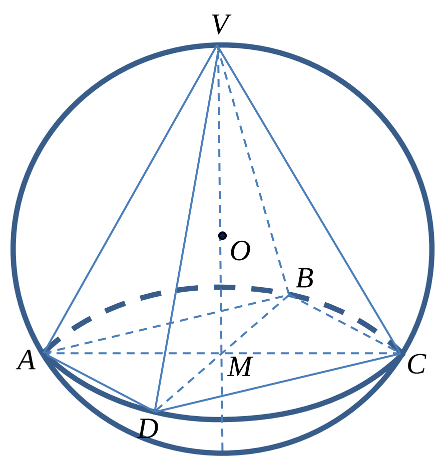
A student wants to construct a model of a roof structure of fixed height h cm from a rectangular piece of cardboard of width k cm. The cardboard is to be bent in such a way that the cross-section $PQRS$ is as shown in the diagram, with $PQ + QR + RS = k$ and with PQ and RS each inclined to the horizontal at an angle α.

(i) Show that $QR = k - 2h \operatorname{cosec} \alpha$ and that the area $A\,\text{cm}^2$ of the cross-section $PQRS$ is given by $A = hk + h^2(\cot\alpha - 2\operatorname{cosec}\alpha)$. [3]

(ii) Use differentiation to find, in terms of k and h, the maximum value of A as α varies. [5]

15. **(ACJC13/Promo/Q7)**

A right pyramid block has a square base $ABCD$ and its vertical height VM is $(a + x)$ where $0 < x < a$. M is the point where the diagonals AC and BD of the square meet. This right pyramid block is inscribed in a sphere of fixed radius a so that the vertices V, A, B, C and D of the block just touch the interior of the sphere with the vertical height VM passing through the centre O of the sphere.



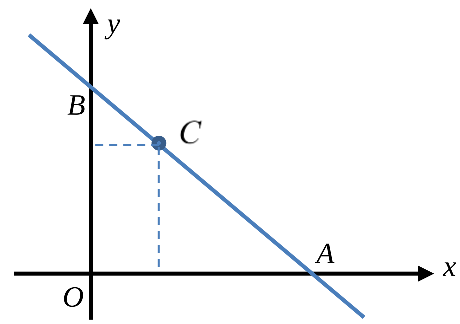
(i) Show that the length of the side of the square base $ABCD$ is $\sqrt{2(a^2 - x^2)}$. [2]

(ii) Hence, find the maximum volume of the block in terms of a. [4]

[Volume of a pyramid $= \frac{1}{3} \times \text{base area} \times \text{height}$]

16. (RI13/Promo/Q8)

The line L passes through the fixed point, C, with coordinates $(2,3)$ and the variable point A, with coordinates $(t,0)$ where $t>2$. The line L meets the y-axis at point B as shown in the diagram.

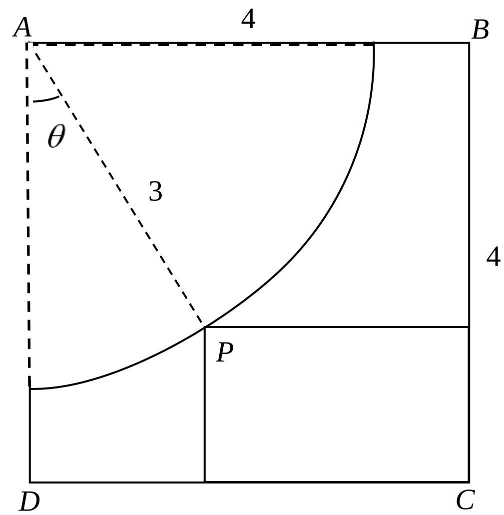


(i) Show that the y-coordinate of B can be expressed as $\frac{3t}{t-2}$. [2]

(ii) Use differentiation to find the value of t, for which the area of triangle OAB is a minimum. [5]

17. (NYJC13/Promo/Q12b)

The diagram shows the floor plan of a storeroom. The floor plan consists of a square $ABCD$ of side 4 units from which a quadrant of a circle with centre A and radius 3 units has been removed. The owner intends to store a rectangular crate with one corner of the base at C, and the opposite corner of the base at P against the curved wall. The base of the crate has area y unit^2 and angle DAP is θ radians, where $0\le\theta\le\frac{\pi}{4}$.



Show that

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=3\left(\sin\theta-\cos\theta\right)\left(4-3\sin\theta-3\cos\theta\right).\qquad [2]$$

Hence, find the least possible value of y. [5]

18. (SAJC20/Promo/Q11)

In geometry, a spherical cap is the **smaller** part of a sphere when the sphere is cut horizontally as shown in Figure 1.

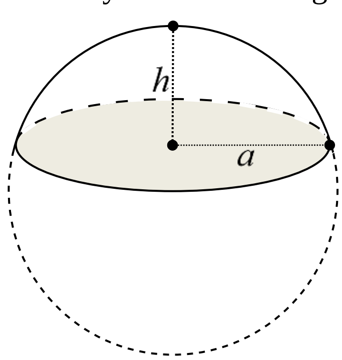


Figure 1

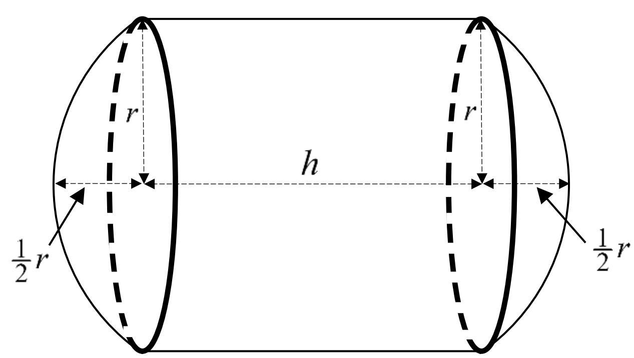


Figure 2

It is given that a spherical cap with height h and base radius of the cap a has curved surface area $\pi(a^2+h^2)$ and volume $\frac{1}{6}\pi h(3a^2+h^2)$.

A pharmaceutical company wants to manufacture a capsule shell to contain their new supplement as shown in Figure 2. The capsule shell consists of two identical spherical caps each joined to one end of a cylinder. The spherical caps have base radius r cm and height $\frac{1}{2}r$ cm while the cylinder has radius r cm and height h cm. The capsule shell has a fixed volume of v cm^3 and is assumed to have a negligible thickness.

[A cylinder with radius r and height h has curved surface area $2\pi rh$ and volume $\pi r^2 h$.]

(i) Show that the external surface area of the capsule shell, A cm^2, is given by

$$A = \frac{17}{12}\pi r^2 + \frac{2v}{r}.$$ [4]

(ii) The pharmaceutical company wants to reduce the external surface area of the capsule shell in order to lower manufacturing cost. Using differentiation, find, in terms of v, the exact value of r that gives the minimum surface area of the capsule shell. [5]

Given that $v = 2$,

(iii) find the range of values of r such that $h > 0$; [1]

(iv) sketch the graph showing the surface area of the capsule as the radius varies. [3]

19. (HCI19/Promo/Q9)

The parametric equations of a curve are given by

$$x = \sin t \cos t, \quad y = \cos\left(t + \frac{\pi}{4}\right), \text{ where } 0 \le t \le \frac{\pi}{2}.$$

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{2}(\sin t - \cos t)}$. [2]

(ii) Hence find the equation of the tangent parallel to the y-axis. [2]

The curve cuts the y-axis at the points P and Q.

(iii) Find the exact coordinates of P and Q. [2]

(iv) The point R on the curve has coordinates $\left(\sin\theta\cos\theta, \cos\left(\theta + \frac{\pi}{4}\right)\right)$.

Show that the area of triangle PQR is given by $\frac{\sqrt{2}}{4}\sin 2\theta$. Hence find the value of θ for which the area of triangle PQR is a maximum. (You need not prove that it is a maximum.) [2]

20. (SAJC19/Promo/Q11)

[The volume of a cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$ and the arc length of a sector of radius r and angle θ radians is $r\theta$.]

Figure 1 shows a sector AOB of θ radians which is cut from a circular card of fixed radius a metres with centre O. A cup in the shape of an inverted right circular cone with radius r and height h is then formed by joining the two radii, OA and OB, of the sector together, without overlap (as shown in Figure 2).

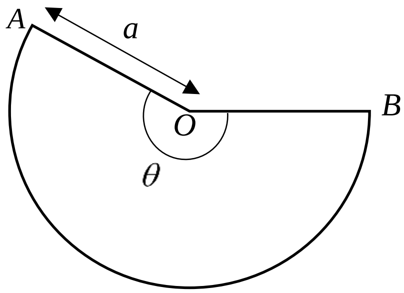


Figure 1

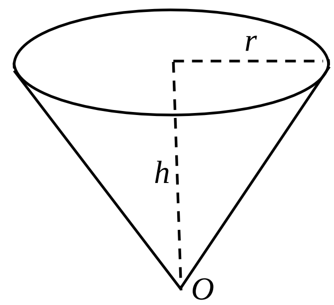


Figure 2

(i) Show that the volume of the cup in Figure 2, V cubic metres is given by

$$V = \frac{a^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}\ . \qquad [4]$$

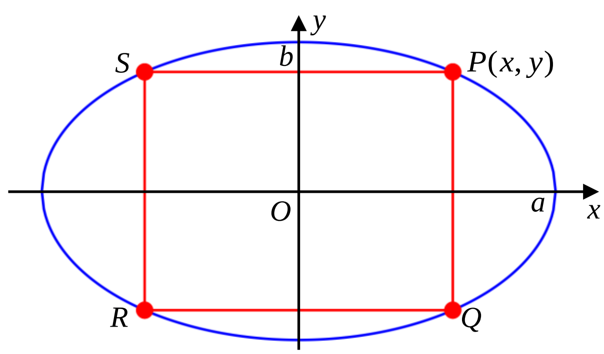
(ii) Use differentiation to find, in terms of a, the exact maximum volume of the cup as θ varies. You are not required to justify that the volume of the cup is a maximum. [5]

(iii) Hence, sketch the graph showing the volume of the cup, V as the angle of the sector AOB, θ, varies. [3]

21. (RI16/Promo/Q5)

The diagram below shows an ellipse, centre O, with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b>0$.

The point P on the ellipse is in the first quadrant with coordinates (x, y) and Q, R and S are three other points on the ellipse such that $PQRS$ forms a rectangle with sides parallel to the axes.



Let A denote the area of the rectangle $PQRS$.

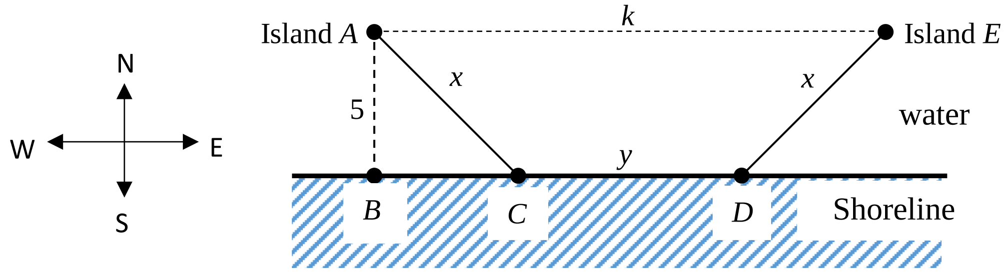
(i) Show that $A^2 = \frac{16b^2}{a^2}x^2(a^2-x^2)$. [2]

(ii) Use differentiation to show that the maximum area of the rectangle is $2ab$. [5]

(iii) Write down, in terms of r, the maximum area of a rectangle inscribed in a circle with centre O and radius r. [1]

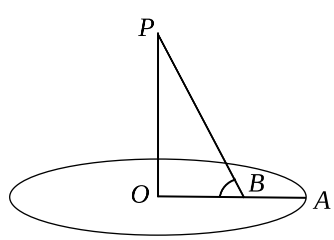
22. **(MJC16/Promo/Q10 modified)**

Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that the birds travel at a slower speed over water than land because hot air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island A that is 5 km north of point B on a straight shoreline, flies to a point C on the shoreline which is due east of point B, and then flies east along the shoreline to a point D before flying to island E which is at a fixed distance k km due east of island A. It is known that island A and point C are x km apart, points C and D are y km apart and point D and island E are x km apart.



It is given that the bird travels at a speed of 65 km/h over water and travels at a speed of 90 km/h along the shoreline.
Using differentiation, determine the value of x such that the total time taken for the bird to fly from island A to island E via the route $ACDE$ is the minimum.
Hence determine the minimum value of k, to the nearest integer, in order for the bird to choose the route $ACDE$ over flying from island A to island E directly. [11]

23. **(CJC18/Promo/Q10b modified)**



During a performance, a drone is positioned at the center of a circular stage, O, and a performer is standing at the circumference of the stage, A, as seen in the diagram above. The drone starts to ascend vertically at a constant speed of $3\,\mathrm{ms}^{-1}$ as the performer starts to walk towards O with a constant speed of $2\,\mathrm{ms}^{-1}$. At time t, the drone is at point P and the performer is at point B. It is given that the angle OBP is α radians and the radius of the circular stage is $20\,\mathrm{m}$.

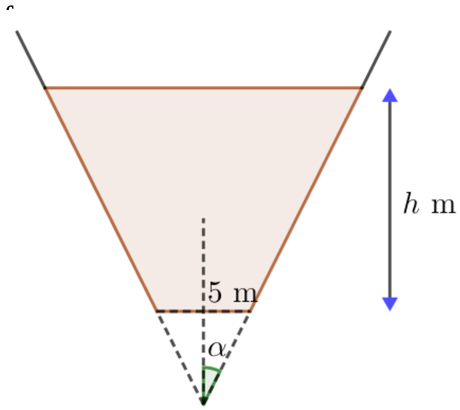
(i) Show that $\alpha = \tan^{-1}\left(\frac{3t}{20-2t}\right)$. [1]

(ii) Find $\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ in terms of t. [3]

(iii) The drone will hover once the rate of change of α is at its maximum. State the value of t at this instant. [1]

24. (NJC19/Promo/6)

A vessel is formed by removing a smaller cone radius 5 m from a bigger cone whose semi-vertic angle is α, where $\tan\alpha = 0.5$. Water flows out of t vessel at a rate of $k\sqrt{h}$ m^3 per minute, where k is positive constant. At time t minutes, the height of t water surface from the hole is h m (see diagram).



(i) Show that the volume of the water V, in m^3, given by $\frac{1}{12}\pi\left[(h+10)^3 - 1000\right]$. [4]

(ii) Find the rate of change of h, in terms of k, wh $V = 120\pi$. [4]

25. (RI18/Promo/Q2)

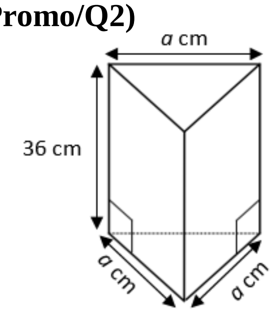


Fig. 1

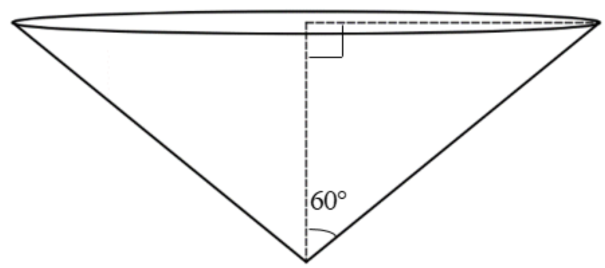


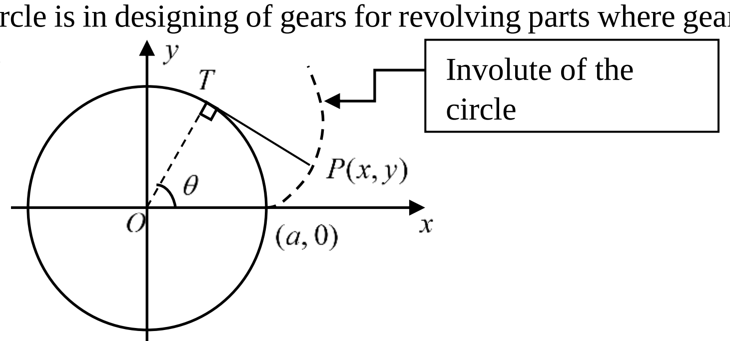
Fig. 2

An equilateral triangle with sides of length a cm is used to construct the base of a triangular prism container of height 36 cm (see Fig. 1). Initially this triangular prism container is half filled with water. This amount of water is then poured at a constant rate into a hollow cone of semi-vertical angle 60° held with its axis vertical and vertex downwards (see Fig. 2), without overflowing. After t seconds, the depth of water in the conical container is h cm. Given that it takes 3 minutes to empty the triangular prism container, find in terms of a and π, the rate at which h is increasing at the instant when $h = 2.5$. [5]

[It is given that the volume of a circular cone with radius r and height h is $\frac{1}{3}\pi r^2 h$.]

26. (VJC19/Promo/Q12)

The diagram shows a string that is unwound from a circle while being held taut. The curve traced by the end point P of the string is called the *involute* of the circle. One of the major applications of involute of circle is in designing of gears for revolving parts where gear tooth follow the shape of involute.



A circle has fixed radius a units and centre O and the initial position of P is at $(a, 0)$.
The parameter θ, $(0 \le \theta \le \frac{\pi}{2})$, is the angle measured from the positive x-axis to OT in the anti-clockwise direction, where T is the point on the circle such that PT is tangential to the circle.
Show that the involute has parametric equations

$$x = a(\cos\theta + \theta\sin\theta),\ \ y = a(\sin\theta - \theta\cos\theta),\ \text{for}\ \ 0 \le \theta \le \tfrac{\pi}{2}. \qquad [3]$$

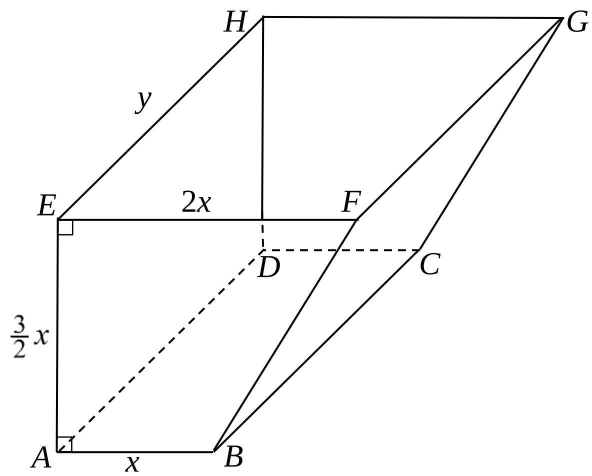
The point W on the involute has parameter $\theta = \frac{\pi}{3}$.

(i) Show that the equation of the normal to the involute at W is

$$\sqrt{3}y = 2a - x. \qquad [5]$$

(ii) At W, x increases at a rate of 0.3 units per second. Given that $z = xy$, determine, in terms of a, the rate of change of z at W. [4]

27. **(NYJC19/Promo/Q12)**
The diagram shows an open tank made of metal with negligible thickness and a fixed capacity of 108 m^3. The open top $EFGH$, base $ABCD$, sides $ADHE$ and $BCGF$ are rectangles, while $ABFE$ and $DCGH$ are trapeziums with AE perpendicular to AB and EF. The dimensions of the tank are
$AB = x$ m, $EF = 2x$ m,
$AE = \frac{3}{2}x$ m and $EH = y$ m.

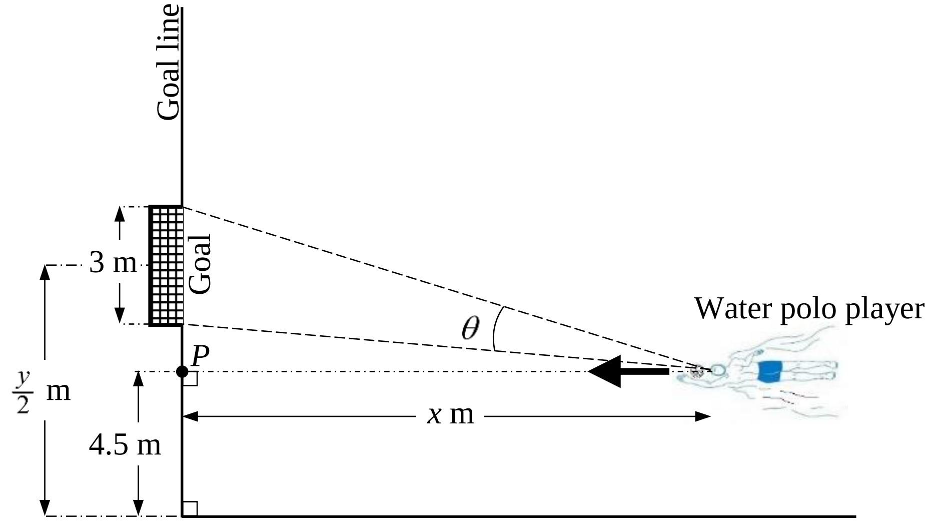


(i) The interior of the tank needs to be treated to prevent rusting. The cost of treating the interior base is \$3 per m^2 and the cost of treating the interior sides is \$4 per m^2. Show that the total cost of treatment is $\$\left(\frac{K}{x} + 18x^2\right)$, where K is an exact constant to be determined. [3]

(ii) Find, by differentiation, the values of x and y such that the cost of treatment is minimum and show that it is less than \$420. [5]

(iii) The tank is mounted on a wall and filled with a liquid to maximum capacity. However, due to poor construction, the liquid begins to leak and spread in a circle of uniform thickness
2 mm on the floor. Assuming that the liquid is leaking from the tank at a constant rate of 8.4×10^{-7} m^{3}s^{-1} and the leak is discovered one minute after it started, find the rate at which the radius of the circle of liquid is increasing at this instant, giving your answer in ms^{-1}. [4]

28. (HCI19/Promo/Q12)



The diagram (not drawn to scale) shows part of a rectangular water polo pool with a fixed width of y m. A goal 3 m wide is placed on the goal line at one end of the pool with the centre of the goal $\frac{y}{2}$ m from the side of the pool. A water polo player at a distance of x m perpendicular to the goal line and a distance of 4.5 m away from the side of the pool swims directly towards a point P on the goal line. A visual angle θ of the goal is the angle subtended at the eye of the water polo player by the goal.

(i) Show that $\tan\theta = \frac{12x}{4x^2+A}$ where $A=(y-6)(y-12)$. [3]

(ii) The desired visual angle of the goal is obtained when θ is a maximum. Find by differentiation, the value of x such that the desired visual angle of the goal is obtained. Leave your answer in exact form in terms of A. [4]

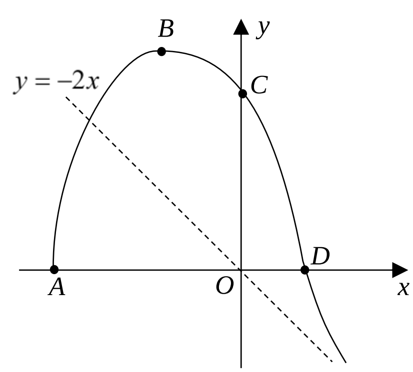
For the rest of the question, let $y=20$.

(iii) Show that the water polo player needs to be $2\sqrt{7}$ m away from the goal line in order to obtain the desired visual angle. [2]

(iv) The water polo player swims at a constant speed of 50 m per minute. Find the rate of change of θ at the instant when the water polo player is 15 m away from the goal line. [3]

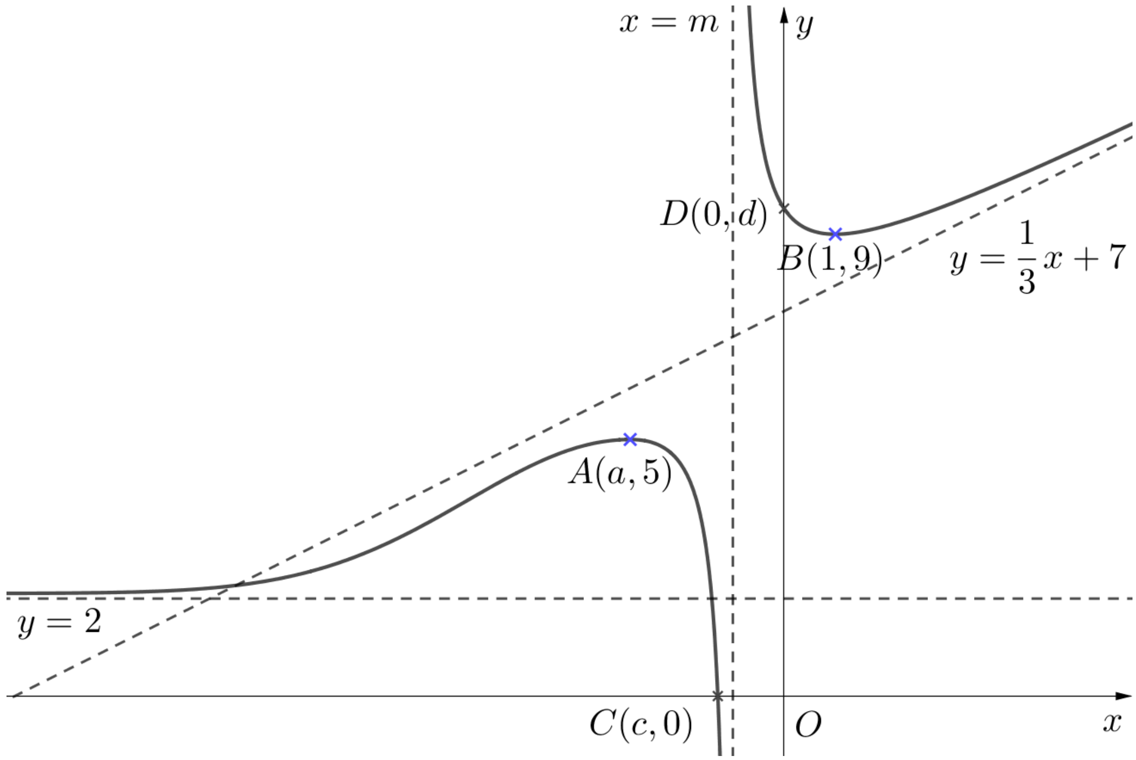
29. (DHS16/Promo/Q8ii modified)

The diagram shows the curve $y=\mathrm{f}(x)$ where the line $y=-2x$ is an asymptote to the curve. Four points $A(-4,0)$, $B(-2,5)$, $C(0,4)$ and $D(1,0)$ on the curve are shown in the diagram. Point D is a non-stationary point of inflexion. Tangents to the curve at A and B are parallel to the y-axis and x-axis respectively. Sketch the graph of $y=\mathrm{f}'(x)$. [3]



30 **(NJC19/Promo/Q3b)**

The diagram below shows the curve with equation $y = \mathrm{f}(x)$. It has turning points $A(a, 5)$ and $B(1, 9)$ and asymptotes with equations $y = 2$, $x = m$ and $y = \frac{1}{3}x + 7$. The curve also crosses the axes at the points $C(c, 0)$ and $D(0, d)$. The gradient of the curve at D is -3.



Sketch the following curve.

(b) $y = \mathrm{f}'(x)$. [3]

Label the coordinates of the points corresponding to *A*, *B*, *C*, and *D* (where applicable), the points where the curve crosses the axes, and the equations of any asymptotes.

Answers

1. (a) $\dfrac{(1-6x^2)\ln x+2x^2-1}{(\ln x)^2}$ (b) -1

(c) $\dfrac{ye^{xy}}{4y(1+y^2)-xe^{xy}}$ or $\dfrac{y(1+y^2)}{4y-x(1+y^2)}$ (d) $\dfrac{dy}{dx}=\dfrac{y-\sec x\tan x}{3^y\ln 3-x}$

(e) $\dfrac{dy}{dx}=e^{\sqrt{2t-t^2}}(t-1)$ (f) $-\dfrac{\ln 2}{x(\ln x)^2}$ (g) $2x^{2x}(\ln x+1)$

(h) $\dfrac{1}{x}-2\tan x$ (i) $\left(\dfrac{y}{x}\right)^2$ (j) $\tan^{-1}\sqrt{x}+\dfrac{\sqrt{x}}{2(1+x)}$

2. tangent to $C \rightarrow x=0$

3. (i) $\dfrac{dy}{dx}=-\dfrac{1}{6t^5}$; $y=-\dfrac{1}{6t^5}x+\dfrac{5}{3t^2}$ (ii) $t=0.829$ or -0.963 or 0.534

4. $\dfrac{dy}{dx}=\dfrac{6x^2y^2}{1-y^2e^y}$; $y=0.703$ 6. $(0.795,-0.622)$

7. (iii) $y=x-\dfrac{1}{2}+\dfrac{\pi}{4}$; $B(-1.54,-1.26)$ 8. (ii) $y=-16x+33$ (iii) 33

9. (a)(i) $\dfrac{dy}{dx}=\dfrac{2t}{3e^{3t}}$; $t=0$ (b) $\dfrac{dy}{dx}=\dfrac{y-x}{2y-x}$; $k>0$; $(-2\sqrt{2},-\sqrt{2})$, $(2\sqrt{2},\sqrt{2})$

11. (ii) $r=\dfrac{2}{\sqrt{2}+1}$ 12. θ is maximum when $x=10\sqrt{3}$

13. (ii) $\dfrac{5}{2}-\dfrac{5\sqrt{3}}{6}$ 14. Max $A=hk-\sqrt{3}h^2$

15. $\dfrac{64a^3}{81}$ unit^3 16. $t=4$

17. Min $y = 3.50$

18. (ii) A is minimum when $r=\sqrt[3]{\dfrac{12v}{17\pi}}$ (iii) $0<r<1.06$

19. (ii) $x=\frac{1}{2}$ (iii) $P\left(0,\dfrac{\sqrt{2}}{2}\right)$, $Q\left(0,-\dfrac{\sqrt{2}}{2}\right)$ (iv) $\theta=\frac{\pi}{4}$

20. (ii) $\dfrac{2\sqrt{3}\pi a^3}{27}$ m^3 21. (iii) $2r^2$

22. $x=7.23$ (3 s.f); minimum value of k is 25

23. (ii) $\dfrac{60}{13t^2-80t+400}$ (iii) 3.08s 24. (ii) $-0.0131k$ m/min

25. $\dfrac{\sqrt{3}}{750\pi}a^2$ cms^{-1} 26. (ii) $0.834a$

27. (ii) $x=2.79$, $y=6.19$ (iii) 7.46×10^{-4} m/s

28. (ii) $x=\frac{\sqrt{A}}{2}$ (iv) 0.447 rad/min

6

Integration Skill Set

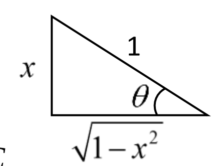
Integration Techniques

s/n	Skills	Examples of questions involving the skills
1.	Recognise the form $[\mathrm{f}(x)]^n\ \mathrm{f}'(x)$ where $n \neq -1$, and use the standard result $\int \left[\mathrm{f}(x)\right]^n \mathrm{f}'(x)\,\mathrm{d}x = \frac{1}{(n+1)}\left[\mathrm{f}(x)\right]^{n+1} + C$ to evaluate the integral.	• Direct application of standard result Lecture notes Example 1(ii): $\int \frac{(\ln x - 1)}{x}\,\mathrm{d}x = \int (\ln x - 1)^1 \left(\frac{1}{x}\right)\mathrm{d}x = \frac{1}{2}(\ln x - 1)^2 + C$ • Rewrite function in $[\mathrm{f}(x)]^n\ \mathrm{f}'(x)$ form Tutorial 6A Q1(a)(ii): $\int x\left(x^2+1\right)^3 \mathrm{d}x = \frac{1}{2}\int 2x\left(x^2+1\right)^3 \mathrm{d}x = \frac{1}{8}\left(x^2+1\right)^4 + C$ Lecture notes Example 1(iii): $\int \frac{t^2}{\sqrt{2t^3-1}}\,\mathrm{d}t = \frac{1}{6}\int \left(2t^3-1\right)^{-\frac{1}{2}}\left(6t^2\right)\mathrm{d}t$ $= \frac{1}{6}\frac{\left(2t^3-1\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$ $= \frac{1}{3}\left(2t^3-1\right)^{\frac{1}{2}} + C$
2.	Recognise the form $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$ and use the standard result $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\left\|\mathrm{f}(x)\right\| + C$ to evaluate the integral.	• Direct application of standard result Lecture notes Example 2(iii): $\int \frac{1}{t\ln t}\,\mathrm{d}t = \int \frac{\frac{1}{t}}{\ln t}\,\mathrm{d}t = \ln\left\|\ln t\right\| + C$ • Rewrite function in $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$ form Lecture notes Example 2(i):

s/n	Skills	Examples of questions involving the skills
		$\int \frac{1}{1-3x}\,\mathrm{d}x = -\frac{1}{3}\int \frac{-3}{1-3x}\,\mathrm{d}x = -\frac{1}{3}\ln\lvert 1-3x\rvert + C$ Lecture notes Example 4(i): $\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x$ $= -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x$ $= -\ln\lvert\cos x\rvert + C$ Tutorial 6A Q3(j): $\int \frac{1}{2+\mathrm{e}^{-x}}\,\mathrm{d}x = \int \frac{1}{2+\frac{1}{\mathrm{e}^{x}}}\,\mathrm{d}x$ $= \int \frac{\mathrm{e}^{x}}{2\mathrm{e}^{x}+1}\,\mathrm{d}x$ $= \frac{1}{2}\int \frac{2\mathrm{e}^{x}}{2\mathrm{e}^{x}+1}\,\mathrm{d}x$ $= \frac{1}{2}\ln(2\mathrm{e}^{x}+1) + C$
3.	Recognise the form $\mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$ and $\mathrm{f}'(x)a^{\mathrm{f}(x)}$, and use the standard results $\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}\,\mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + C$ and $\int \mathrm{f}'(x)a^{\mathrm{f}(x)}\,\mathrm{d}x = \frac{1}{\ln a}a^{\mathrm{f}(x)} + C$ respectively to evaluate the integral.	• Rewrite function in $\mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$ or $\mathrm{f}'(x)a^{\mathrm{f}(x)}$ form Lecture notes Example 5(ii): $\int x\mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{1}{-2}\int(-2x)\mathrm{e}^{-x^2}\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x^2} + C$ Lecture notes Example 5(iii): $\int 3^{1-x}\,\mathrm{d}x = -\int(-1)3^{1-x}\,\mathrm{d}x = -\frac{1}{\ln 3}3^{1-x} + C$

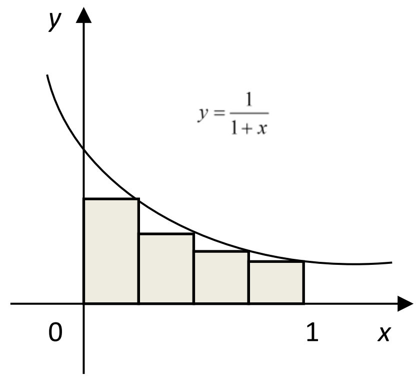
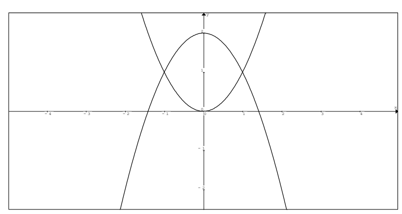
s/n	Skills	Examples of questions involving the skills
4.	Use trigonometry identities.	• Sine and Cosine double-angle formulae Tutorial 6A Q1(d)(i): $\int \sin^2 x \mathrm{d}x = \int \frac{1-\cos 2x}{2} \mathrm{d}x = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$ Tutorial 6A Q1(d)(ii): $\int 3\cos 3x \sin 3x \ \mathrm{d}x = \frac{3}{2}\int \sin 6x \,\mathrm{d}x = -\frac{3}{2}\left(\frac{\cos 6x}{6}\right) + C$ $= -\frac{1}{4}\cos 6x + C$ Further example: $\int \frac{1}{1+\cos x}\mathrm{d}x = \int \frac{1}{2\cos^2 \frac{x}{2}} \mathrm{d}x = \frac{1}{2}\int \sec^2 \frac{x}{2} \mathrm{d}x$ $= \tan \frac{x}{2} + C$ • Sum-to-product (or factor) formulae Tutorial 6A Q1(d)(iii): $\int 7\sin 4x \cos 3x \ \mathrm{d}x = \frac{7}{2}\int 2\sin 4x \cos 3x \ \mathrm{d}x$ $= \frac{7}{2}\int \sin 7x + \sin x \ \mathrm{d}x$ $= -\frac{1}{2}\cos 7x - \frac{7}{2}\cos x + C$ • Identity $1 + \tan^2 x = \sec^2 x$ Example: $\int \tan^2 2\theta \,\mathrm{d}\theta = \int (\sec^2 2\theta - 1)\,\mathrm{d}\theta$ $= \frac{1}{2}\tan 2\theta - \theta + C$

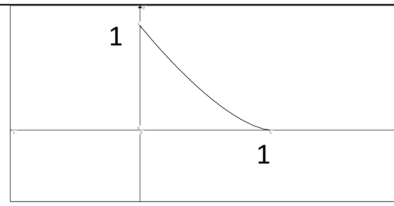
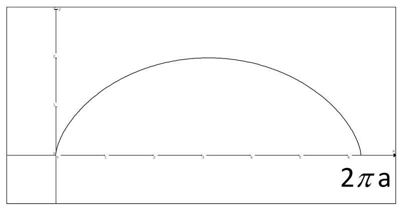
s/n	Skills	Examples of questions involving the skills
5.	Simplify improper rational function by performing long division first before attempting integration.	Tutorial 6A Q3(b): $\int \frac{x^3+x^2-x+5}{x^2+x-2}\mathrm{d}x = \int x+\frac{x+5}{x^2+x-2}\mathrm{d}x$
6.	Use partial fractions to simplify proper rational functions with linear factors in the denominator.	Tutorial 6A Q3(b): $\int \frac{x^3+x^2-x+5}{x^2+x-2}\mathrm{d}x = \int x+\frac{x+5}{x^2+x-2}\mathrm{d}x$ $= \int x+\frac{x+5}{(x+2)(x-1)}\mathrm{d}x$ $= \int x+\frac{-1}{x+2}+\frac{2}{x-1}\mathrm{d}x$
7.	For rational functions of the form $\frac{1}{px^2+qx+r}$, $p<0$ or $p>0$; and $\frac{1}{\sqrt{px^2+qx+r}}$, $p<0$; complete the square for the quadratic expression in the denominator before applying standard integration results found in MF26.	Tutorial 6A Q3(d): $\int \frac{1}{2y^2+4y+5}\mathrm{d}y = \frac{1}{2}\int \frac{1}{y^2+2y+\frac{5}{2}}\mathrm{d}y$ $= \frac{1}{2}\int \frac{1}{(y+1)^2+\frac{3}{2}}\mathrm{d}y$ Tutorial 6A Q3(f): $\int \frac{1}{\sqrt{3-t^2+2t}}\mathrm{d}t = \int \frac{1}{\sqrt{2^2-(t-1)^2}}\mathrm{d}t$
8.	For rational functions of the form $\frac{sx+t}{px^2+qx+r}$ [but not $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$], rewrite the function as $\frac{sx+t}{px^2+qx+r} = A\cdot\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(px^2+qx+r\right)}{px^2+qx+r}+B\cdot\frac{1}{px^2+qx+r}$ before applying standard integration results.	Lecture notes Example 14: $\int \frac{x-1}{x^2+x+1}\mathrm{d}x = \int \frac{\frac{1}{2}(2x+1)}{x^2+x+1}-\frac{\frac{3}{2}}{x^2+x+1}\mathrm{d}x$

s/n	Skills	Examples of questions involving the skills
9.	For integration involving the use of a substitution, change the original variable to the new variable using differentiation and direct substitution before integration. Remember to express final answer in terms of the original variable.	Lecture notes Example 16: $u = 3x+1$ $\frac{\mathrm{d}u}{\mathrm{d}x} = 3$ $\int \frac{x}{(3x+1)^3}\,\mathrm{d}x = \int \frac{\frac{1}{3}(u-1)}{u^3}\left(\frac{1}{3}\mathrm{d}u\right)$ $= \frac{1}{9}\int u^{-2} - u^{-3}\mathrm{d}u$ $= \frac{1}{9}\left(-\frac{1}{u}+\frac{1}{2u^2}\right)+C$ $= \frac{1}{9}\left(-\frac{1}{3x+1}+\frac{1}{2(3x+1)^2}\right)+C$
10.	For substitution involving trigonometry, draw a right angled triangle to obtain other simple trigonometric functions sine, cosine or tangent. This is useful in expressing the answer in terms of the original variable.	Lecture notes Example 17: $x = \sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta$ $\therefore \int \sqrt{1-x^2}\,\mathrm{d}x$ $= \frac{1}{2}\int \cos 2\theta + 1\,\mathrm{d}\theta$ $= \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C$ $= \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C$ $= \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$ 
11.	For integration of definite integrals by substitution, change the limits from values of the original variable to the corresponding values of the new variable.	Tutorial 6A Q4(c): $x = 2\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$

s/n	Skills	Examples of questions involving the skills
		When $x=1,\ 2\sin\theta=1\Rightarrow\theta=\frac{\pi}{6}$ When $x=\sqrt{3},\ 2\sin\theta=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$ $\int_1^{\sqrt{3}}\frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{2\sin\theta}{\sqrt{4-4\sin^2\theta}}2\cos\theta\,\mathrm{d}\theta$
12.	For integration of a single function that cannot be integrated directly, let $v'=1$ and apply integration by parts.	Examples include: $\int 1\cdot\tan^{-1}x\,\mathrm{d}x=x\tan^{-1}x-\int\frac{x}{1+x^2}\,\mathrm{d}x$ $\int 1\cdot\cos^{-1}x\,\mathrm{d}x=x\cos^{-1}x-\int x\left(-\frac{1}{\sqrt{1-x^2}}\right)\mathrm{d}x$ $\int 1\cdot\sin^{-1}x\,\mathrm{d}x=x\sin^{-1}x-\int x\left(\frac{1}{\sqrt{1-x^2}}\right)\mathrm{d}x$ $\int 1\cdot\ln x\,\mathrm{d}x=x\ln x-\int x\left(\frac{1}{x}\right)\mathrm{d}x$
13.	The same integral reappearing on the RHS after integration by parts	Tutorial 6A Q3(g): $\int \mathrm{e}^x\sin x\,\mathrm{d}x$ $=\mathrm{e}^x\sin x-\int\mathrm{e}^x\cos x\,\mathrm{d}x$ $=\mathrm{e}^x\sin x-(\mathrm{e}^x\cos x+\int\mathrm{e}^x\sin x\,\mathrm{d}x)$ $\therefore\ 2\int\mathrm{e}^x\sin x\,\mathrm{d}x=\mathrm{e}^x\sin x-\mathrm{e}^x\cos x+C'$ $\int\mathrm{e}^x\sin x\,\mathrm{d}x=\frac{1}{2}\mathrm{e}^x(\sin x-\cos x)+C$

Applications of Integration

s/n	**Skills**	**Examples of questions involving the skills**
1.	Approximate the area under a curve by the sum of area of rectangles drawn under the curve.	Tutorial 6B Q3 
2.	Find the area of a region bounded by a curve and lines parallel to the coordinate axes	Tutorial 6B Q6 Area bounded by $y = x^2$, the x–axis and the lines $x = 1$, $x = 2$.
3.	Find the area bounded by 2 curves	Tutorial 6B Q2 Area bounded by the 2 curves $y = x^2$ and $y = 2 - x^2$  [Visualise the area bounded before you decide if you use the formula $\pi\int y \, \mathrm{d}x$ or $\pi\int x \, \mathrm{d}y$, or perhaps even need to modify from either of the formulae.]
4.	Find the area under a curve defined parametrically	Tutorial 6B Q4 Find the area under the curve $x = \cos^2 t, \quad y = \sin^3 t, \quad 0 \le t \le \frac{\pi}{2}.$

s/n	Skills	Examples of questions involving the skills
		 Tutorial 6B Q5 Area enclosed by $x=a\left(t-\sin t\right)$, $y=a\left(1-\cos t\right)$ for $0\le t\le 2\pi$. 
5.	Find the volume of revolution about the *x*– or *y*–axis	Tutorial 6B Q9 The region *R* is enclosed by the curve $y=\left(4-x^2\right)^{\frac{-1}{2}}$ and the lines $x=-1$ and $x=1$. (ii) Find the volume generated when R is rotated through 4 right angles about the *x*–axis. (iii) Find the volume generated when R is rotated through two right angles about the y-axis. [Visualise the volume generated before you decide if you need to modify the formula $\pi\int y^2\mathrm{d}x$ (for rotation about *x*-axis) or $\pi\int x^2\mathrm{d}y$ (for rotation about *y*-axis). It may involve addition of volumes generated (e.g. Q2), subtraction of volumes generated (e.g. Q7), involve volume of a cylinder (e.g. Q9), etc.]

6

Integration

1. **(MJC18/Promo/Q2)**

Find the following integrals:

(a) $\displaystyle\int \frac{e^{2x}}{e^{2x}-3}\,dx$ [2]

(b) $\displaystyle\int \frac{3x^2+2}{\sqrt{\left(x^3+2x-8\right)}}\,dx$ [2]

(c) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$ [2]

(d) $\displaystyle\int x^2 \ln x\,dx$ [3]

2. **(VJC16/Promo/Q11)**

(a) (i) Find $\displaystyle\int \frac{2-x}{\sqrt{4x-x^2-1}}\,dx$. [2]

(ii) Find $\displaystyle\int \frac{1}{\sqrt{4x-x^2-1}}\,dx$. [2]

(iii) Using the results of part **(i)** and **(ii)**, find $\displaystyle\int \frac{x}{\sqrt{4x-x^2-1}}\,dx$. [2]

(b) (i) By using the substitution $u=\sqrt{x}$, find $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$. [4]

(ii) Find the exact value of $\displaystyle\int_0^9 \ln\left(1+\sqrt{x}\right)dx$, leaving your answer in the form $a\ln b-\dfrac{c}{b}$, where a, b and c are integers to be determined. [4]

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3. **(NJC18/Promo/Q6)**

(a) Find $\int \tan^{-1} 2x \, \mathrm{d}x$. [3]

(b) Find $\int \frac{\sqrt{1-x}}{x} \, \mathrm{d}x$ by using the substitution $x = \sin^2 u$, where $0 < u \le \frac{\pi}{2}$. [4]

4. **(ACJC17/Promo/Q7)**

(a) **(i)** Find $\int \frac{x-3}{6x - x^2 - 3} \, \mathrm{d}x$. [2]

(ii) Find $\int \frac{1}{6x - x^2 - 3} \, \mathrm{d}x$. [2]

(iii) Using the results of part **(i)** and **(ii)**, find $\int \frac{2x}{6x - x^2 - 3} \, \mathrm{d}x$. [2]

(b) Find $\int x \tan^{-1} x \, \mathrm{d}x$. [3]

(c) Use the substitution $x = (1 + \cos\theta)$, where $0 < \theta < \pi$, to find $\int \frac{x}{\sqrt{2x - x^2}} \, \mathrm{d}x$. [4]

5. **(AJC17/Promo/Q4)**

(a) Find $\int \frac{x^2 + x + 1}{x^2 - x + 1} \, \mathrm{d}x$. [4]

(b) By using the substitution $x = 2\cos u$, find the exact value of $\int_0^1 \sqrt{4 - x^2} \, \mathrm{d}x$. [5]

6. (MJC16/Promo/Q7)

(a) Use the substitution $u=\sqrt{(x+1)}$ to find $\int \frac{x^2}{\sqrt{(x+1)}}\,\mathrm{d}x$. [4]

(b) Find $\int \frac{\mathrm{e}^{5x}}{\left(\mathrm{e}^{5x}-\mathrm{e}\right)^4}\,\mathrm{d}x$. [3]

(c) Show that $\frac{x^2}{1+x^2}=1-\frac{1}{1+x^2}$. [1]

Hence find the exact value of $\int_0^{\sqrt{3}} x\tan^{-1}x\,\mathrm{d}x$. [4]

7. (ACJC16/Promo/Q9)

(a) By using the substitution $x=\frac{1}{t}$, find $\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x$, leaving your answer in exact form. [4]

(b) Show that $\int \frac{x+1}{x^2+4x+7}\,\mathrm{d}x=\frac{1}{2}\ln\left(x^2+4x+7\right)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}}+c$. [4]

Hence find $\int_{-2}^{1} \frac{|x+1|}{x^2+4x+7}\,\mathrm{d}x$ exactly. [3]

8. (DHS17/Promo/Q8(a))

Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan x^2\right)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(\sec x^2)\right)$. Hence evaluate $\int_0^{\frac{1}{2}\sqrt{\pi}} x^3\sec^2(x^2)\,\mathrm{d}x$ exactly, simplifying your answer.

[4]

9. **(ACJC18/Promo/Q10)**

(a) Find $\int \frac{3x^2+3x-2}{(1-x)(1+3x^2)}\,\mathrm{d}x$. [3]

(b) Using the substitution $x = 2(1+\sin^2\theta)$, evaluate $\int_2^{\frac{7}{2}} \sqrt{\frac{x-2}{4-x}}\,\mathrm{d}x$ exactly. [4]

(c) Find $\int \cos 3kx \cos kx \,\mathrm{d}x$, where k is a positive integer.

Hence, given that k is odd, evaluate

$$\int_{\frac{\pi}{2}}^{\pi} x\left(\cos 3kx \cos kx\right)\mathrm{d}x,$$

leaving your answer in the form $\frac{p}{k^2}$, where p is a real constant to be determined. [5]

10. **(AJC18/Promo/Q3)**

(a) Find $\int \cos x \cos \frac{x}{2}\,\mathrm{d}x$ and hence evaluate the exact value of

$$\int_0^{\pi} \left|\cos x\right| \cos\frac{x}{2}\,\mathrm{d}x.$$ [5]

(b) Find exact value of k such that

$$\int_0^{\frac{1}{2k}} \frac{1}{\sqrt{1-k^2x^2}}\,\mathrm{d}x = \int_0^1 \ln\left(2x+1\right)\mathrm{d}x.$$ [7]

11. **(DHS17/Promo/Q4(b))**

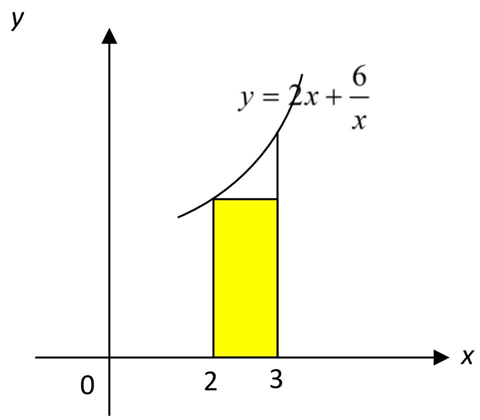
By means of the substitution $x = a\cos\theta$, evaluate $\int \frac{1}{\sqrt{\left(a^2-x^2\right)}}\,\mathrm{d}x$ in terms of a, where a is a positive constant. [3]

12. **(NJC08/Promo/Q13(i))**

By using integration by parts, show that

$\int \mathrm{e}^{-t}\cos nt\,\mathrm{d}t = \frac{\mathrm{e}^{-t}}{1+n^2}\left(n\sin nt - \cos nt\right) + B$, where $n \in \mathbb{Z}^+$ and B is an arbitrary constant. [4]

13. **(ACJC08/Promo/Q11)**



The diagram shows a sketch of part of the graph of $y = 2x + \frac{6}{x}$. By considering the shaded rectangle and the area of the region between the graph and the x-axis for $2 \le x \le 3$, show that $\int_2^3 \left(2x + \frac{6}{x}\right) \mathrm{d}x > 7$. [1]

Show also that $\int_2^3 \left(2x + \frac{6}{x}\right) \mathrm{d}x < 8$. [1]

Hence deduce that $\frac{1}{p} < \ln 1.5 < \frac{1}{q}$, where p and q are positive integers to be determined.

[4]

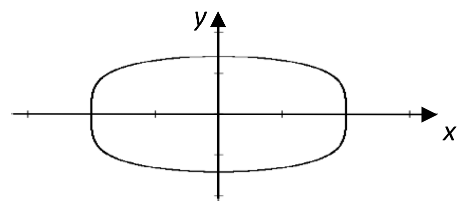
14. **(ACJC16/Promo/Q2)**

Find the area of the region bounded by the curve with equation $x^2 - y^2 = 1$ and the line with equation $y = 7 - 2x$. [4]

15. (AJC16/Promo/Q3)

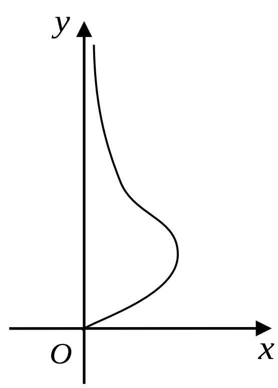
By considering integration by parts, or otherwise, show that

$$\int \sqrt{4-x^2}\ \mathrm{d}x = \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + c.$$ [4]



The diagram shows the curve C with equation $y^4 = 4 - x^2$. The region enclosed by C where $y \geq 0$ is rotated through 360° about the x-axis. Find the exact value of the volume of solid thus formed. [3]

16. (DHS16/Promo/Q7)

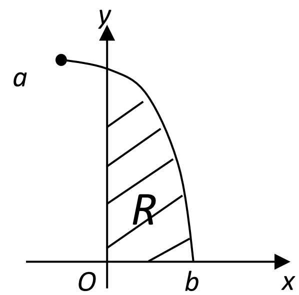


The diagram shows the graph of curve C with equation $y\mathrm{e}^{-y} - \frac{1}{2}x = 0$ for $y \geq 0$. The y-axis is an asymptote to C.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y. Hence find the equation of the tangent to C which is parallel to the y-axis. [4]

Obtain a formula for $\int_0^n x\mathrm{e}^{-x}\ \mathrm{d}x$ in terms of n, where $n > 0$. Hence find the area of the region between C and the positive y-axis. [5]

17. (NJC08/Promo/Q13)



The diagram below shows the curve C with equation given by

$$x = \mathrm{e}^{-t}\cos 2t, \quad y = \sin t,$$

where $\frac{\pi}{2} \le t \le \pi$.

Write down the exact values of a and b. By considering the integral $\int_0^a x \, \mathrm{d}y$, show that the exact area of region R is $\frac{3}{10}\left[\left(\sqrt{2}\right)\mathrm{e}^{-\frac{3\pi}{4}} - \mathrm{e}^{-\pi}\right]$. [7]

18. (MIPU216/Promo/Q12)

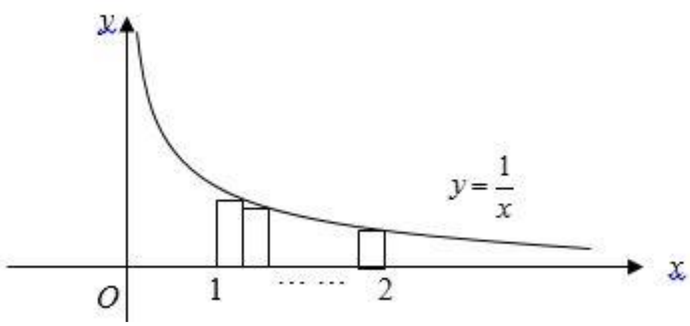
(a) A curve C has parametric equations

$$x = t^3, \quad y = \mathrm{e}^t.$$

(i) Find the exact area of the region bounded by C, the x-axis and the lines $x = 1$ and $x = 8$. [4]

(ii) By obtaining the Cartesian equation of C, find the volume of revolution when the region bounded by C, the y-axis and line $y = 3$ is rotated completely about the y-axis. Give your answer correct to 2 decimal places. [3]

(b) The diagram below shows the curve $y = \frac{1}{x}, x > 0$.



(i) Show that the total area of n rectangles, each of equal width, under the curve between $x = 1$ and $x = 2$ is equal to $\sum_{r=1}^{n} \frac{1}{n+r}$. [2]

(ii) Give an interpretation of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r}$ and find its exact value. [2]

19. (NYJC15/Promo/Q6)

The curve C has equation $y=(x-3)\ln x$. The region R is bounded by the curve C and the x-axis.

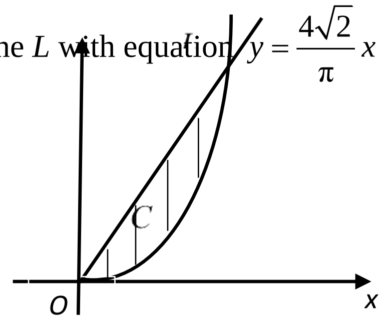
(i) Sketch the curve C, stating clearly the equations of any asymptotes and coordinates of any points of intersection with the axes. [2]

(ii) Find the exact area of R. [5]

(iii) Find the numerical value of the volume of the solid formed when R is rotated completely about the x-axis. [2]

20. (NJC16/Promo/Q7)

The diagram below shows a shaded region which is bounded by the curve C with equation $y=\sec x\tan^3 x$ and the line L with equation $y=\dfrac{4\sqrt{2}}{\pi}x$.



(i) Verify that the point $\left(\dfrac{\pi}{4},\sqrt{2}\right)$ lies on both C and L. [1]

(ii) Find the exact volume of the solid generated by revolving the shaded region about the x-axis through 2π. Express your answer in the form $\dfrac{1}{42}\left(a\pi^2-b\pi\right)$, where a and b are integers. [6]

21. (NJC16/Promo/Q11)

The curve C has parametric equations $x = t + e^t$, $y = 2t$, for $t \in \mathbb{R}$.

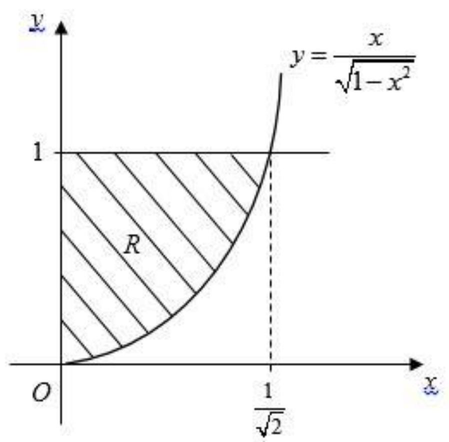
(i) Sketch C, labelling any intersections between C and the axes. [3]

(ii) Show that the tangent to the curve at $t = 1$, denoted by l, has equation $y = \frac{2}{1+e}x$. [3]

(iii) Find the exact value of $\int_{-1}^{1} w\left(1+e^w\right) dw$.

Hence, find the exact area bounded by the curve C, the lines l and $x = -1 + e^{-1}$. [8]

22. (PJC16/Promo/Q8)



The diagram shows the region R bounded by the curve $y = \frac{x}{\sqrt{1-x^2}}$, the line $y = 1$ and the y-axis.

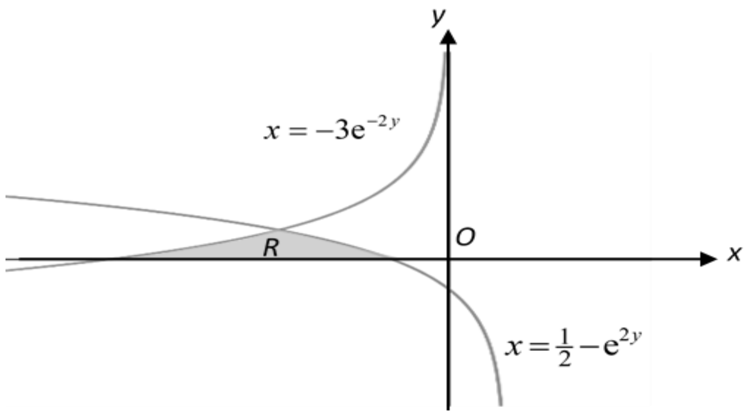
(i) Find the exact area of R. [2]

(ii) Find the numerical value of the volume of revolution obtained when R is rotated completely about the x-axis. Give your answer correct to 4 significant figures. [3]

(iii) Find the exact value of the volume of revolution obtained when R is rotated completely about the y-axis. [5]

23. (DHS19/Promo/Q6)

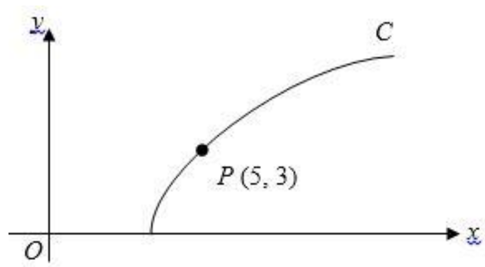
The shaded region R is bounded by the curves $x=-3\mathrm{e}^{-2y}$, $x=\frac{1}{2}-\mathrm{e}^{2y}$ and the x-axis as shown in the diagram below.



(i) Find the exact area of the region R. [5]

(ii) Find the volume of the solid of revolution formed when R is rotated through 4 right angles about the y-axis, giving your answer correct to 2 decimal places. [2]

24. (TPJC16/Promo/Q17)



The diagram shows the curve C with parametric equations

$$x=4t+\frac{1}{t},\quad y=4t-\frac{1}{t},\quad \text{for } t\geq\frac{1}{2}.$$

Region R is bounded by C, the x-axis and the line $x=5$

(i) Show that the area of R is given by $\int_a^b\left(16t-\frac{8}{t}+\frac{1}{t^3}\right)\mathrm{d}t$, where a and b are constants to be determined, and evaluate this integral exactly. [6]

A point P on C has coordinates (5, 3). Region S is bounded by C, the x-axis and the line OP, where O is the origin.

(ii) Given that the Cartesian equation of C is $x^2 - y^2 = 16$, find the volume of revolution when S is rotated completely about the x–axis. [3]

25. NJC18/Promo/1

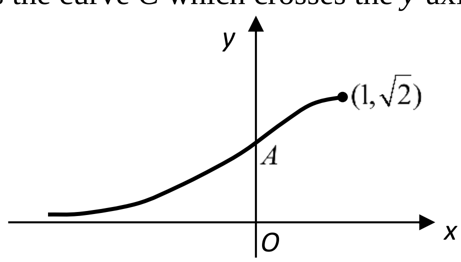
The region R is enclosed by the ellipse with equation $\frac{(x-2)^2}{4} + (y-1)^2 = 1$ and the axes.

Find the volume of revolution obtained when R is rotated through $360°$ about the x-axis. [4]

26. (ACJC18/Promo/Q11)

A curve C has parametric equations $x = 1 + \tan t$ and $y = \sqrt{2}\,\cos t$, for $-\frac{\pi}{2} < t \le 0$.

The diagram below shows the curve C which crosses the y-axis at the point A.



(i) Find $\frac{dy}{dx}$ and hence find the equation of the normal to C at A. [3]

(ii) Find the exact area of the region bounded by the curve C, the line $x = 1$ and the normal to C at A. [4]

(iii) Show that the Cartesian equation of C is given by $y^2 = \frac{a}{1+(x-1)^2}$, $x \le 1$, $x \in \mathbb{R}$, where a is a real constant to be determined. [2]

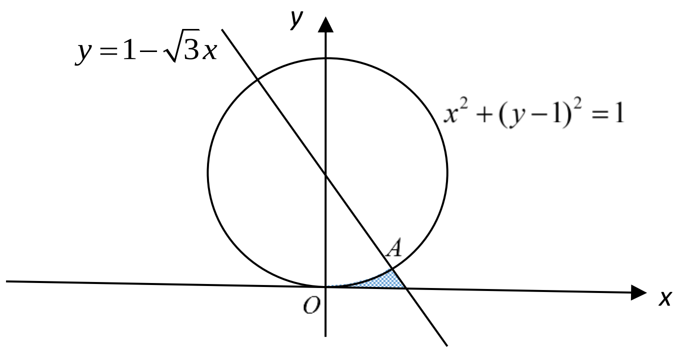
The region bounded by C and the lines $x = 1$ and $y = 1$ is rotated through 2π radians about the y-axis. Hence or otherwise, find the volume of solid formed. [4]

27. AJC18/Promo/Q8

(i) Using the substitution $x = \sin\theta$, show that

$$\int_0^{\frac{1}{2}} \sqrt{1-x^2}\ \mathrm{d}x = \frac{\pi}{12} + \frac{\sqrt{3}}{8}\ . \qquad [4]$$

(ii) The diagram shows the circle $x^2 + (y-1)^2 = 1$ and the line $y = 1 - \sqrt{3}x$.
The circle and the line intersect at point A with coordinates $\left(\frac{1}{2},\ 1-\frac{\sqrt{3}}{2}\right)$.



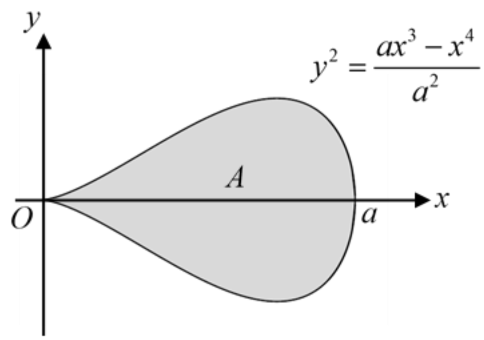
Using the result in part (**i**), find the exact area of the shaded region bounded by the circle, the line and the x-axis. [4]

28. (RVHS Promo 9758/2021/Q11)

A chef is intending to prepare a pear mousse dessert to be served on a leaf-shaped plate.

(a) The leaf-shaped plate has a top surface area of at least 75 cm^2. The diagram below shows the curve C_1 with equation $y^2 = \dfrac{ax^3 - x^4}{a^2}$ for $x \geq 0$ and $a \geq 12$.

The curve is symmetrical about the x-axis with x-intercepts at $(0,0)$ and $(a,0)$. In the diagram, 1 unit represents 1 cm.



The top surface of the plate can be modelled by the region A that is bounded by C_1.

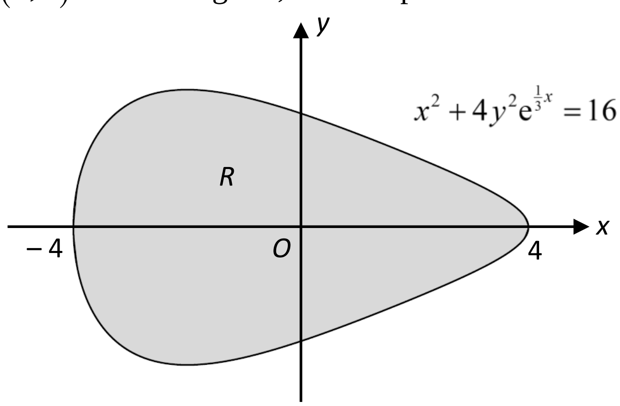
(i) Show that the area of region A is given by the integral

$$\int_0^a \frac{\sqrt{4ax^3 - 4x^4}}{a}\,\mathrm{d}x\,. \qquad [3]$$

(ii) Given that $a = 12$, state the area of region A. [1]

(iii) Find the least integer value of a such that the area of the plate is at least 75 cm^2. [2]

(b) The shape of the pear mousse dessert can be modelled by a solid of revolution. The diagram below shows the curve C_2 with equation $x^2 + 4y^2 e^{\frac{1}{3}x} = 16$. The curve is symmetrical about the x-axis, with x-intercepts at $(-4, 0)$ and $(4, 0)$. In the diagram, 1 unit represents 1 cm.



The region bounded by C_2 is denoted by R. The shape of the pear mousse dessert is obtained by rotating the region R by π radians about the x-axis.

(i) Find $\int x^2 e^{-\frac{1}{3}x} \, dx$. [3]

Hence, or otherwise, find the exact volume of the pear mousse dessert in cm^3. Express your answer in the form $A\pi e^{\frac{4}{3}} + B\pi e^{-\frac{4}{3}}$, where A and B are constants to be determined. [5]

Answers

1(a) $\frac{1}{2}\ln\left|e^{2x}-3\right|+C$

(b) $2\left(x^3+2x-8\right)^{\frac{1}{2}}+C$

(c) $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$

(d) $\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$

2(a)(i) $\sqrt{4x-x^2-1}+C$

(ii) $\sin^{-1}\left(\frac{x-2}{\sqrt{3}}\right)+C$

(iii) $2\sin^{-1}\left(\frac{x-2}{\sqrt{3}}\right)-\sqrt{4x-x^2-1}+C$

(b)(i) $x-2\sqrt{x}+2\ln\left(1+\sqrt{x}\right)+C$

(ii) $16\ln 2-\frac{3}{2}$

3(a) $x\tan^{-1}2x-\frac{1}{4}\ln\left(1+4x^2\right)+c$

(b) $-2\ln\left(\frac{1}{\sqrt{x}}+\sqrt{\frac{1-x}{x}}\right)+2\sqrt{1-x}+c$

4(a)(i) $-\frac{1}{2}\ln\left|6x-x^2-3\right|+C$

(ii) $\frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{6}+x-3}{\sqrt{6}-x+3}\right|+C$

(iii) $-\ln\left|6x-x^2-3\right|+\frac{3}{\sqrt{6}}\ln\left|\frac{\sqrt{6}+x-3}{\sqrt{6}-x+3}\right|+C$

(b) $\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+C$

(c) $-\cos^{-1}(x-1)-\sqrt{x(2-x)}+C$

5(a) $x+\ln\left|x^2-x+1\right|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$ **(b)** $\frac{\pi}{3}+\frac{\sqrt{3}}{2}$

6(a) $\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{4}{3}(x+1)^{\frac{3}{2}}+2\sqrt{x+1}+C$ **(b)** $-\frac{1}{15}\left(e^{5x}-e\right)^{-3}+C$

(c) $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$

7(a) $\frac{\pi}{12}$ **(b)** $\ln\left(\frac{3}{2}\right)$

8 $2x\sec^2(x^2)$; $2x\tan x^2$; $\frac{1}{8}\pi-\frac{1}{4}\ln 2$

9(a) $-\ln|1-x|-\sqrt{3}\tan^{-1}\sqrt{3}x+C$ **(b)** $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$

(c) $\frac{1}{4k}\left[\frac{\sin 4kx}{2}+\sin 2kx\right]+C$; $p=\frac{1}{4}$

10(a) $\frac{1}{3}\sin\frac{3x}{2}+\sin\frac{x}{2}+c$; $\frac{4\sqrt{2}}{3}-\frac{2}{3}$ **(b)** $k=\frac{\pi}{9\ln 3-6}$

11 $-\cos^{-1}\left(\frac{x}{a}\right)+C$

13 $p=3,\ q=2$ **14** 13.8 square units

15 $2\pi^2$

16(i) $\frac{dy}{dx}=\frac{e^y}{2(1-y)}$; $x=\frac{2}{e}$ **(ii)** $-ne^{-n}-e^{-n}+1$; 2

17(ii) $a=\frac{1}{\sqrt{2}}$, $b=e^{-\pi}$

18(a)(i) $3e(2e-1)\text{units}^2$ **(ii)** $x=(\ln y)^3$; 2.28 units^3

(b)(ii) $\ln 2$

19(ii) $\frac{9}{2}\ln 3-4$ **(iii)** 0.534π or 1.68

20(ii) $\frac{1}{42}\left(7\pi^2-6\pi\right)$, where $a=7, b=6$

21(iii) $\frac{2}{e}$; $1+e-\frac{\left(-1+e^{-1}\right)^2}{1+e}-\frac{4}{e}$

22(i) $\left(\sqrt{2}-1\right)\text{units}^2$ **(ii)** 1.674

(iii) $\pi\left(1-\frac{\pi}{4}\right)$ **23** (i) $\frac{\ln 2+1}{4}$ (ii) 4.24

24(i) $a=\frac{1}{2}$, $b=1$; $\frac{15}{2}-8\ln 2$ **(ii)** $\frac{32}{3}\pi$ or 33.5

25. 0.602

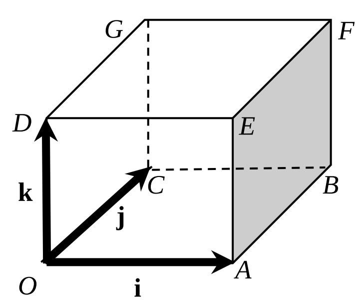
26(i) $-\sqrt{2}\sin t\cos^2 t$; $y=-2x+1$ **(ii)** $-\sqrt{2}\ln\left|\sqrt{2}-1\right|$ units^2 **(iii)** $a=2$; 1.01 units^3

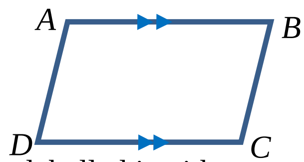
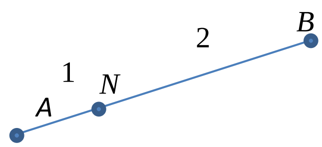
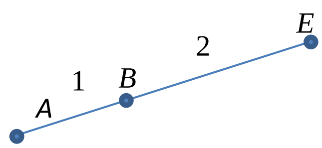
27(ii) $\frac{1}{2\sqrt{3}}-\frac{\pi}{12}$

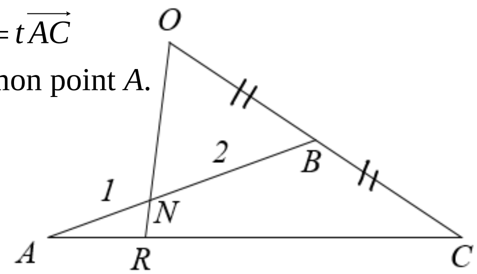
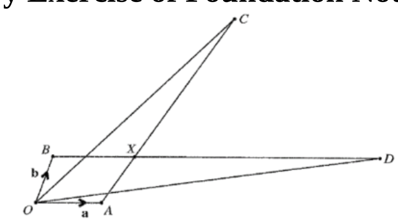
28. 0.602

7

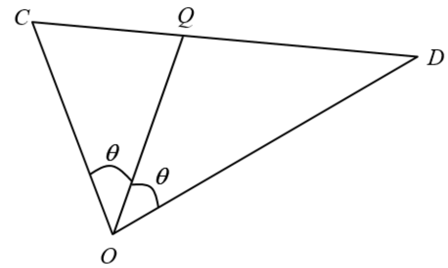
Vectors Skill Set

No.	Skill	Examples of questions involving the skill
1	**(a)** Finding **magnitude** of a vector	**Tutorial 7A Q2a** Finding the length of the line segment *AB* $\lvert \underset{\sim}{a} \rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$ where $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$
	(b) Unit Vector	We need **magnitude of a vector** to find **the unit vector** of **a**, i.e. $\hat{\underset{\sim}{a}}$, where $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{\lvert \underset{\sim}{a} \rvert}$. **Tutorial 7A Q6(ii)+(iii), Q7(iii)+(iv) and Q8(iii)** An application for the unit vector would be to find the **length of projection** and the **projection vector**. **Tutorial 7B Q8** Finding $\overrightarrow{ST} = (\text{magnitude})(\text{unit vector})$
	(c) Stating the **position vectors** of points in a 3-D diagram	**Tutorial 7A Q1**  $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\overrightarrow{OF} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ Please note that you may be asked for **coordinates** in some other questions: E.g. $A(1,0,0)$, $B(1,1,0)$, $C(1,1,1)$

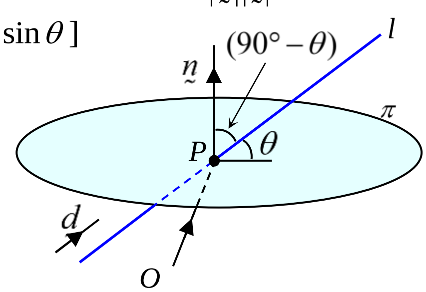
No.	Skill	Examples of questions involving the skill
2	Showing that **2 vectors are equal** requires concept of **parallel vectors** and vectors with **same magnitude** Using **equal vectors** to **show that *ABCD* is a parallelogram**	**Notes Examples 1(a) & 2** To show that 2 non-zero vectors, say $\underset{\sim}{p}$ and $\underset{\sim}{q}$, are equal, we need to show that $\underset{\sim}{p} \mathbin{//} \underset{\sim}{q}$ (i.e. $\underset{\sim}{p} = k\underset{\sim}{q}$) and that $\lvert \underset{\sim}{p} \rvert = \lvert \underset{\sim}{q} \rvert$. **Tutorial 7A Q2b** To show that *ABCD* is a parallelogram, we need to show that $\overrightarrow{AB} = \overrightarrow{DC}$.  [Note that the vertices in a polygon must be labelled in either clockwise or anticlockwise manner.]
3	Using **Ratio Theorem** to find the position vector of a point lying on a line	**Tutorial 7A Q3, 2c, 1e,** For example, if the point *N* on *AB* is such that *AN* : *NB* = 1 : 2, then  $\overrightarrow{ON} = \dfrac{2\overrightarrow{OA} + \overrightarrow{OB}}{3}$ Note that if the point, say *E*, lies on *AB* **produced** such that $\overrightarrow{AE} = 3\overrightarrow{AB}$ then the point *E* lies outside of the line *AB* as shown below  $\overrightarrow{AE} = 3\overrightarrow{AB} \Rightarrow \dfrac{AE}{AB} = \dfrac{3}{1}$ Then *AB* : *BE* = 1 : 2 And we have $\overrightarrow{OB} = \dfrac{2\overrightarrow{OA} + \overrightarrow{OE}}{3}$ **Tutorial 7A Q1d** To find position vector of the **midpoint of *BD***, *M*, use $\overrightarrow{OM} = \dfrac{\overrightarrow{OB} + \overrightarrow{OD}}{2}$. Note that $\overrightarrow{OM} \neq \dfrac{1}{2}\overrightarrow{BD}$. In fact, $\overrightarrow{BM} = \dfrac{1}{2}\overrightarrow{BD}$
4	**Collinear points**	**Tutorial 7A Q3(ii), (iii)** Point *P* lies on *BA* produced. Hence *AP* is parallel to *AB*. Since *A* is the common point, points *A*, *B* and *P* are collinear. $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b} - \underset{\sim}{a} = (1-\lambda)(\underset{\sim}{b} - \underset{\sim}{a}) = (1-\lambda)\overrightarrow{AB}$ **Note: Do not divide two vectors** i.e. for $\overrightarrow{AP} = (1-\lambda)\overrightarrow{AB}$, do not write $\dfrac{1}{1-\lambda} \neq \dfrac{\overrightarrow{AB}}{\overrightarrow{AP}}$. Instead write $\dfrac{1}{1-\lambda} = \dfrac{AB}{AP}$.

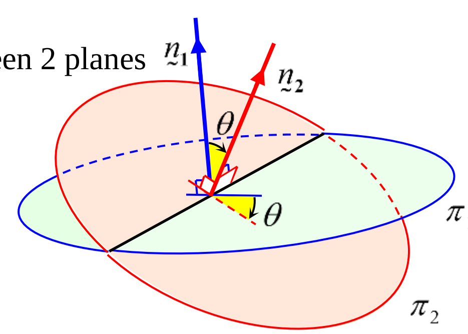
No.	Skill	Examples of questions involving the skill
		Tutorial 7A Q3(ii) *A*, *R* and *N* are collinear, $\overrightarrow{AR} = t\overrightarrow{AC}$ $\overrightarrow{AR} // \overrightarrow{AC}$ and there is a common point *A*.  Note that *R* is actually the point of intersection of the lines *ON* and *AC*. Hence it is easier to solve using equations of lines in **Tutorial 7B, Q10(i)** **Tutorial 7B Q11(ii)** For the case when $b = 1$, find the position vectors of the points *P* on l_1 and *Q* on l_2 such that *O*, *P* and *Q* are collinear, where *O* is the origin.
5	**Non-Parallel Vectors**	**Supplementary Exercise of Foundation Notes Q28**  With reference to the origin *O*, the points A, B, C and D are such that $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = 2\underset{\sim}{a} + 4\underset{\sim}{b}$ and $\overrightarrow{OD} = \underset{\sim}{b} + 5\underset{\sim}{a}$. The lines BD and AC cross at X (see diagram). **(i)** Using $\overrightarrow{AX} = \lambda\overrightarrow{AC}$ where λ is a constant, show that $\overrightarrow{OX} = (\lambda + 1)\underset{\sim}{a} + 4\lambda\underset{\sim}{b}$. **(ii)** Using $\overrightarrow{BX} = \mu\overrightarrow{BD}$ where μ is a constant, find $\overrightarrow{OX}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and μ. **(iii)** Hence find $\overrightarrow{OX}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Note that *X* is actually the point of intersection of the lines *BD* and *AC*. Hence it is easier to solve using equations of lines as in **Tutorial 7B, Q10**
6	**(a)** Applying **definition of the dot (scalar) product** $\underset{\sim}{a} \cdot \underset{\sim}{b} = \lvert\underset{\sim}{a}\rvert \lvert\underset{\sim}{b}\rvert \cos\theta$ to find angle between vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ or	**Tutorial 7A Q4** When finding angle between 2 vectors, do not include the modulus. As seen in Tutorial 7A Q4(b), the dot product of two vectors is negative since the angle between them is obtuse. **Hence we include modulus only when we need to find the acute angle.**

<table>
<tr><th>No.</th><th>Skill</th><th>Examples of questions involving the skill</th></tr>
<tr><td></td><td>
acute angle between 2 lines with direction vectors $\underset{\sim}{d}_1$ and $\underset{\sim}{d}_2$
or acute angle between a line with direction vectors $\underset{\sim}{d}$ and a plane with normal vector $\underset{\sim}{n}$
or acute angle between 2 planes with normal vectors $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$
$\cos\theta = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|}$ OR
$\cos\theta = \left|\dfrac{\underset{\sim}{d}_1\cdot\underset{\sim}{d}_2}{\left|\underset{\sim}{d}_1\right|\left|\underset{\sim}{d}_2\right|}\right|$ OR
$\cos(90^\circ-\theta) = \dfrac{\left|\underset{\sim}{d}\cdot\underset{\sim}{n}\right|}{\left|\underset{\sim}{d}\right|\left|\underset{\sim}{n}\right|}$ OR
$\cos\theta = \dfrac{\left|\underset{\sim}{n}_1\cdot\underset{\sim}{n}_2\right|}{\left|\underset{\sim}{n}_1\right|\left|\underset{\sim}{n}_2\right|}$
</td><td>
Tutorial 7B Q9(i)
Find the acute angle between a vector and a line

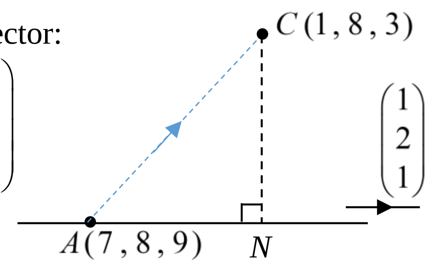
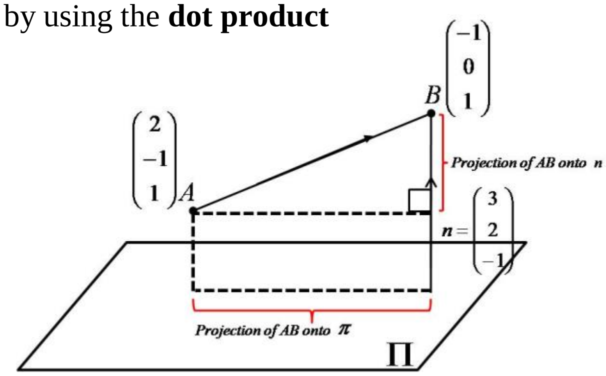
Tutorial 7B Q4, Q5(ii), Q7(ii), Q11(i)
Finding angle between 2 lines or a line and axis
Supplementary Exercise Q6 and Q27
Finding angle bisector.
Bisecting the angle may not necessarily result in equidistance on the opposite side. i.e. $CQ \neq QD$


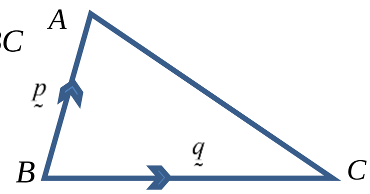
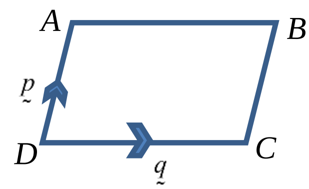
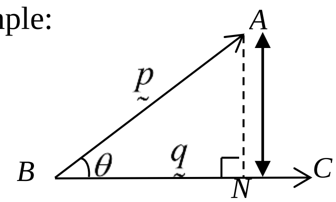
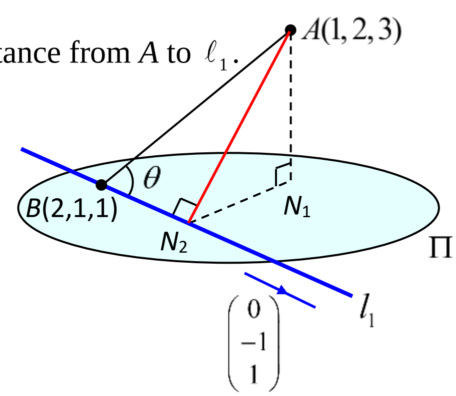
Tutorial 7A Q5(ii)
$\underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \Leftrightarrow \underset{\sim}{a}\perp\underset{\sim}{b}\ \ (\because \cos 90^\circ = 0)$

Tutorial 7C Q12(ii)
Finding the acute angle between line and plane
$\cos(90^\circ-\theta) = \dfrac{\left|\underset{\sim}{d}\cdot\underset{\sim}{n}\right|}{\left|\underset{\sim}{d}\right|\left|\underset{\sim}{n}\right|}$ or $\sin\theta = \dfrac{\left|\underset{\sim}{d}\cdot\underset{\sim}{n}\right|}{\left|\underset{\sim}{d}\right|\left|\underset{\sim}{n}\right|}$
[since $\cos(90^\circ-\theta) = \sin\theta$]


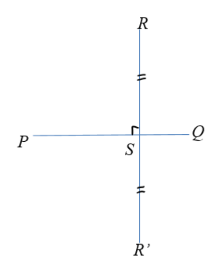
Tutorial 7C Q4(c), 7(i)
Finding the acute angle between 2 planes
$\theta = \cos^{-1}\dfrac{\left|\underset{\sim}{n}_1\cdot\underset{\sim}{n}_2\right|}{\left|\underset{\sim}{n}_1\right|\left|\underset{\sim}{n}_2\right|}$


</td></tr>
</table>

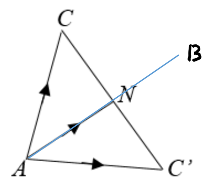
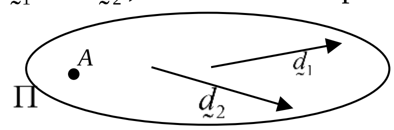
No.	Skill	Examples of questions involving the skill
	(b) Computing dot product $\underset{\sim}{a}\cdot\underset{\sim}{b}=a_1b_1+a_2b_2+a_3b_3$ where $\underset{\sim}{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\underset{\sim}{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$	**Tutorial 7A Q6, 7** (1) Finding the length of projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ = $\left\|\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}\right\|$ (2) Finding the projection vector of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ = $\left(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$ **Tutorial 7A Q10(ii)** (3) $\underset{\sim}{a}\bullet\underset{\sim}{a}=\left\|\underset{\sim}{a}\right\|^2\ (\because \cos 0°=1)$ $(3\underset{\sim}{b}-2\underset{\sim}{c})\bullet(3\underset{\sim}{b}-2\underset{\sim}{c})=\left\|3\underset{\sim}{b}-2\underset{\sim}{c}\right\|^2$ $(3\underset{\sim}{b}-2\underset{\sim}{c})\bullet(3\underset{\sim}{b}-2\underset{\sim}{c})\neq(3\underset{\sim}{b}-2\underset{\sim}{c})^2$ **Tutorial 7B Q12(ii)** Method 2 – Using projection vector: $\overrightarrow{AN}=\left(\overrightarrow{AC}\bullet\hat{\underset{\sim}{d}}\right)\hat{\underset{\sim}{d}}$ where $\underset{\sim}{d}=\begin{pmatrix}1\\2\\1\end{pmatrix}$  **Tutorial 7C Q10** Notice that we can only find the **length of projection** of $\overrightarrow{AB}$ **onto the normal** of π by using the **dot product** i.e. $\left\|\overrightarrow{AB}\bullet\hat{\underset{\sim}{n}}\right\|$.  We use **cross product** when finding the length of projection of $\overrightarrow{AB}$ onto π = $\left\|\overrightarrow{AB}\times\hat{\underset{\sim}{n}}\right\|$
	(c) Geometrical meaning of $\left\|\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}\right\|$	**Tutorial 7A Q8(iii)** It is the length of projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$.
7	**(a)** Applying **definition of the cross (vector)** product $\underset{\sim}{a}\times\underset{\sim}{b}=\left\|\underset{\sim}{a}\right\|\left\|\underset{\sim}{b}\right\|\sin\theta\hat{\underset{\sim}{n}}$	**Tutorial 7A Q5(i)** (1) Finding the vector that is perpendicular to both non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. If $\underset{\sim}{a}\times\underset{\sim}{b}=\underset{\sim}{c}$, then $\underset{\sim}{c}\perp\underset{\sim}{a}$ and $\underset{\sim}{c}\perp\underset{\sim}{b}$. **Note:** To check the accuracy of $\underset{\sim}{c}$, we can check that $\underset{\sim}{c}\bullet\underset{\sim}{a}=0$ and $\underset{\sim}{c}\bullet\underset{\sim}{b}=0$.

No.	Skill	Examples of questions involving the skill
	(b) Computing cross (vector) product $\underset{\sim}{a}\times\underset{\sim}{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ $=\begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix}$	**Tutorial 7A Q6(i), Q8(iv)** (2) Finding the area of a triangle *ABC* $\frac{1}{2}\left\|\underset{\sim}{p}\times\underset{\sim}{q}\right\|$  **Tutorial 7A Q7(i)** (3) Finding the area of a parallelogram *ABCD* $\left\|\underset{\sim}{p}\times\underset{\sim}{q}\right\|$  **Tutorial 7A Q9** (4) Finding cross products of vectors $\underset{\sim}{m}\times\underset{\sim}{n}=[\lambda\underset{\sim}{a}+(1-\lambda)\underset{\sim}{b}\,]\times[2(1-\lambda)\underset{\sim}{a}-\lambda\underset{\sim}{b}\,]$ **Tutorial 7B Q6** (5) Finding the perpendicular (shortest) distance from a point *A* to a line *BC* is $\left\|\underset{\sim}{p}\times\hat{\underset{\sim}{q}}\right\|$ Example:  **Tutorial 7C Q5c** (6) Find the perpendicular distance from *A* to ℓ_1. $BN=\left\|\overrightarrow{BA}\bullet\hat{\underset{\sim}{d}}\right\|$ $AN=\left\|\overrightarrow{BA}\times\hat{\underset{\sim}{d}}\right\|$  Note the ⊥ distance from *A* to plane **<u>is different</u>** from the ⊥ distance from *A* to line, though line is on the plane.

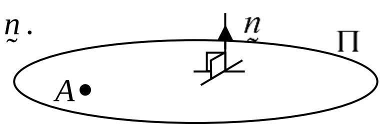
No.	Skill	Examples of questions involving the skill
	(c) Geometrical Meaning of $\left\lvert \underset{\sim}{a} \times \underset{\sim}{p} \right\rvert$	**Tutorial 7A Q8(iv)** $\left\lvert \underset{\sim}{a} \times \underset{\sim}{p} \right\rvert$ is the area of the parallelogram with sides *OA* and *OP*
8	Find **(a)** the **vector equation** of a line $l: \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d},\ \lambda \in \mathbb{R}$ **(b)** the **Cartesian equation** of a line $\dfrac{x-a_1}{d_1} = \dfrac{y-a_2}{d_d} = \dfrac{z-a_3}{d_3}$	**Tutorial 7B Q1(b)** The line passing through (2, 1, 0) and (1, 1, −3) $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ and $\underset{\sim}{d} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ Vector equation: $l: \underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix},\ \lambda \in \mathbb{R}$ Cartesian equation: $\dfrac{x-2}{1} = \dfrac{z}{3},\ y = 1$
9	Determine if a point lies on a line	**Tutorial 7B Q2, 4** Given that the coordinates of the point is (3, 2, 1). <u>Method 1: Using vector equation of the line</u> Say $l: \underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},\ \lambda \in \mathbb{R}$, then we check if there exist a value of λ such that $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. <u>Method 2: Using Cartesian equation</u>, Say $x - 2 = 3 - y = \dfrac{z+1}{2}$, then we substitute 3, 2 and 1 into *x*, *y* and *z* respectively and check if it holds.
10	Solving a system of 3 simultaneous equations to determine **if 2 lines intersect** or they are **skew lines** **Note:** We need to check whether the 2 lines are parallel to each other , i.e. $\underset{\sim}{b} = k\underset{\sim}{d}$. Only when l_1 is not parallel to l_2 then we proceed to check if l_1 and l_2 are intersecting or skew lines.	**Tutorial 7B Q3(a)** <u>Step 1</u> If we have Cartesian equations of a line, we first convert them to vector equations. <u>Step 2</u> We equate the vector equations of the two lines: $l_1: \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b},\ \lambda \in \mathbb{R}$ and $l_2: \underset{\sim}{r} = \underset{\sim}{c} + \mu \underset{\sim}{d},\ \mu \in \mathbb{R}$ to obtain $\underset{\sim}{a} + \lambda \underset{\sim}{b} = \underset{\sim}{c} + \mu \underset{\sim}{d}$ <u>Step 3</u> We solve the system of 3 simultaneous equations for λ and μ. If there is no solution, then we conclude that the 2 equations **are skew lines.** If we can find a value of λ and μ, then we can substitute back to $\underset{\sim}{a} + \lambda \underset{\sim}{b}$ or $\underset{\sim}{c} + \mu \underset{\sim}{d}$ to find the point of intersection.

No.	Skill	Examples of questions involving the skill
		Tutorial 7B Q3(b)(iii) In this example, students are guided to finding the "common perpendicular" that meets two perpendicular lines (non-parallel and do not intersect i.e. l_1 and l_2 are skew lines). This common perpendicular is in fact the shortest distance between the two skew lines l_1 and l_2.
11	Find the foot of perpendicular from a point to the line	**Tutorial 7B Q12(ii)** Consider a line l with equation $l_1 : \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}, \lambda \in \mathbb{R}$. A is a point on l with position vector $\underset{\sim}{a}$, P is a point not on l and N is the foot of the perpendicular from P to l. <u>Method 1: using dot product of perpendicular vectors</u> Since N lies on l, $\therefore$ $\overrightarrow{ON} = \underset{\sim}{a} + \lambda \underset{\sim}{d}$ for some $\lambda \in \mathbb{R}$. … (1) Express $\overrightarrow{PN} = \overrightarrow{ON} - \overrightarrow{OP}$ in terms of λ. Since $\overrightarrow{PN} \perp l$, $\therefore$ $\overrightarrow{PN} \cdot \underset{\sim}{d} = 0$ … (2) Solve (2) for λ. Substitute value of λ into (1) to obtain $\overrightarrow{ON}$. <u>Method 2: using projection vector</u> Select a point on l. We use the point A with position vector $\overrightarrow{OA} = \underset{\sim}{a}$ since it is given. Find $\overrightarrow{AP}$ using $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$. Find $\overrightarrow{AN}$, the projection vector of $\overrightarrow{AP}$ onto l, using $\overrightarrow{AN} = (\overrightarrow{AP} \cdot \hat{\underset{\sim}{d}})\, \hat{\underset{\sim}{d}}$. Obtain $\overrightarrow{ON}$ using $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$.
12	Using equal vectors or Ratio Theorem to **find the point of reflection in the line**	**Tutorial 7B Q8** Find coordinates of R', the point obtained when R is reflected in the line PQ. Find S, then use $\overrightarrow{RS} = \overrightarrow{SR'}$ (i.e. equal vectors). Alternatively, use Ratio Theorem $\overrightarrow{OS} = \frac{1}{2}\overrightarrow{OR} + \frac{1}{2}\overrightarrow{OR'}$ 

<table>
<tr><th>No.</th><th>Skill</th><th>Examples of questions involving the skill</th></tr>
<tr><td></td><td>Find the equation of line reflected in another line</td><td>Tutorial 7B Q13
Find the Cartesian equation of the line which is a reflection of the line AC in the line AB.

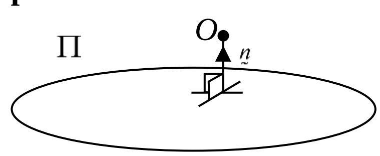
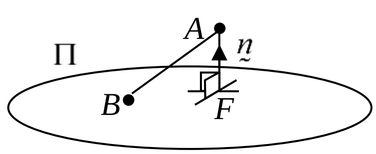
“reflection of line AC with line AB” means there is a point C’ which is a reflection of C in the line AB and find the vector equation of AC’

</td></tr>
<tr><td>13</td><td>Defining equations of planes in 3 forms:
(a) Parametric form
(b) Scalar Product form
(c) Cartesian form
and convert from one form to another</td><td>Tutorial 7C Q1, 2
(a) Parametric form
We need a fixed point A, lying on the plane Π, and 2 non-parallel vectors $\underset{\sim}{d}_1$ and $\underset{\sim}{d}_2$, which are both parallel to Π.


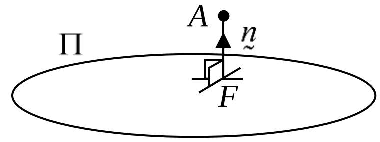
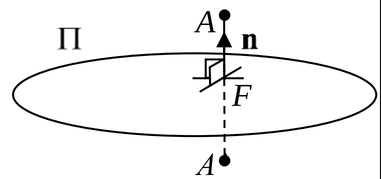
We have $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d}_1 + \mu \underset{\sim}{d}_2, \lambda, \mu \in \mathbb{R}$ and $\underset{\sim}{a}$ is the position vector of the point A with respect to the origin.

(b) Scalar Product form
We need a fixed point A lying on the plane Π, and the normal vector to Π, denoted by $\underset{\sim}{n}$.
We have $\underset{\sim}{r} \bullet \underset{\sim}{n} = p$ where $p = \underset{\sim}{a} \bullet \underset{\sim}{n}$.


(c) Cartesian form
We let $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\underset{\sim}{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ and we have
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = p \Rightarrow n_1 x + n_2 y + n_3 z = p$.

<u>* Note 1</u>
We need to know how to convert from one form to another, i.e.
Parametric form $\leftrightarrow$ Scalar product form $\leftrightarrow$ Cartesian form
$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d}_1 + \mu \underset{\sim}{d}_2, \lambda, \mu \in \mathbb{R} \leftrightarrow \underset{\sim}{r} \bullet \underset{\sim}{n} = p$ where $\underset{\sim}{n} = \underset{\sim}{d}_1 \times \underset{\sim}{d}_2 \leftrightarrow$
$n_1 x + n_2 y + n_3 z = p$
<u>* Note 2</u>
$\underset{\sim}{a}$, $\underset{\sim}{d}_1, \underset{\sim}{d}_2$ and $\underset{\sim}{n}$ are not unique.</td></tr>
</table>

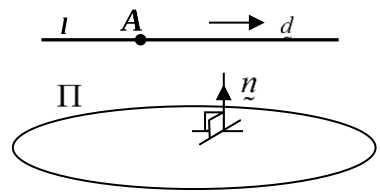
No.	Skill	Examples of questions involving the skill
14	Relationship between a point A and a plane Π	**To check if the point A lies on the plane Π** **Tutorial 7C Q11(ii)** All we need to do is to substitute the position vector of the point A to the equation of the plane Π. If the equation holds, A lies on Π. Otherwise, A doesn't lie on Π. Example: To verify that $A(2, 0, 1)$ lies on $\Pi_1 : \underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1$, we substitute $A(2, 0, 1)$ into Π_1 and we have $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1$.
		To find the perpendicular (shortest) distance between A & Π **Tutorial 7C Q8(ii)** (a) If point A is the **point of origin**, O, then the perpendicular distance between O and Π can be found by dividing both sides of the vector equation of a plane (in scalar product form i.e $\underset{\sim}{r} \bullet \underset{\sim}{n} = p$) by the magnitude of the normal vector, $\lvert \underset{\sim}{n} \rvert$, i.e. $\underset{\sim}{r} \bullet \frac{\underset{\sim}{n}}{\lvert \underset{\sim}{n} \rvert} = \frac{p}{\lvert \underset{\sim}{n} \rvert}$ where $\left\lvert \frac{p}{\lvert \underset{\sim}{n} \rvert} \right\rvert$ is the **perpendicular distance** between O and Π.  **Tutorial 7C Q5(b)** (b) To find the **perpendicular distance** between a point A and the plane Π, is similar to finding **the length of projection** of BA on $\underset{\sim}{n}$, i.e. $\left\lvert \overrightarrow{BA} \bullet \hat{\underset{\sim}{n}} \right\rvert$, where B is a point that lies on the plane Π.  We find the position vector of the point B by observation. For example, equation of the plane Π is given as $\underset{\sim}{r} \bullet \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = -1$. By observation, $B(0, 0, 1)$ lies on Π.

No.	Skill	Examples of questions involving the skill
		To find the foot of perpendicular, *F*, from a point *A* to Π  **Tutorial 7C Q5(f), Q12(iv)** <u>Method 1</u> Finding the foot of perpendicular is equivalent to finding the point of intersection of the line *AF* and the plane Π. <u>Method 2</u> We will find the projection vector of $\overrightarrow{BA}$ onto $\underset{\sim}{n}$, where *B* is a point that lies on the plane Π, to obtain $\overrightarrow{FA}$ and hence find $\overrightarrow{OF}$.
		To find the point of reflection of a point *A* in a plane Π. **Tutorial 7C Q11(iii)**  We can find *A'* by first finding the foot of perpendicular of *A* on Π, *F*, then use ratio theorem $\overrightarrow{OF}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ to find $\overrightarrow{OA'}$.
		To determine if 2 points are on the same side of a plane Π. The equation of a plane Π is given by $\mathbf{r}\bullet\underset{\sim}{n}=p$. We substitute the position vectors of the 2 points into $\mathbf{r}\bullet\underset{\sim}{n}=p$ and observe the value of *p*. <u>Case 1</u> If one point gives a value < *p* and the other point gives a value > *p*, then the 2 points are on opposite sides of the plane. <u>Case 2</u> If both points gives a value that are both < *p* or both > *p*, then both points are on the same side of the plane. For example, given that the equation of the plane is $\underset{\sim}{r}\bullet\begin{pmatrix}1\\2\\-2\end{pmatrix}=3$, we want to find out if the point $A(1,3,1)$ and the point $B(0,0,2)$ are on the same side of the plane, we will do the following: $\begin{pmatrix}1\\3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\2\\-2\end{pmatrix}=5>3$ and $\begin{pmatrix}0\\0\\2\end{pmatrix}\bullet\begin{pmatrix}1\\2\\-2\end{pmatrix}=-4<3$ Since one is greater than 3 and the other is less than 3, point *A* and *B* lie on opposite sides of the plane.

<table>
<tr><th>No.</th><th>Skill</th><th>Examples of questions involving the skill</th></tr>
<tr><td rowspan="2">15</td><td rowspan="2">Relationship between a line l and a plane Π</td><td>

Given a line *l* and a plane Π, 3 scenarios can happen:

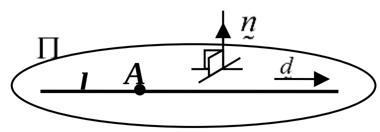
Tutorial 7C Q10
(a) Scenario 1: *l* is parallel to Π but doesn't lie on Π



If *l* is parallel to Π, then $\underset{\sim}{d}$ is perpendicular to **n**. Hence to show that *l* is parallel to Π, we just need to show that $\underset{\sim}{d} \cdot \underset{\sim}{n} = 0$.

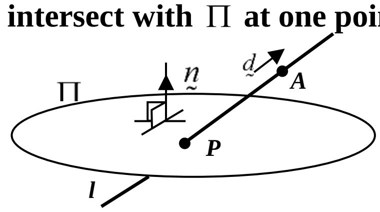
To show that *l* doesn't lie on Π, we just need to show that the point *A* on *l* doesn't lie on Π, i.e. the equation doesn't hold if we substitute the position vector of *A* into the equation of Π.

Tutorial 7C Q5(a)
(b) Scenario 2: *l* is contained in Π



We just need to show that $\underset{\sim}{d} \cdot \underset{\sim}{n} = 0$ and that the point *A* on *l* also lies on Π (there are infinite number of solutions).

(c) Scenario 3: *l* intersect with Π at one point



If *l* is not parallel to Π, i.e. $\underset{\sim}{d} \cdot \underset{\sim}{n} \neq 0$, then *l* intersects Π at one point.

</td></tr>
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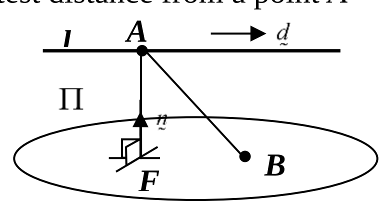
To find the shortest distance between a line *l* and a plane Π.

Tutorial 7C Q10
This is equivalent to finding the shortest distance from a point *A* (which lies on *l*) to Π.



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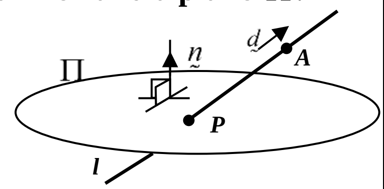
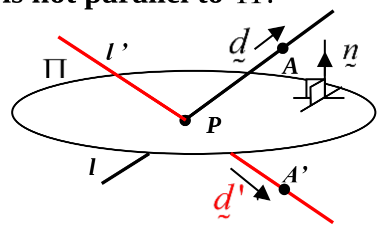
</table>

No.	Skill	Examples of questions involving the skill
		<u>* Note</u> Finding the shortest distance from l to Π is equivalent to finding the length of projection of $\overrightarrow{BA}$ on $\underset{\sim}{n}$, i.e. $\left\lvert\overrightarrow{BA}\bullet\hat{\underset{\sim}{n}}\right\rvert$.
		To find the point of intersection of a line l and a plane Π.  **Tutorial 7C Q4(a), Q5(d)** Let the point of intersection be P. Step 1: Since the point P lies on l, $\overrightarrow{OP}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$. Step 2: Since P also lies on Π, we substitute $\overrightarrow{OP}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$ into $\Pi:\underset{\sim}{r}\bullet\underset{\sim}{n}=p$. For example, we substitute $\overrightarrow{OP}=\begin{pmatrix}3\\5\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\6\\-1\end{pmatrix}$ into $\mathbf{r}\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix}=7$ and we get $\begin{pmatrix}3+2\lambda\\5+6\lambda\\1-\lambda\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix}=7$. Step 3: We solve for λ. Step 4: We substitute the value of λ back into the equation of l: $\underset{\sim}{r}=\begin{pmatrix}3\\5\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\6\\-1\end{pmatrix}$ to find the position vector of the point P, the point of intersection of l and Π. <u>* Note</u> We use the above 4 steps to find the foot of perpendicular from a point A to a plane Π.
		To find the acute angle between a line l and a plane Π. **Tutorial 7C Q4(a)** We let θ to denote the angle between l and Π. <u>Method 1</u> $\theta=90°-\alpha$ where α is the acute angle between l and **n**, i.e. $\theta=90°-\cos^{-1}\dfrac{\lvert\underset{\sim}{d}\bullet\underset{\sim}{n}\rvert}{\lvert\underset{\sim}{d}\rvert\lvert\underset{\sim}{n}\rvert}$ <u>Method 2</u> $\theta=\sin^{-1}\dfrac{\lvert\underset{\sim}{d}\bullet\underset{\sim}{n}\rvert}{\lvert\underset{\sim}{d}\rvert\lvert\underset{\sim}{n}\rvert}$.

<table>
<tr><th>No.</th><th>Skill</th><th>Examples of questions involving the skill</th></tr>
<tr><td></td><td></td><td>

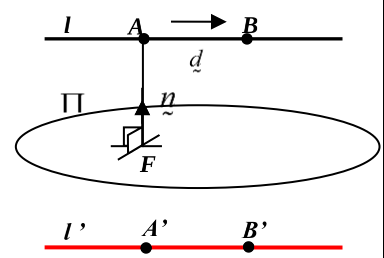
To find the line of reflection, *l'*, of *l* in a plane Π.

Case 1: When *l* is not parallel to Π.



We need to find (i) a point that lies on *l'* and (ii) the direction vector $\underset{\sim}{d}'$ to find the vector equation of the line of reflection *l'*.
Step 1: Notice that the point *P*, the point of intersection of *l* and Π, also lies on *l'*.
Step 2: To find $\underset{\sim}{d}'$, we need 2 points that lie on *l'*. We will take the point *P* as one of the 2 points. The other point will be the point of reflection of *A* (*A'*) in Π. Hence, $\underset{\sim}{d}' = \overrightarrow{PA'}$, i.e.
$l' : \underset{\sim}{r} = \overrightarrow{OP} + \lambda \overrightarrow{PA'}$.

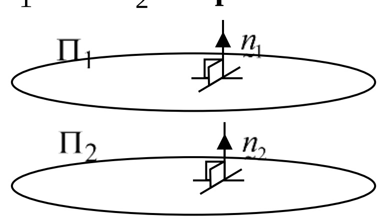
Case 2: When *l* is parallel to Π.



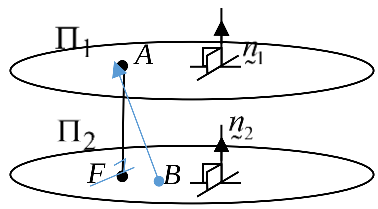
We need to find 2 points that lies on *l'* to find the vector equation of the line of reflection *l'*. These 2 points can be the point of reflection of *A* and *B* that lie on the line *l*.
</td></tr>
<tr><td>16</td><td>Relationship between 2 planes Π_1 and Π_2</td><td>

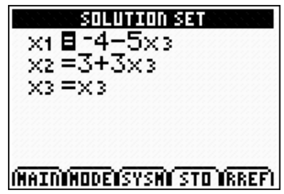
Given 2 planes Π_1 and Π_2, 2 scenarios can happen:

(a) Scenario 1: Π_1 and Π_2 are parallel to each other



To show Π_1 and Π_2 are parallel to each other, we just need to show that $\underset{\sim}{n}_2 = k\underset{\sim}{n}_1$.
</td></tr>
</table>

No.	Skill	Examples of questions involving the skill
		Tutorial 7C Q8(iii) (i) **To find the shortest (perpendicular) distance between** Π_1 and Π_2 where $\prod_1 : \underset{\sim}{r}\bullet\underset{\sim}{n}_1 = D_1$ and $\prod_2 : \underset{\sim}{r}\bullet\underset{\sim}{n}_2 = D_2$ <u>Method 1</u> This is equivalent to finding the shortest distance between the point *A* (which lies on Π_1) and Π_2.  Find a point *B* that lies on Π_2 and use $\left\|\overrightarrow{BA}\bullet\hat{\underset{\sim}{n}}_1\right\|$ to find the shortest distance *AF*. <u>Method 2</u> Compare the scalar product form of the equations of the two planes as follows: Suppose $\underset{\sim}{n}_2 = k\underset{\sim}{n}_1$, $k \in \mathbb{R}$ where $k \neq 0$ Step 1**:** Convert $\prod_2 : \underset{\sim}{r}\bullet\underset{\sim}{n}_2 = D_2$ $\prod_2 : \underset{\sim}{r}\bullet k\underset{\sim}{n}_1 = D_2 \Rightarrow \prod_2 : \underset{\sim}{r}\bullet\underset{\sim}{n}_1 = \dfrac{D_2}{k}$ Step 2: Divide by $\left\|\underset{\sim}{n}_1\right\|$ throughout for equations of both planes. $\prod_1 : \underset{\sim}{r}\bullet\dfrac{\underset{\sim}{n}_1}{\left\|\underset{\sim}{n}_1\right\|} = \dfrac{D_1}{\left\|\underset{\sim}{n}_1\right\|}$ and $\prod_2 : \underset{\sim}{r}\bullet\dfrac{\underset{\sim}{n}_1}{\left\|\underset{\sim}{n}_1\right\|} = \dfrac{\frac{D_2}{k}}{\left\|\underset{\sim}{n}_1\right\|}$ Observe that if D_1 and $\dfrac{D_2}{k}$ are both positive or both negative, the two planes lie on the same side as compared to origin. If D_1 and $\dfrac{D_2}{k}$ are of opposite signs, then the two planes are on opposite sides of the origin. Distance between $\prod_1$ and $\prod_2 = \left\|\dfrac{D_1}{\left\|\underset{\sim}{n}_1\right\|} - \dfrac{\frac{D_2}{k}}{\left\|\underset{\sim}{n}_1\right\|}\right\|$

No.	Skill	Examples of questions involving the skill
		(b) Scenario 2: Π_1 and Π_2 intersect each other in a line *l*. We first convert the equations of Π_1 and Π_2 into Cartesian equations and use GC to solve the system of 2 equations. For example, we have $\Pi_1 : \underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 2 \Rightarrow x + 2y - z = 2$ and $\Pi_1 : \underset{\sim}{r} \bullet \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 1 \Rightarrow 2x + 3y + z = 1$. We solve and get  . We will interpret the screenshot as $\underset{\sim}{r} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$, which is the equation of the line of intersection by writing x_3 as λ. <u>* Note</u> (1) If 2 planes are perpendicular to each other, then their normal vectors are also perpendicular to each other. (2) The direction vector of the line of intersection, $\underset{\sim}{d}$, can be found by $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$.
		To find the angle between the two planes Π_1 and Π_2 **Tutorial 7C Q7(i)** This is equivalent to finding the angle between $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$, i.e., $\theta = \cos^{-1} \dfrac{\underset{\sim}{n}_1 \bullet \underset{\sim}{n}_2}{\lvert \underset{\sim}{n}_1 \rvert \lvert \underset{\sim}{n}_2 \rvert}$.

No.	Skill	Examples of questions involving the skill
17	Geometrical Interpretation	**Tutorial 7C Q6(i)** The points P, Q and R have position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively. The points P and Q are fixed and R varies. Given that $\underset{\sim}{q}$ is non-zero and $\left(\underset{\sim}{r}-\underset{\sim}{p}\right)\times\underset{\sim}{q}=\underset{\sim}{0}$, describe geometrically the set of all possible positions of the point R. $\left(\underset{\sim}{r}-\underset{\sim}{p}\right)\times\underset{\sim}{q}=\underset{\sim}{0}$ $\underset{\sim}{r}-\underset{\sim}{p}=\lambda\underset{\sim}{q}$ $\underset{\sim}{r}=\lambda\underset{\sim}{q}+\underset{\sim}{p}$ R is the set of points on the line parallel to $\underset{\sim}{q}$ and passes through the point with position vector $\underset{\sim}{p}$. **Tutorial 7C Q6(b)(ii)** Given instead that $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\mathbf{p}=\begin{pmatrix}-1\\2\\4\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}3\\-5\\2\end{pmatrix}$ and that $(\mathbf{r}-\mathbf{p})\bullet\mathbf{q}=0$, find the relationship between x, y and z. Describe the set of all possible positions of the point R in this case. Given $(\underset{\sim}{r}-\underset{\sim}{p})\bullet\underset{\sim}{q}=0$ $\therefore\ \begin{pmatrix}x+1\\y-2\\z-4\end{pmatrix}\bullet\begin{pmatrix}3\\-5\\2\end{pmatrix}=0$ $3x-5y+2z=-5$ Also, $(\underset{\sim}{r}-\underset{\sim}{p})\bullet\underset{\sim}{q}=0 \Rightarrow (\underset{\sim}{r}-\underset{\sim}{p})$ is perpendicular to $\underset{\sim}{q}$ Hence $\overrightarrow{PR}=\underset{\sim}{r}-\underset{\sim}{p}$ is always perpendicular to $\overrightarrow{OQ}=\underset{\sim}{q}$ $\therefore$ R is the set of points representing a plane that passes through P $(-1,2,4)$ with normal $\begin{pmatrix}3\\-5\\2\end{pmatrix}$.
18	Real life Applications of Vectors	**Tutorial 7C Q12 and Q13**

7
Vectors

1. **(A Level N84/P1/Q5)**

The position vectors **a**, **b**, **c**, **d** of the points *A*, *B*, *C*, *D* respectively, are given by $\mathbf{a}=-\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+\mathbf{j}+4\mathbf{k}$, $\mathbf{c}=-2\mathbf{i}+5\mathbf{j}-4\mathbf{k}$, $\mathbf{d}=5\mathbf{i}+10\mathbf{j}+7\mathbf{k}$. The point *X* on $\overrightarrow{BC}$ is such that $\overrightarrow{BX}=\frac{1}{3}\overrightarrow{XC}$ and the point *Y*, on $\overrightarrow{AX}$ produced, is such that $\overrightarrow{AX}=\frac{1}{4}\overrightarrow{XY}$.

(a) Find the position vectors of *X* and *Y*. [3]

(b) Determine the position vector of the point *L* on $\overrightarrow{XY}$ such that $\overrightarrow{DL}$ is perpendicular to $\overrightarrow{XY}$, and find the value of the ratio $\frac{XL}{LY}$. [6]

2. (**CJC18/Promo/Q5)**

Relative to the origin O, the points *A* and *B* have position vectors **a** and **b** respectively, where **a** and **b** are non-parallel. It is given that $|\mathbf{a}|=5$, $|\mathbf{b}|=10$ and $|2\mathbf{a}+\mathbf{b}|=18$.

(i) By using the result $\mathbf{v}\bullet\mathbf{v}=|\mathbf{v}|^2$, or otherwise, show that $\mathbf{a}\cdot\mathbf{b}=31$. [2]

(ii) State the geometrical meaning of $\left|\mathbf{a}\bullet\frac{\mathbf{b}}{|\mathbf{b}|}\right|$ and find its numerical value. [2]

(iii) The points *P*, Q and *R* have position vectors $3\mathbf{a}-2\mathbf{b}$, $4\mathbf{a}+3\mathbf{b}$ and $7\mathbf{a}+\lambda\mathbf{b}$ respectively. Given that these three points are collinear, find the value of λ. [3]

3. (**MJC18/Promo/Q5)**

The position vectors of the points *P* and *Q* are given as $\overrightarrow{OP}=\mathbf{p}$ and $\overrightarrow{OQ}=\mathbf{q}$. The point *R* on *PQ* is such that $PR:RQ=2:3$.

(i) Find $\overrightarrow{OR}$ in terms of **p** and **q** and show that the area of triangle *OQR* can be written as $k|\mathbf{p}\times\mathbf{q}|$, where k is a real constant to be found. [4]

(ii) The point *S* on *OP* is such that $OS:SP=1:2$. The position vector of point U is given as $\overrightarrow{OU}=\mathbf{u}$ such that $\overrightarrow{US}\bullet\overrightarrow{OP}=\frac{1}{4}$. Given that **p** is a unit vector and the angle between **p** and **u** is $\frac{\pi}{3}$, find the exact value of $|\mathbf{u}|$. [5]

4. **(A Level N01/P2/Q15)**

(a) The points P and Q have position vectors $3\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $9\mathbf{i}-7\mathbf{j}-2\mathbf{k}$ respectively. Show that $PQ=9$. Find the unit vector in the direction of $\overrightarrow{PQ}$, and find also a Cartesian equation for the line PQ. The line l, which passes through P, has equation $\dfrac{x-3}{-2}=\dfrac{y+1}{1}=\dfrac{z-1}{2}$. Find

(i) the length of the projection of PQ onto l,

(ii) the length of the perpendicular from Q to l. [10]

(b) By expanding $(\mathbf{b}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{c})$, simplify $|\mathbf{b}|^2+|\mathbf{c}|^2-(\mathbf{b}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{c})$. By taking $\mathbf{b}=\overrightarrow{AC}$ and $\mathbf{c}=\overrightarrow{AB}$, deduce the cosine formula for triangle ABC. [6]

5. **(MI16/Promo/Q4)**

Relative to the origin O, the points A, B and C have position vectors $\mathbf{a} = 3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$ and $\mathrm{c} = 4\mathbf{i} - 10\mathbf{j} + 7\mathbf{k}$ respectively.

(i) Show that A, B and C are collinear and state the ratio AB: AC. [3]

(ii) Determine if C is on the line segment AB, giving a reason for your answer. [2]

(iii) Give geometrical interpretations of $|\mathbf{a}\bullet\hat{\mathbf{b}}|$ and $|\mathbf{a}\times\hat{\mathbf{b}}|$ in terms of O, A and B only. [2]

(iv) Evaluate $|\mathbf{a}\times\mathbf{b}|$ exactly and hence find the exact area of the triangle OAB. [3]

(v) Using the answer in part **(iv)** and the result in part **(i)**, deduce the exact perpendicular distance from O to the line AC. [3]

6. **(EJC20/Promo/Q5)**

(a) The variable vector $\mathbf{u}$ satisfies the following equations:

$$\mathbf{u}\bullet\begin{pmatrix}4\\1\\-2\end{pmatrix}=-6 \text{ and } \mathbf{u}\times\begin{pmatrix}1\\2\\3\end{pmatrix}=k\begin{pmatrix}1\\-2\\1\end{pmatrix}, \text{ for } k\in\mathbb{R}, k\neq 0.$$

(i) Explain why $\mathbf{u}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=0$. [1]

(ii) Hence or otherwise, find the set of vectors $\mathbf{u}$ and describe this set geometrically. [3]

(b) The points A, B and C have distinct non-zero position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively and the vectors satisfy the equation $\mathbf{c}=\lambda\mathbf{a}+(1-\lambda)\mathbf{b}$ where $\lambda\in\mathbb{R}, \lambda\neq 0, \lambda\neq 1$. Prove that the points A, B and C are collinear. [2]

7. **(PJC16/Promo/Q11)**

Referred to an origin *O*, the points *A* and *B* are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point *C* is on *AB* produced such that $2AB = 3BC$.

(i) Find $\overrightarrow{OC}$ in terms of **a** and **b**. [2]

(ii) Show that the area of triangle *OBC* can be written as $k|\mathbf{a} \times \mathbf{b}|$, where k is a constant to be found. [2]

(iii) The point *D* is on *OB* such that the ratio of the area of triangle *OAD* to the area of triangle *OBC* is 4 : 3. Find $\overrightarrow{OD}$ in terms of **b**. [3]

(iv) Given that **a** and **b** are unit vectors and $\measuredangle AOB = 60°$, find the length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OA}$. [3]

(c) Given that the point *P* has a non-zero position vector **p** and that the plane Π has equation $\mathbf{r} \bullet \mathbf{n} = 0$, where **n** is a unit vector, state the geometrical meaning of $|\mathbf{p} \bullet \mathbf{n}|$ in relation to the point *P* and the plane Π. [1]

8. **(CJC14/Promo/Q7)**

Referred to the origin *O*, the points *A* and *B* have position vectors **a** and **b** such that $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.

The line l passes through point A and is parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

(i) Find a vector equation of line l. [1]

(ii) Find the coordinates of the foot of the perpendicular from *B* to the line *l*. [3]

(iii) Hence, or otherwise, find the shortest distance between *B* and the line *l*. [2]

(iv) Find a Cartesian equation of the line which is a reflection of the line *AB* in the line *l*. [4]

9. **(AJC10/Promo/Q11)**

Relative to an origin *O*, the position vectors of the points *A* and *B* are $\begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$ respectively. The line l_1 passes through the points *A* and *B*. Another line l_2 has Cartesian equations given by $x - 1 = 2 - z,\ y = 4$. The lines l_1 and l_2 intersect at the point *A*. Find

(i) a vector equation of the line l_2, and state the value of α, [2]

(ii) the acute angle between the lines l_1 and l_2, [2]

(iii) the position vector of the foot of perpendicular, *N*, from *B* to the line l_2. [3]

The point *S* lies on the line segment *BN* and is such that $\dfrac{\text{Area of } \Delta ABS}{\text{Area of } \Delta ABN} = \dfrac{3}{4}$.

Find the position vector of *S*. [3]

10. **(HCI10/Promo/Q8)**

The position vector of *A*, relative to the origin O, is $\mathbf{j}+7\mathbf{k}$. The line l through *A* is parallel to the vector $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and the foot of perpendicular from *O* to l is denoted by S.

(a) Find the acute angle between l and $\overrightarrow{OA}$. [2]

(b) Find the coordinates of *S*. [3]

Point *B* is on *OA* such that $3OB = OA$ and point *C* is the mid-point of *OS*. *T* is a point on l such that *B*, *C* and *T* are collinear.

(c) Find the position vector of *T*. [4]

11. **(IJC10/Promo/Q13)**

The lines l_1 and l_2 have equations $\mathbf{r}=\begin{pmatrix}6\\5\\-6\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}-7\\19\\14\end{pmatrix}+\mu\begin{pmatrix}1\\7\\-2\end{pmatrix}$ respectively, where $\lambda,\ \mu\in\mathbb{R}$. The point *P* on l_1 and the point *Q* on l_2 are such that *PQ* is perpendicular to both l_1 and l_2. The point *H* on the line segment *PQ* is such that $3PH = HQ$.

(i) Find the acute angle, to the nearest degree, between the lines l_1 and l_2. [3]

(ii) By considering $\overrightarrow{PQ}$, find the position vectors of *P* and *Q* with respect to the origin *O*. [7]

(iii) Show that the position vector of *H* is $4\mathbf{i}+6\mathbf{j}+\mathbf{k}$. [2]

(iv) Find the length of projection of $\overrightarrow{AH}$ onto the line l_1, where point *A* has coordinates $(6, 5, -6)$. [3]

12. **(NYJC10/Promo/Q8(modified))**

Relative to the origin *O*, the points *A*, *B* and *C* have position vectors $-3\mathbf{i}+8\mathbf{j}+\mathbf{k}$, $7\mathbf{i}+3\mathbf{j}+6\mathbf{k}$ and $\alpha\mathbf{i}+\beta\mathbf{j}-6\mathbf{k}$ respectively, where α and β are real numbers.

(i) Given that the vector equation of line *BC* is $\mathbf{r}=(7+3\lambda)\mathbf{i}+3\mathbf{j}+(6+4\lambda)\mathbf{k}$, find the values of α and β. [2]

(ii) Find the length of projection of $\overrightarrow{AB}$ onto the line *BC*. [2]

(iii) Hence find the position vector of the foot of perpendicular from *A* to the line *BC*. [3]

(iv) Find the exact area of the triangle *ABC*. [3]

13. **(DHS19/Promo/Q7)**

With respect to the origin O, the position vectors of two points A and B are given by $\mathbf{a}$ and $\mathbf{b}$ respectively where $|\mathbf{a}|=1,\ |\mathbf{b}|=2$ and $\mathbf{a}\bullet\mathbf{b}=1$. Point C lies on AB such that $AC=2CB$. Point N is the foot of perpendicular from C to the line OB.

(i) Find the position vector of the point C in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(ii) Show that the length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OB}$ is $\frac{3}{2}$ and deduce the position vector of N. [4]

(iii) Find the value of λ such that C, D and N are collinear where $\overrightarrow{OD}=\mathbf{a}+\lambda\mathbf{b}$. [3]

14. **(HCI12/Promo/Q13)**

With respect to the origin O, the points A, B and C have position vectors $\mathbf{a}=\begin{pmatrix}1\\ \sin t\\ 0\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}\cos t\\ 2\\ 1\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}2\\ -1\\ 1\end{pmatrix}$ respectively, where t is a real parameter such that $0\le t<\pi$.

(a) Find the exact value of t given that $\mathbf{a}$ is perpendicular to $\mathbf{b}$. [3]

(b) The point X is on AB produced such that $AB:BX$ is $1:4$ and the point Y is such that $ACXY$ is a parallelogram. Take $t=\frac{\pi}{2}$ for the rest of the question.

(i) Find the position vectors of X and Y. [4]

(ii) Find the area of $ACXY$. Hence, find the shortest distance from X to the line that passes through the points A and C. [3]

(iii) The line l_1 has equation $\frac{x-2}{6}=\frac{3-y}{h}$, $z=k$, where h and k are constants. The line that passes through A and B intersects l_1 at a right angle. Find the values of h and k. [4]

15. **(EJC19/Promo/Q4)**

Referred to the origin O, the points A, B, and C have position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively.

(i) Given that non-zero numbers λ and μ are such that $\lambda\mathbf{a}+\mu\mathbf{b}+\mathbf{c}=\mathbf{0}$ and $\lambda+\mu+1=0$ with $\mu>0$. Show that A, B, and C are collinear and find the ratio $CA:CB$ in terms of μ. [4]

(ii) F is another point such that the line passing through A, B, and C does not contain it. Find $\frac{|\overrightarrow{BF}\times\overrightarrow{BC}|}{|\overrightarrow{AF}\times\overrightarrow{AC}|}$ in terms of μ. [2]

16. **(IJC13/PrelimP1/Q12)**

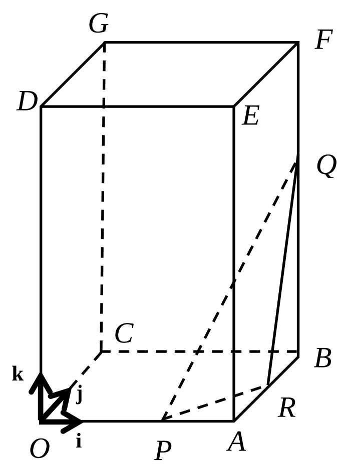
The plane p_1 has equation $2x-3y+z=8$ and it meets the x- and z- axes at the points A and B respectively. State the position vectors of A and B, relative to the origin O, and show that a Cartesian equation of the line AB is $4-x=\frac{z}{2}$, $y=0$. [4]

The point C lies on AB such that $\frac{AC}{CB}=\frac{1}{3}$. The plane p_2 passes through C and is parallel to the vector $\mathbf{i}-2\mathbf{j}$. Given that p_2 is also perpendicular to p_1, find

(i) the Cartesian equation of p_2, [5]

(ii) the perpendicular distance from A to p_2. [3]

17. **(HCI12/PrelimP1/Q9)**



The diagram above shows a cuboid $OABCDEFG$ with horizontal base $OABC$ where $OA=6$ cm and $AB=4$ cm. OD, AE, BF and CG are vertical with height 10 cm. The point O is taken as the origin and vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions OA, OC and OD respectively. The point P on OA and the point Q on BF are such that $OP:PA=2:1$ and $BQ:QF=3:2$. Point R is the midpoint of AB.

(i) Show that the equation of the plane PQR is $3x-3y+z=12$. [3]

(ii) Find the distance from the point C to the plane PQR. [2]

(iii) Find the acute angle between the plane PQR and the plane $OABC$. [2]

(iv) Find the position vector of the point of intersection M of the line PQ and the plane $OCGD$. Hence write down the distance from M to the plane $OABC$. [3]

(v) Find a vector equation of the plane, in the form $\mathbf{r}=\mathbf{u}+\lambda\mathbf{v}+\mu\mathbf{w}$, when the plane PQR is reflected about the plane $OABC$. [2]

18. **(DHS13/PrelimP1/Q10)**

The lines l_1 and l_2 have equations $\mathbf{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}b\\1\\-1\end{pmatrix}$, where $b>1$, and $\mathbf{r}=\begin{pmatrix}4\\0\\1\end{pmatrix}+\mu\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ respectively.

(i) Given that the acute angle between l_1 and l_2 is 30^{o}, find the value of b, giving your answer correct to 2 decimal places. [3]

For the rest of the question, use $b = 3$.

(ii) Find the coordinates of the points A and B where l_1 and l_2 meet the xy–plane respectively. [3]

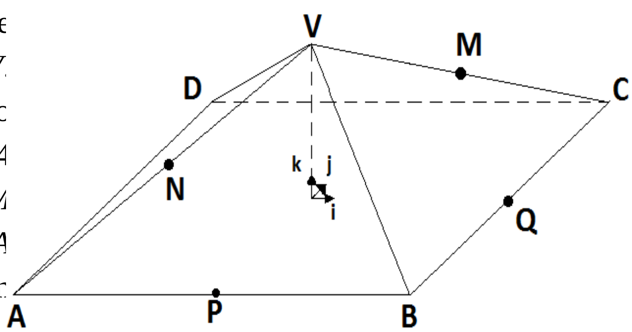
(iii) The point C has position vector 2i + 7j + 3k. Show that the equation of plane p_1 which passes through the points A, B, and C is given by $x-2y+5z=3$. [3]

(iv) Another plane p_2 with equation $x+2y+3z=2$ meets p_1 in the line l_3. Find the vector equation of l_3. [2]

(v) Explain whether the lines l_2 and l_3 are parallel, skew, coincident or intersecting. [1]

19. **(A Level J94/P1/Q14)**

In the diagram, O is the centre of the square base $ABCD$ of a right pyramid, vertex V. Perpendicular unit vectors **i**, **j**, **k** are parallel to AB, AD, OV respectively. The length of AB is 4 units and the length of OV is $2h$ units. P, Q, M and N are the mid-points of AB, BC, CV and VA respectively. The point O is taken as the origin for position vectors.



(a) Show that the equation of the line PM may be expressed as $\mathbf{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+t\begin{pmatrix}1\\3\\h\end{pmatrix}$, where t is a parameter. [3]

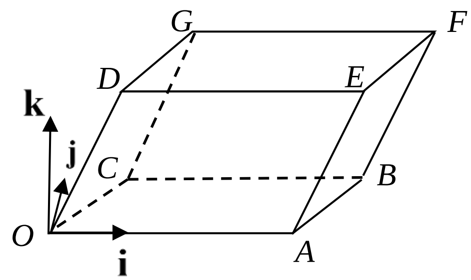
(b) Find an equation for the line QN. [3]

(c) Show that the lines PM and QN intersect, and that the position vector $\overrightarrow{OX}$ of their point of intersection is $\begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{h}{2}\end{pmatrix}$. [2]

(d) Given that OX is perpendicular to VB, find the value of h and calculate the acute angle between PM and QN, giving your answer correct to the nearest 0.1°. [4]

20. (HCI13/Promo/Q12)

An art structure, which is a parallelpiped (made of 6 faces of parallelograms) has a horizontal base *OABC*, with *OA*, *OC* and *OD* as its three sides and remaining vertices are *B*, *E*, *F*, and *G* as shown in the diagram below.



It is given that $\overrightarrow{OA} = 5\mathbf{i}$ and $\overrightarrow{OC} = \mathbf{i} + 7\mathbf{j}$.

The lines l_1 and l_2 have equations given by

$l_1:\ \underset{\sim}{r} = (5+\lambda)\mathbf{i} + (7\lambda - 14)\mathbf{j} + 6\mathbf{k}$, where λ is a real parameter and

$l_2: 3x = z + 15,\ y = 0$.

E and *F* are on line l_1 and *A* and *E* are on line l_2.

(i) Find the position vector of *E*. [2]

(ii) Find the equation, in scalar product form, of the plane *ABFE*. [3]

(iii) Find the projection vector of $\overrightarrow{AE}$ onto the base *OABC*. Hence, or otherwise, find the area of the projection of the plane *ABFE* onto the base. [2]

(iv) Find the equation of the line l_3, which is the reflection of line *AE* about the base *OABC*. [2]

(v) An architect wants to add a shelter which has the plane equation $x + ay + bz = c$, where *a*, *b* and *c* are unknown constants. He wants the shelter to meet the plane *ABFE* at *EF*. What can be said about the values of *a*, *b* and *c* ? [2]

21. (EJC20/Promo/Q12)

Stereophotogrammetry is a method of determining coordinates of points in the three-dimensional (3-D) replication of physical scenes. It relies on using multiple images taken by digital cameras from different positions.

In a simplistic model for this process, the camera sensors are represented by planes with finite size. A *ray of sight* for a particular point in the physical scene is defined as the line passing through its image point on the camera sensor and the focal point of the camera. The 3-D coordinates of a particular point is the intersection of the rays of sight from the different cameras.

For a particular set up, Camera 1 has focal point, F_1 at $(20, 5, 10)$ and Camera 2 has focal point, F_2 at $(-20, 5, 10)$. The image point of a point *A*, the highest tip of a flag pole, on Camera 1 is $A_1\left(\frac{61}{3}, 0, \frac{28}{3}\right)$, and the image point of *A* on Camera 2 is $A_2\left(-\frac{67}{3}, 0, \frac{28}{3}\right)$.

(i) Find the vector equations of l_1 and l_2, the rays of sight for point *A* from the Cameras 1 and 2 respectively, and hence find the coordinates of point *A*. [4]

The base of the flag pole is known to be on the plane *P* that contains the point $D(75, 90, -50)$ and the line *L* with equation $\mathbf{r} = \begin{pmatrix} 23 \\ 90 \\ -46 \end{pmatrix} + s\begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}, s \in \mathbb{R}$.

(ii) Show that the vector equation of the plane *P* is $\mathbf{r} \bullet \begin{pmatrix} 1 \\ -2 \\ 13 \end{pmatrix} = -755$. [3]

The flag pole was erected perpendicular to the base plane *P*.

(iii) Find the coordinates of the point *B*, the base of the flag pole. [3]

(iv) Hence or otherwise, find the length of the flag pole. [2]

(v) Given that the horizontal plane in this model is the *x*-*y* plane, find the angle of incline for the plane *P* from the horizontal plane. [2]

22 GCEA 2017/P1/Q6

(i) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where **a** and **b** are constant vectors and *t* is a parameter. [2]

(ii) Interpret geometrically the vector equation $\mathbf{r} \bullet \mathbf{n} = d$, where **n** is a constant unit vector and *d* is a constant scalar, stating what *d* represents. [3]

(iii) Given that $\mathbf{b} \bullet \mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} \bullet \mathbf{n} = d$ to find **r** in terms of **a**, **b**, **n** and *d*. Interpret the solution geometrically. [3]

Answers

1. **(a)** $\overrightarrow{OY} = \begin{pmatrix} 5 \\ 14 \\ 6 \end{pmatrix}, \overrightarrow{OX} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ **(b)** $\overrightarrow{OL} = \begin{pmatrix} 4 \\ 11 \\ 5 \end{pmatrix}$; 3 : 1

2. **(ii)** 3.1 **(iii)** $\lambda = 18$

3. **(i)** $\overrightarrow{OR}=\dfrac{2\mathbf{q}+3\mathbf{p}}{5}$; $k=\dfrac{3}{10}$ **(ii)** $\dfrac{1}{6}$

4. **(a)** $\dfrac{1}{3}\begin{pmatrix}2\\-2\\-1\end{pmatrix}$, $\dfrac{x-3}{2}=\dfrac{y+1}{-2}=\dfrac{z-1}{-1}$ **(i)** 8 **(ii)** $\sqrt{17}$

5. **(i)** 2:1 **(iv)** $2\sqrt{710}$, $\sqrt{710}$ units^2 **(v)** $\dfrac{\sqrt{41890}}{59}$

6. **(a)(ii)** $\left\{\mathbf{u}=\begin{pmatrix}x\\y\\z\end{pmatrix}\middle|\, x=-\dfrac{4}{3}+\lambda,\ y=-\dfrac{2}{3}+2\lambda,\ z=3\lambda,\ \lambda\in\mathbb{R}\right\}$

7. **(i)** $\overrightarrow{OC}=\dfrac{5}{3}\mathbf{b}-\dfrac{2}{3}\mathbf{a}$ **(iii)** $\overrightarrow{OD}=\dfrac{8}{9}\mathbf{b}$ **(iv)** $\dfrac{1}{6}$

8. **(i)** $\underset{\sim}{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix},\lambda\in\mathbb{R}$ **(ii)** $\left(\dfrac{17}{3},\dfrac{11}{3},\dfrac{23}{3}\right)$ **(iii)** $\dfrac{1}{3}\sqrt{42}$

(iv) $\dfrac{x-1}{10}=\dfrac{y+1}{13}=\dfrac{z-3}{19}$

9. **(i)** $\underset{\sim}{r}=\begin{pmatrix}1\\4\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix},\lambda\in\mathbb{R}$; $\alpha=4$ **(ii)** $\theta=39.2°$ **(iii)** $\begin{pmatrix}-1\\4\\4\end{pmatrix}$; $\dfrac{1}{2}\begin{pmatrix}-3\\7\\7\end{pmatrix}$

10. **(a)** 60.7° **(b)** (2, 3, 5) **(c)** $\overrightarrow{OT}=\begin{pmatrix}4\\5\\3\end{pmatrix}$

11. **(i)** 66° (to nearest degree) **(ii)** $\overrightarrow{OP}=\begin{pmatrix}8\\4\\-4\end{pmatrix}$; $\overrightarrow{OQ}=\begin{pmatrix}-8\\12\\16\end{pmatrix}$ **(iv)** 3 units

12. **(i)** $\alpha=-2$, $\beta=3$ **(ii)** 10 **(iii)** $\begin{pmatrix}1\\3\\-2\end{pmatrix}$ **(iv)** $\dfrac{75\sqrt{2}}{2}$ units^2

13. **(i)** $\overrightarrow{OC}=\dfrac{1}{3}(\mathbf{a}+2\mathbf{b})$ **(ii)** $\overrightarrow{ON}=\dfrac{3}{4}\mathbf{b}$ **(iii)** $\lambda=\dfrac{1}{2}$

14. **(a)** $t=\pi-\tan^{-1}\left(\frac{1}{2}\right)$ **(b)(i)** $\overrightarrow{OX}=\begin{pmatrix}-4\\6\\5\end{pmatrix};\overrightarrow{OY}=\begin{pmatrix}-5\\8\\4\end{pmatrix}$

(b)(ii) $\sqrt{350}$ square units; 7.64 units **(b)(iii)** $h=-6;\ k=\frac{1}{2}$

15. **(i)** $CA:CB=\mu:1+\mu$ **(ii)** $\frac{\text{area }\Delta BFC}{\text{area }\Delta AFC}=\frac{1+\mu}{\mu}$

16. $\vec{OA}=\begin{pmatrix}4\\0\\0\end{pmatrix}$, $\vec{OB}=\begin{pmatrix}0\\0\\8\end{pmatrix}$ **(i)** $2x+y-z=4$ **(ii)** $\frac{4}{\sqrt{6}}$

17. **(ii)** 5.51 cm **(iii)** $\theta=76.7°$ **(iv)** $\overrightarrow{OM}=\begin{pmatrix}0\\-8\\-12\end{pmatrix}$ 12 cm

(v) $\mathbf{r}=\begin{pmatrix}4\\0\\0\end{pmatrix}+u\begin{pmatrix}2\\2\\0\end{pmatrix}+v\begin{pmatrix}2\\4\\-6\end{pmatrix}$

18. **(i)** $b=3.07$ (2 d.p.) **(ii)** $A(9,3,0)$, $B(5,1,0)$

(iv) $l_3:\ \mathbf{r}=\begin{pmatrix}2.5\\-0.25\\0\end{pmatrix}+\beta\begin{pmatrix}-8\\1\\2\end{pmatrix},\ \beta\in\mathbb{R}$ **(v)** l_2 and l_3 are skew lines

19. **(b)** $\mathbf{r}=\begin{pmatrix}2\\0\\0\end{pmatrix}+s\begin{pmatrix}-3\\-1\\h\end{pmatrix}$, s $\in\mathbb{R}$ **(d)** $\sqrt{2}, 70.5^\circ$

20. **(i)** $\begin{pmatrix}7\\0\\6\end{pmatrix}$ **(ii)** $\underset{\sim}{\mathbf{r}}.\begin{pmatrix}21\\-3\\-7\end{pmatrix}=105$ **(iii)** $\begin{pmatrix}2\\0\\0\end{pmatrix}$, 14

(iv) $l_3:\underset{\sim}{r}=\begin{pmatrix}5\\0\\0\end{pmatrix}+\beta\begin{pmatrix}1\\0\\-3\end{pmatrix},\beta\in\mathbb{R}$ **(v)** $a=-\frac{1}{7},\ 7+6b=c.$

21 **(i)** $l_1:\underset{\sim}{r}=\begin{pmatrix}20\\5\\10\end{pmatrix}+\lambda\begin{pmatrix}-1\\15\\2\end{pmatrix},\lambda\in\mathbb{R}$ and $l_2:\underset{\sim}{r}=\begin{pmatrix}-20\\5\\10\end{pmatrix}+\mu\begin{pmatrix}7\\15\\2\end{pmatrix},\mu\in\mathbb{R}$; $A(15,80,20)$

(iii) $(10,90,-45)$ **(iv)** 66.0 units **(v)** 9.8°

Modified RI 2021 JC1 Promo (3 hours, 100 marks)

1 **[JPJC21/Promo/Q10]**

Find $\displaystyle\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}\,dx.$ [2]

2 A curve C has equation $y=\dfrac{ax+b}{cx-2}$, where a, b and c are constants. It is given that C passes through the points with coordinates $(1,\ 5)$ and $(-8,\ 0.5)$. The curve C is translated 1 unit in the positive x-direction. The new curve passes through the point with coordinates $(0,\ -0.2)$. Find the values of a, b and c. [4]

3 The curve C has equation $y^3=4-\dfrac{xy^2}{2}$.

(i) Show that $\dfrac{dy}{dx}=-\dfrac{y}{6y+2x}$. [2]

(ii) Find the equation of the normal to C at the point P where $y=1$. [3]

4 The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

(a) Show that the area of triangle ABC is $\frac{1}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|$. Hence show that the shortest distance from B to AC is

$$\frac{\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|}{\left|\mathbf{c}-\mathbf{a}\right|}.$$ [4]

(b) Given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors such that $\left|\mathbf{a}-\mathbf{b}\right|=\left|\mathbf{a}+\mathbf{b}\right|$, find the value of $\mathbf{a}\cdot\mathbf{b}$. [2]

5 A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_n = \sum_{r=1}^{n} (2r + n + 1).$$

Another sequence $v_1, v_2, v_3, \ldots$ is given by $v_n = \dfrac{2}{u_n}$, where $n \in \mathbb{Z}^+$.

(i) Find u_n in terms of n. [2]

(ii) Show that $v_n = \dfrac{1}{n} - \dfrac{1}{n+1}$. [1]

(iii) Describe the behaviour of the sequence $v_1, v_2, v_3, \ldots$. [1]

(iv) Find the sum, S_N, of the first N terms of the sequence $v_1, v_2, v_3, \ldots$. [2]

(v) Give a reason why the series S_N converges, and write down the value of the sum to infinity. [2]

6 **(i)** On the same axes, sketch the graphs of $y = x + 2 + \dfrac{1}{x-1}$ and $y = |2x + 2|$, stating the coordinates of any points of intersections with the axes, turning points and the equations of any asymptotes. [5]

(ii) Hence solve the inequality

$$x + 2 + \frac{1}{x-1} > |2x + 2|,$$

giving your answers in exact form. [5]

7 **(a)** An arithmetic sequence $a_1,\ a_2,\ a_3, ...$ has common difference d, where $d < 0$. The sum of the first n terms of the sequence is denoted by S_n. Given that $|a_8| = |a_{13}|$, find the value of n for which S_n is maximum. [4]

(b) The terms $u_1,\ u_2$ and u_3 are three consecutive terms of a geometric progression. It is given that

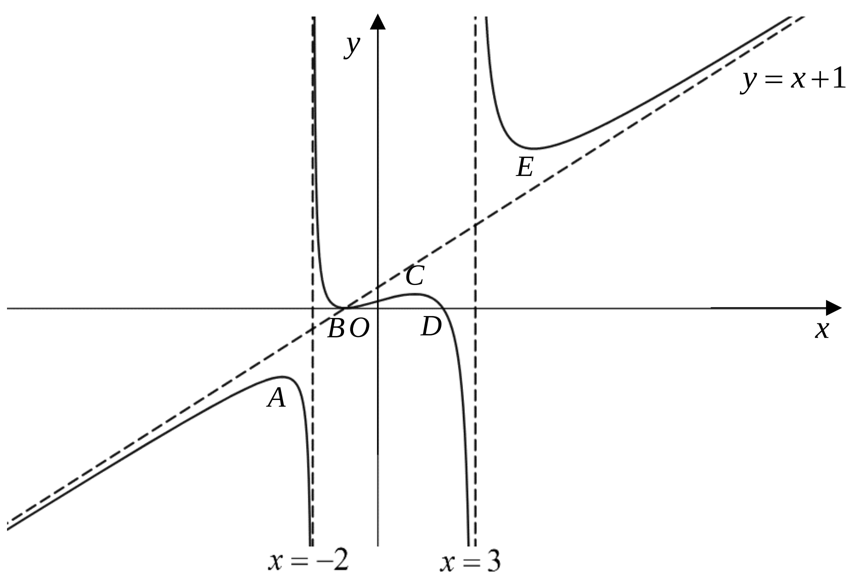
$$u_1,\ u_2 \text{ and } u_3 - 32$$

form an arithmetic progression, and that

$$u_1,\ u_2 - 4 \text{ and } u_3 - 32$$

form another geometric progression. Find the possible values of u_1, u_2 and u_3. [6]

8 The diagram below shows the graph of $y = \mathrm{f}(x)$ with asymptotes $x = -2$, $x = 3$ and $y = x + 1$. The curve intersects the x-axis at points B and D, and has turning points at points A, B, C and E. The coordinates of A, B, C, D and E are $(-3, -3)$, $(-1, 0)$, $\left(\frac{3}{2}, \frac{3}{4}\right)$, $(2, 0)$ and $(5, 8)$ respectively.



(i) By showing clearly the equations of asymptotes and the coordinates of any turning points and the points where the curve crosses the axes, where possible, sketch, on **separate diagrams**, the graphs of

(a) $y = \mathrm{f}\left(|x|\right)$, [4]

(b) $y = \dfrac{1}{\mathrm{f}\left(x\right)}$. [4]

(ii) By drawing another suitable graph on the same diagram in part **(i)(b)**, determine the number of solutions to the equation

$$\frac{x^2}{36} + \frac{1}{16\left[\mathrm{f}\left(x\right)\right]^2} = 1.$$ [2]

9 Distances in this question are in metres.

Harry and Tom's model airplanes are taking off from the horizontal ground, which is the *x*-*y* plane. Tom's airplane takes off after Harry's. The position of Harry's airplane t seconds after it takes off is given by $\mathbf{r} = (5\mathbf{i} + 6\mathbf{j}) + t(-4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$. The position of Tom's airplane s seconds after it takes off is given by $\mathbf{r} = (-39\mathbf{i} + 44\mathbf{j}) + s(4\mathbf{i} - 6\mathbf{j} + 7\mathbf{k})$.

(i) State the height of Harry's airplane two seconds after it takes off and find its distance travelled in the two seconds. [3]

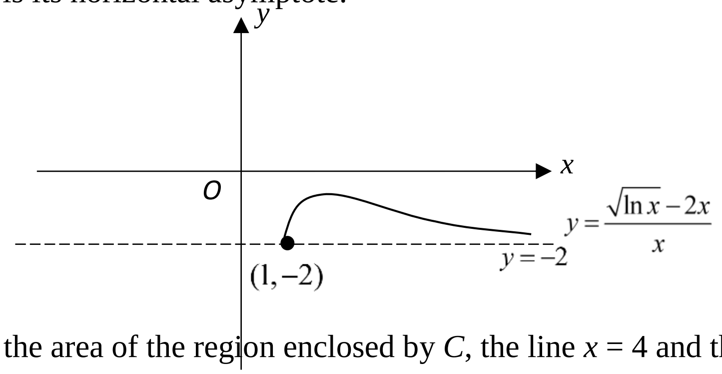
(ii) Find the acute angle between the path of Harry's airplane and the ground. [2]

(iii) Show that the paths of the airplanes are perpendicular. [1]

(iv) Given that the two airplanes collide, find the coordinates of the point of collision. How long after Harry's airplane takes off does Tom's airplane take off? [3]

10 **(i)** Using integration by parts, show that $\int \frac{1}{x^2} \ln x \, dx = -\frac{1}{x} \ln x - \frac{1}{x} + B$, where B is an arbitrary constant. [2]

The graph of the curve C with equation $y = \frac{\sqrt{\ln x} - 2x}{x}$, $x \geq 1$ is as shown below. The line $y = -2$ is its horizontal asymptote.



(ii) Find the area of the region enclosed by C, the line $x = 4$ and the line $y = -2x$. [4]

(iii) The region A is enclosed by C and the lines $x = 4$ and $y = -2$. By using a suitable transformation, show that the exact volume when A is rotated about the $y = -2$ through 2π radians is $\frac{\pi}{4}(p \ln 2 + q)$, where p and q are constants to be determined. [4]

11 Functions f and g are defined by

$$\mathrm{f} : x \mapsto 4x - 2k \quad \text{for } x \in \mathbb{R}, \text{ where } k \text{ is a constant,}$$

$$\mathrm{g} : x \mapsto \frac{9}{2 - x} \quad \text{for } x \in \mathbb{R},\ x \neq 2.$$

(i) Explain why gf does not exist. [1]

(ii) Find the range of values of k for which the equation $\text{fg}(x)=x$ has real roots. [4]

For the rest of the question, let $k=5$.

(iii) Sketch the graph of $y=\text{fg}(x)$ for $x<2$. Hence sketch the graph of $y=(\text{fg})^{-1}(x)$ on the same diagram, showing clearly the relationship between the two
graphs. [4]

The function h represents the height in metres of an object at time t seconds and is defined for the domain $0 \le t \le b$ by

$$\text{h}(t)=\begin{cases} \dfrac{30}{7}+\text{g}(t+9) & \text{for } 0 \le t \le a, \\ 2-\text{f}(t) & \text{for } a < t \le b, \end{cases}$$

where a and b are constants. At $t=0$, the object was thrown up from 3 metres above the ground level. When $t=a$, the object started to descend and finally reached the ground at $t=b$.

(iv) Find the values of a and b. [2]

(v) Sketch the graph of $y=\text{h}(t)$ for $0 \le t \le b$. [1]

12 **[SAJC21/Promo/Q10]**

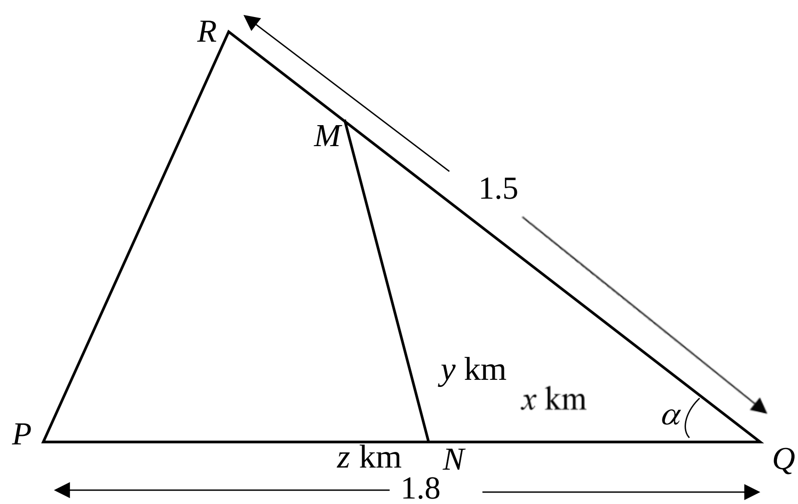
A property developer wants to develop a triangular plot of land PQR as shown in the diagram below.

One section, NQM, is to be used for residential development and the other section,

$PNMR$, is to be used for commercial development where M is on RQ and N is on PQ.

It is given that $NQ = x$ km, $QM = y$ km, $MN = z$ km, RQ = 1.5 km, PQ = 1.8 km, and

a fixed angle $\angle NQM = \alpha$ radians where $\alpha \in \left(0, \dfrac{\pi}{2}\right)$.



(i) To achieve the requirements set out by the government on the use of the plot, the developer plans the use such that the residential development and commercial development takes up the same area each in the plot PQR.

Show that $z^2 = x^2 + \dfrac{1.8225}{x^2} - 2.7\cos\alpha$. [4]

The developer wants to build a fence on the boundary MN. In order to minimize the construction costs, he decides that the boundary MN should be of minimum length.

(ii) Using differentiation, find the value of x which will minimise the length MN, giving your answers correct to 3 decimal places. [7]

(iii) Given that $\angle NQM = \alpha = \dfrac{\pi}{3}$, sketch the graph showing the relationship of the square of the length MN as the length of NQ varies. [3]

Modified RI 2022 JC1 Promo (3 hours, 100 marks)

1 Elly runs a home bakery business and sells three different flavors of cakes. A 0.5 kg Strawberrylicious, Cheeseburst and Chocofanatic cake is priced at x, y and z dollars respectively. A 1 kg cake costs 40% more than a 0.5 kg cake of the same flavour.

Customer A bought one 0.5 kg Strawberrylicious cake, one 0.5 kg Cheeseburst cake and one 0.5 kg Chocofanatic cake. She paid a total of $175.

Customer B bought one 1 kg Strawberrylicious cake, three 1 kg Cheeseburst cakes and two 0.5 kg Chocofanatic cakes. She paid a total of $463.

During a 11.11 sale, Customer C bought one 1 kg Strawberrylicious cake, one 1 kg Cheeseburst cake and two 1 kg Chocofanatic cakes at 10% off the total bill. He paid a total of $296.10 after the discount.

(i) Find the values of x, y and z. [3]

(ii) Find the total amount paid by Customer D, who bought three 0.5 kg Strawberrylicious cakes, two 0.5 kg Cheeseburst cakes and two 1 kg Chocofanatic cakes at 25% off the total bill during a 12.12 sale. [1]

2 Without using a calculator, solve the inequality $3x+1\leq\frac{x^2+2}{x}$. [5]

3 The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OC}=\mathbf{c}$.

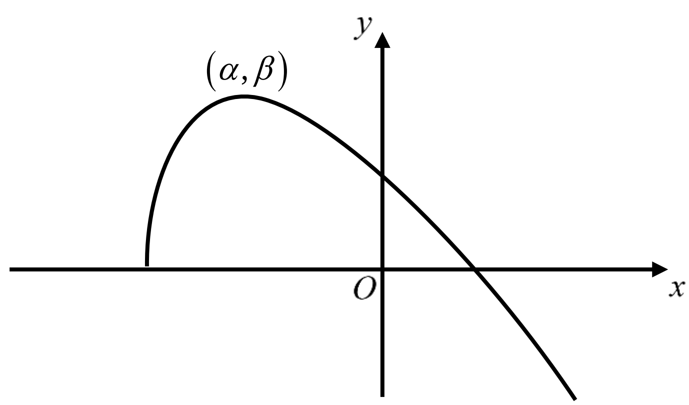
(a) Show that if $\mathbf{c}=|\mathbf{a}|\mathbf{b}+|\mathbf{b}|\mathbf{a}$, then OC bisects the angle AOB. [3]

(b) Given that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, show that $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$. [2]

4 **(i)** Find $\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)}$, where $n \geq 3$. (There is no need to express your answer as a single algebraic fraction.) [5]

(ii) Explain why $\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$ is a convergent series, and state the value of the sum to infinity. [2]

5 The diagram shows the curve with equation $y = (2-x)\sqrt{x+5}$. The curve has a turning point at (α, β). The finite region R is bounded by the curve and the x-axis.



(i) Using the substitution $u = \sqrt{x+5}$, show that the exact area of R is $\frac{a}{b}\sqrt{7}$, where a and b are integers to be found. [7]

(ii) Use your calculator to find the values of α and β and hence find the area bounded by the curve, the line $y = \beta$ and the y-axis. [3]

The finite region S is bounded by the curve, the line $x = 4$ and the x-axis. A solid sculpture is formed by rotating the regions R and S through 2π radians about the x-axis and joining them together.

(iii) Find the volume of the sculpture, correct to 3 decimal places. [2]

6 Find

(i) $$\int \frac{5x}{\left(x^2-1\right)^{10}}\,\mathrm{d}x,$$ [2]

(ii) $$\int \frac{1}{\sec x - 2\tan x}\,\mathrm{d}x.$$ [3]

7 **(a)** Sketch on the same diagram the graphs of $y = |x-1|$ and $y = x^2 - 3$.

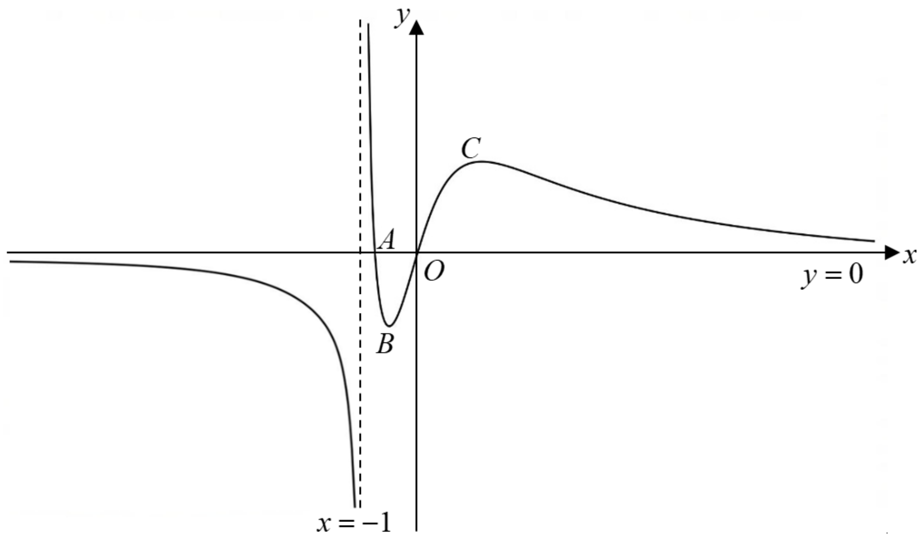
Hence solve the inequality $|x-1| < x^2 - 3$. [3]

(b) A curve C has equation $y = \dfrac{2x^2 + x + a}{x}$.

(i) For $a > 1$, sketch C**,** labelling clearly the equations of the asymptotes and the coordinates of the stationary points in terms of a where appropriate. [4]

(ii) Given that the solution set of the inequality $\dfrac{2x^2 + x + a}{x} \le 0$ is $(-\infty, 0)$, deduce the set of values of a. [2]

8



The diagram shows the curve of $y = \mathrm{f}\left(x\right)$, where $x = -1$ and $y = 0$ are asymptotes. The curve crosses the *x*-axis at the point *A* and the origin, and has turning points at *B* and *C*. The coordinates of *A*, *B* and *C* are $\left(-\dfrac{3}{4}, 0\right), \left(-\dfrac{1}{2}, -\dfrac{3}{2}\right)$ and $\left(1, 2\right)$ respectively. It is given that $\mathrm{f}'\left(0\right) = 2$ is the maximum gradient of the curve.

(a) Describe fully a sequence of two transformations which would transform the curve of $y = \mathrm{f}\left(x\right)$ onto the curve $y = \mathrm{f}\left(2x-1\right)$, and state the coordinates of the point on $y = \mathrm{f}\left(2x-1\right)$ corresponding to *A*. [3]

By showing clearly the equations of asymptotes and the coordinates of the points where the curve crosses the axes and turning points where appropriate, sketch, on separate diagrams, the curves of

(b) $y = \mathrm{f}(-|x|)$, [3]

(c) $y = \mathrm{f}'(x)$. [3]

9 The cartesian equation of the plane π_1 is given by $4x-5y+z=5$.

(i) Find the perpendicular distance from the origin O to π_1. [1]

(ii) Find the acute angle between π_1 and the y-axis. [2]

The cartesian equation of the plane π_2 is given by $2x+5y+3z=5$.

(iii) Find a vector equation of l, the line of intersection of π_1 and π_2. [2]

The cartesian equation of the plane π_3 is given by $3x-3y+z=0$.

(iv) Describe the geometrical relationship between l and π_3, justifying your answer. [3]

(v) Find an equation of the plane p which contains the point $(3,-5,1)$ and is perpendicular to π_1, π_2 and π_3. [2]

10 The function f is defined by

$$\mathrm{f}: x \mapsto \ln\left(x^2-6x+10\right)-5, \quad x \in \mathbb{R}, \quad x \leq 3.$$

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Sketch on the same diagram the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, giving the coordinates of the axial intercepts. [3]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{1}{x-5}+x-2, \quad x \in \mathbb{R}, \quad x \neq 5.$$

(iii) Find the range of gf^{-1}. [2]

(iv) Solve the inequality $\mathrm{f}(x)<\mathrm{g}(x)$. [2]

11 [It is given that a right circular cone of radius r and height h has total surface area $\pi r^2 + \pi r\sqrt{r^2 + h^2}$ and volume $\frac{1}{3}\pi r^2 h$.

An ice-cream company plans to launch a new product called Tasty Cone into the market. Each Tasty Cone is in the shape of a right circular cone with a base radius r cm and height h cm as shown in Fig. 1.

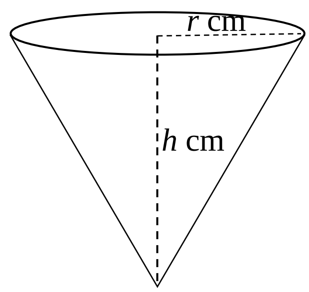


Fig. 1

(i) The volume of each Tasty Cone is set by the company to be 100 ml and the total surface area, S cm^2, should be as small as possible to reduce the cost of packaging. It is given that $1 \text{ ml} = 1 \text{ cm}^3$.

Show that $S = \pi r^2 + \frac{1}{r}\sqrt{\pi^2 r^6 + 90000}$. [2]

(ii) Sketch the graph of S for $r > 0$. Hence write down the value of r and find the corresponding value of h which will give the smallest S, giving your answers correct to 3 significant figures. [3]

A special edition of the Tasty Cone will include a mystery flavour in the shape of a sphere inscribed in the cone as shown in Fig. 2. The sphere has centre O and radius 2 cm. The cross-section of the cone is shown in Fig. 3, where A is a point of contact of the sphere with the cone.

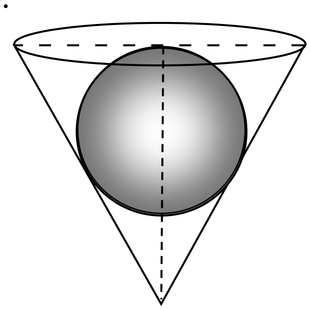


Fig. 2

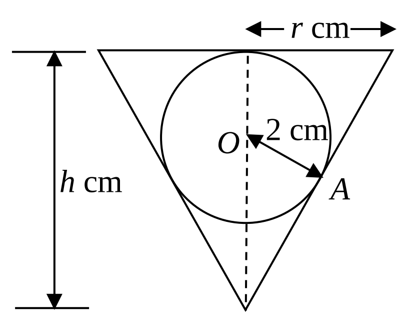


Fig. 3

(iii) Show that $r^2 = \dfrac{4h}{h-4}$. [2]

(iv) Use differentiation to find the exact values of h and r which will give the smallest volume of the cone in the special edition. [5]

12 On 1 Jan 2023, Sam takes an interest-free study loan of $28 000 from his parents for his university fees. He starts to repay $$m$ to his parents on 1 Feb 2023 and progressively increases $50 in his repayment amount at the start of each subsequent month.

(i) Given that $m = 500$, on which date will Sam finish repaying his parents? [4]

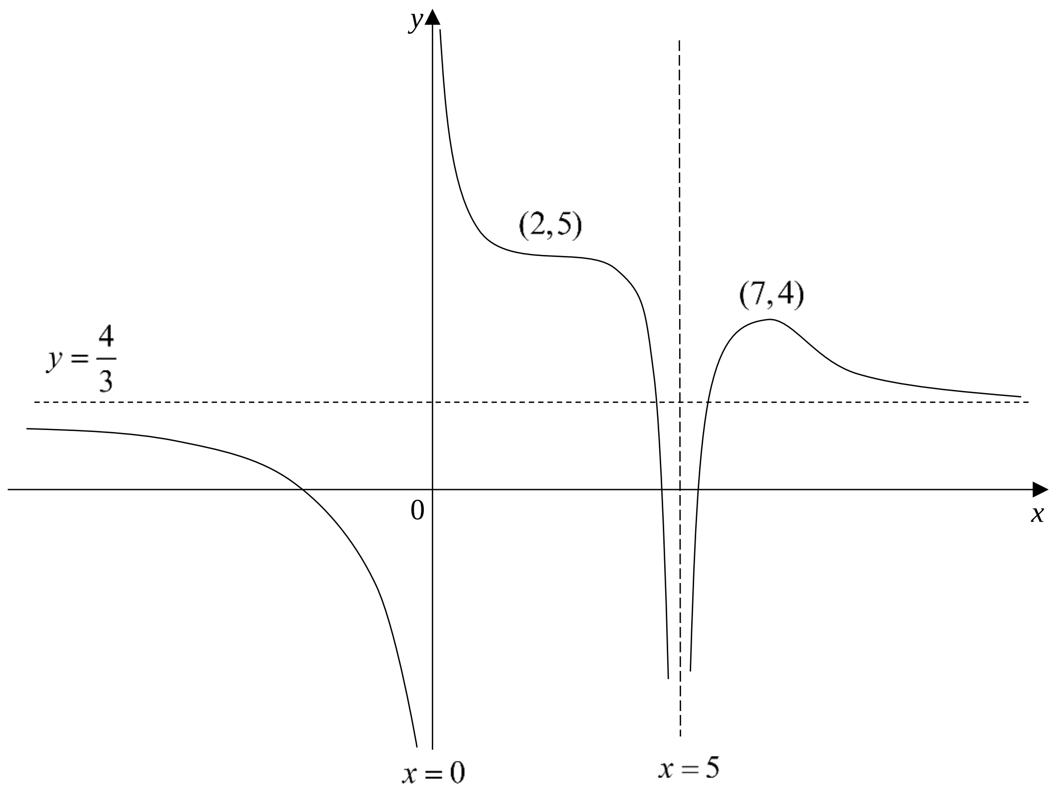
(ii) If Sam wants to finish repaying his parents by the end of 2024, find the minimum value of m, giving your answer to the nearest dollars. [3]

On 1 Jan 2023, Sam's friend, Joshua, takes a study loan of $28 000 from UAB Bank. The bank charges an interest rate of 0.2% per month at the end of each month from the start of the loan period.

(iii) Given that he repays $500 to the bank on the first day of every month beginning from 1 Feb 2023 onwards, on which date will Joshua finish repaying his study loan? [5]

HCI 2021 JC1 Promo (3 hours, 100 marks)

1 The diagram below shows the curve $y = \mathrm{f}(x)$. The curve has a stationary point of inflexion at $(2,5)$ and a maximum point at $(7,4)$. The curve also has vertical asymptotes $x = 0$, $x = 5$ and a horizontal asymptote $y = \dfrac{4}{3}$.



Sketch the curve $y = \mathrm{f}'(x)$, labelling clearly the coordinates of any axial intercepts, turning points and equations of asymptotes where applicable. [3]

2 A local family consisting of adults, children and senior citizens, is planning for a trip to the Jewel Bird Park. The ticket prices are listed as shown below. If all members in the family purchase the Local Resident Discounted tickets, they will need to pay a total price of $301.20. If all members in the family purchase the Wildlife Quest Bundle, they will need to pay a total price of $478.50. Given that the number of adults is four times the number of senior citizens, find the number of adults, children and senior citizens in the family. [3]

Local Resident Discounted Tickets

Adult	\$27.00
Senior Citizen Ages 60 and above	\$15.00
Child Ages 3 to 12	\$18.40

Wildlife Quest Bundle

Adult Admission + Wildlife Quest	\$39.00
Child Admission + Wildlife Quest Ages 3 to 12	\$29.50

3 A curve C has equation $kxe^{y}+ke^{x}=y^{2}+k^{2}$, where k is a positive constant.

(i) Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, y and k. [3]

(ii) Explain why there is no point on C where the tangent is parallel to the x-axis. [2]

4 With respect to origin O, the distinct points P, Q, R and S have position vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ respectively. It is known that $\mathbf{q}$ is a unit vector.

(i) Given that $(\mathbf{r}-\mathbf{p})\times(\mathbf{p}-\mathbf{q})=\mathbf{0}$, state, with justification, the relationship between the points P, Q and R. [2]

(ii) Give a geometrical interpretation of $|\mathbf{q}\cdot(\mathbf{r}-\mathbf{s})|$. [1]

(iii) Given that $\mathbf{p}=\begin{pmatrix}-1\\2\\4\end{pmatrix}$ and that $\mathbf{q}$ is parallel to and in the opposite direction of $\mathbf{p}$, find $\mathbf{q}$. [2]

5 A curve has equation $y = x\mathrm{e}^{x^2}$.

(i) The region R is bounded by the curve, the lines $y = -\mathrm{e}$, $y = \mathrm{e}$ and the y-axis. Find the exact area of R. [4]

(ii) Find the volume generated when R is rotated about the x-axis through $360°$, giving your answer correct to 3 decimal places. [3]

6 The sum of the first n terms of a series is given by the expression $\mathrm{e}^2 - (-2)^n\left(\mathrm{e}^{2-n}\right)$.

(i) State the first term of the series in terms of e. [1]

(ii) By finding the n^{th} term of the series, show that this is a geometric series. [3]

(iii) Explain why the sum to infinity, S, of the series exists, and determine the exact value of S. [3]

7 A curve C has parametric equations

$$x = \theta^2 + 1\ ,\ \ y = 2\sin\theta + 1 \ \text{ where } -\pi \le \theta \le \pi .$$

(i) Find the exact values of θ at which C crosses the x-axis. [1]

(ii) Sketch C, labelling the coordinates of the points at which C crosses the x-axis. [2]

(iii) Find the equation of the tangent to C at the point P with parameter p, where $-\pi \le p \le \pi$. [3]

(iv) Q is a point on C such that the tangent at Q is parallel to the y-axis. Find the area bounded by C, the tangent at Q and the x-axis. [3]

8 A function f is said to be self-inverse if $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all x in the domain of f.
It is given that g is a self-inverse function and is defined by

$$\mathrm{g} : x \mapsto \frac{x+a}{3x+b},\ \text{for } x \in \mathbb{R},\ x > -\frac{b}{3}\ ,$$

where a and b are constants and $\mathrm{g}(1) = 5$.

(i) Find the value of b and show that $a = 9$. [3]

(ii) Find $\mathrm{g}^{2021}(1)$. [1]

The function h is defined by

$$\mathrm{h} : x \mapsto |1-x|(x+5),\ \text{for } x \in \mathbb{R},\ x \ge 2.$$

(iii) Given that h^{-1} exists, find h^{-1} in similar form. [4]

(iv) Show that gh exists and find the exact range of gh. [3]

9 The curve C has equation $y=\dfrac{(1-x)(x-2)}{x^2-2x-8}$.

(i) Find the value of x when $y=-1$. [1]

(ii) Sketch C, showing clearly the equations of any asymptotes and coordinates of any turning points and axial intercepts. [3]

(iii) Solve the inequality [1]

$$\frac{-x^2+3x-2}{x^2-2x-8}\geq -1.$$

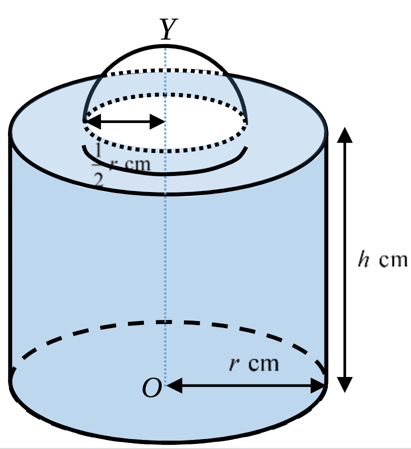
(iv) Using the result in part **(iii)**, solve the inequality [2]

$$\frac{-1+3x-2x^2}{1-2x-8x^2}\geq -1.$$

(v) Find the area bounded by the curve C and the x-axis, leaving your answer in the form $a+\ln\dfrac{b}{c}$, where a, b and c are integers. [5]

$\left[\text{It is given that a sphere of radius } R \text{ has surface area } 4\pi R^2 \text{ and volume } \dfrac{4}{3}\pi R^3.\right]$

10 A perfume maker designs a prototype of a perfume bottle of fixed volume V cm^3 for a new fragrance as shown in the diagram below.



The prototype comprises 2 different segments where the vertical axis of the prototype, OY, is $\left(h+\frac{1}{2}r\right)$ cm. The bottom segment is a glass cylinder of radius r cm and height h cm. The top segment is a chrome-plated plastic hemisphere of radius $\frac{1}{2}r$ cm. It is assumed that the prototype is of negligible thickness and there is no gap between the 2 segments.

(i) Find h in terms of V, r and π. [2]

(ii) Given that the cost of manufacturing the glass cylinder is \$8 per cm^2 and the chrome-plated plastic is \$3 per cm^2, show that the total cost of manufacturing the prototype is \$$\left(\frac{85}{6}\pi r^2+\frac{16V}{r}\right)$. Hence, using differentiation, find the exact value of $\frac{h}{r}$ such that the cost of manufacturing the prototype is minimum. [7]

It is now given that the diameter of the chrome-plated hemisphere is 3 cm and the height of the glass cylinder is 5 cm.

(iii) A crack at the bottom of the prototype causes the perfume to leak out of the prototype at a constant rate of $1.5\ \text{cm}^3/\text{s}$. Given that perfume is initially filled to the brim of the top segment of the prototype, find the exact rate of decrease of the height of the perfume in the prototype 5 seconds after the prototype has cracked. [3]

11 The points A, B and C have coordinates $(-1,-12,4)$, $(5,0,7)$ and $(6,1,4)$ respectively. The line l_1 has equations $\frac{x-1}{2}=\frac{2-y}{3}$, $z=4$ and the line l_2 passes through A and B.

(i) Find the coordinates of the foot of perpendicular from C to l_1. [4]

(ii) Find the acute angle between l_1 and l_2. [3]

(iii) The point D is on l_2 such that the distance from D to A is twice the distance from D to B. Find the possible point(s) D. [4]

(iv) The line l_3 passes through point A and is perpendicular to both l_1 and l_2. Find the equation of l_3. [2]

12 A couple takes up a housing loan of $\$L$ and the interest is charged before each monthly repayments at a fixed rate of $p\%$ per annum. Their monthly repayment commences on 1 September 2021. Monthly repayments of $\$x$ are due and payable on the first day of subsequent months until their housing loan is fully repaid.

(i) State an expression in terms of L and p for the interest charged before their first repayment on 1 September 2021. [1]

(ii) Show that the outstanding loan at the start of the n^{th} month after their monthly repayment is given by

$$\left(1+\frac{p}{1200}\right)^n L-\frac{1200x}{p}\left[\left(1+\frac{p}{1200}\right)^n-1\right].$$

[3]

The couple is taking a housing loan of \$504,000 at a fixed interest rate of 2.6% per annum.

(iii) Calculate the monthly repayment if the couple plans to repay the loan in 30 years. [2]

(iv) Given that the couple decides to pay monthly repayments of \$4000, find the date at which the couple will be able to fully repay their housing loan and the amount that the couple pays for their final monthly repayment. [3]

The couple decides to start adopting a savings plan on 1 September 2021. The couple decides to deposit $\$k$ on 1 September 2021 to the savings plan and for each subsequent month, they will deposit $\$a$ more than the previous month. Each month, the savings plan gives a fixed interest of 0.1% for the amount deposited for that month. The total amount that the couple will have in the savings plan after n months is given by

$$\sum_{r=1}^{n}\left(450.45+50.05r\right).$$

(v) Find the values of k and a. [2]

(vi) Assuming that the couple intends to pay monthly repayments of \$4000 for their housing loan, find the least number of months that is needed so that the couple can use the amount in their savings plan to make a one-time repayment to fully repay their outstanding housing loan. [2]

HCI 2022 JC1 Promo (3 hours, 100 marks)

1 **(i)** Solve algebraically the inequality $x^2+x>\dfrac{4x}{3-x}$. [3]

(ii) Hence solve the inequality $x-x^2<\dfrac{4x}{3+x}$. [2]

2 The equation of a curve G is given by

$$x^2y-\ln y=1\ ,\quad \text{where } x<0 \text{ and } y>0.$$

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2xy^2}{1-x^2y}$. [3]

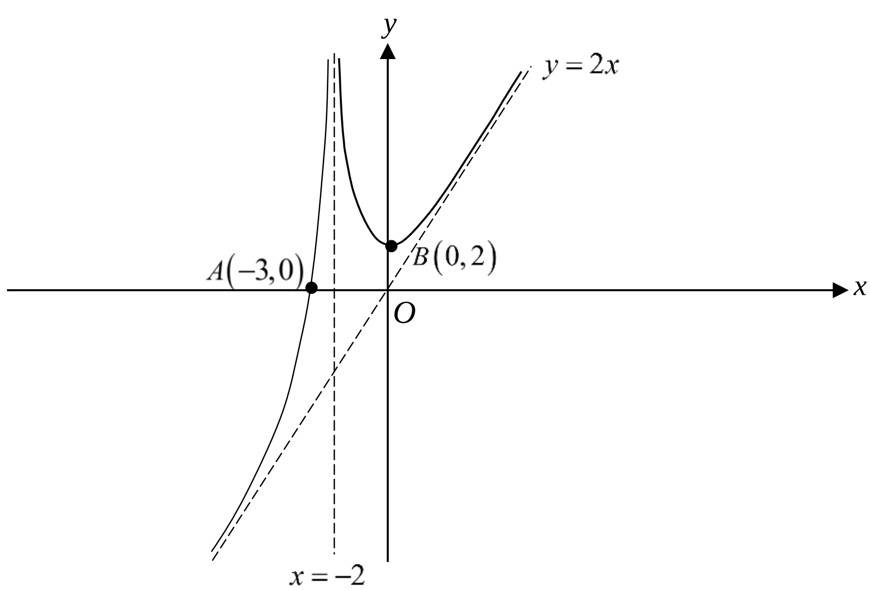
(ii) Find the equation of the tangent to G at the point P where $y=1$. [3]

3 **(i)** Find $\displaystyle\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$ in terms of n and state, with justification, the value of

$\displaystyle\sum_{r=1}^{\infty}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$. [3]

(ii) Using the results in part **(i)**, find $\displaystyle\sum_{r=n+2}^{\infty}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$ in terms of n. [3]

4



In the diagram, the curve $y=\mathrm{f}(x)$ passes through the points A and B with coordinates $(-3,0)$ and $(0,2)$ respectively where B is a minimum point. The asymptotes are $y=2x$ and $x=-2$.

On separate diagrams, stating clearly, where possible, the equations of any asymptotes and coordinates of any turning points and axial intercepts, sketch the graphs

(i) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y=\mathrm{f}(2-x)$. [3]

5 Relative to the origin O, the points M, S and T have coordinates $(3, p, 0.03)$, $(1,1,0.02)$ and $(q, 0, 0.01)$ respectively, where p and q are real constants. The line l passes through the point M and is parallel to the vector $(\mathbf{i}+\mathbf{j})$.

(i) Given that l is perpendicular to the line passing through S and T, show that $q=2$. [2]

(ii) Hence find the area of triangle OST, leaving your answer correct to 6 decimal places. [2]

(iii) It is further given that l and the line passing through S and T do not intersect. Determine the set of possible values of p. [3]

6 The function h is defined as follows.

$$\mathrm{h}: x \mapsto \frac{1}{8}\left(x^3-6x^2+32\right), \qquad x \in \mathbb{R}.$$

(i) Explain why h does not have an inverse. [1]

(ii) If the domain of h is restricted to $0 \le x \le k$, state the largest value of k for which the function h^{-1} exists. [1]

Use the domain of h in part **(ii)** for the rest of this question.

(iii) Sketch the graphs of h and h^{-1} on the same diagram, showing clearly the relationship between the two graphs, and the coordinates of the end points for both graphs. [3]

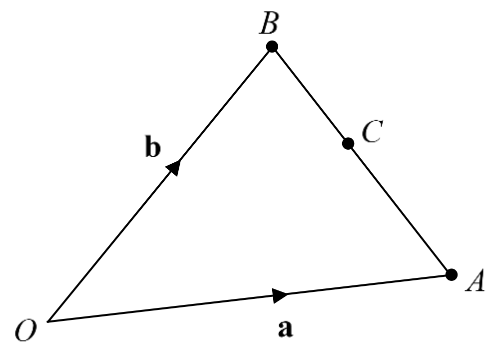
(iv) Deduce the solution(s) of the equation $\mathrm{h}(x)=\mathrm{h}^{-1}(x)$. [2]

(v) The function g is defined as follows.

$$\mathrm{g}: x \mapsto \ln\left((x-3)^2+1\right), \qquad x \in \mathbb{R}.$$

Given that the composite function gh exists, find the exact range of gh. [2]

7



With reference to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ as shown in the diagram above. The point C lies between A and B such that the distance between A and C is λ times the distance between A and B, where $\lambda \in \mathbb{R}$, $0 < \lambda < 1$.

(i) State the position vector of C in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. [1]

(ii) The mid-point M, of the line segment OB is such that $\overrightarrow{MC}$ is parallel to $(2\mathbf{a}+\mathbf{b})$. Find the value of λ. [3]

(iii) Interpret the geometrical meaning of $\dfrac{|\mathbf{b}\times\mathbf{a}|}{|\mathbf{b}|}$. [1]

(iv) Using the value of λ found in part **(ii)**, find an expression for the shortest distance from C to the line segment OB in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

8 **(a)** Find $\int 3t\tan^{-1}(3t)\,\mathrm{d}t$. [4]

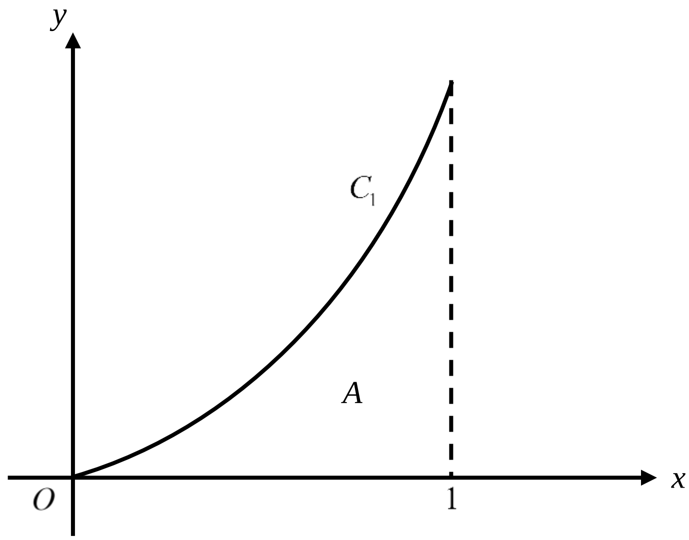
(b) Using the substitution $u = x^2+1$, show that $\int_0^{\sqrt{7}} x^3\left(x^2+1\right)^{\frac{1}{3}}\,\mathrm{d}x$ can be expressed as

$$\frac{1}{2}\int_a^b u^{\frac{4}{3}} - u^{\frac{1}{3}}\,\mathrm{d}u,$$

where a and b are constants to be determined.

Hence find the exact value of $\int_0^{\sqrt{7}} x^3\left(x^2+1\right)^{\frac{1}{3}}\,\mathrm{d}x$. [5]

9



The diagram shows a sketch of the curve C_1 with equation $\mathrm{f}(x) = xe^{x^2}$ for $0 \le x \le 1$. The region under C_1 is given by A.

(i) **(a)** By sketching suitable vertical rectangular strips of equal width $\frac{1}{n}$ on the diagram, determine whether

$$S_n = \frac{1}{n}\left[\mathrm{f}\left(\frac{1}{n}\right) + \mathrm{f}\left(\frac{2}{n}\right) + \mathrm{f}\left(\frac{3}{n}\right) + \ldots + \mathrm{f}\left(\frac{n}{n}\right)\right]$$

is more than or less than the area of A. [2]

(b) Find the exact value of $\lim_{n \to \infty} S_n$. [2]

(ii) The curve C_2 is defined by the parametric equations

$$x = t^2, \quad y = t + 1 \qquad \text{where } t \in \mathbb{R}.$$

(a) Sketch C_2, stating clearly the coordinates of the axial intercepts, and find the cartesian equation of C_2. [3]

(b) The region bounded by C_1, C_2 and the y-axis is rotated through 4 right angles about the x-axis. Find the volume generated, correct to 4 decimal places. [5]

10 A function f is defined by

$$\mathrm{f} : x \mapsto x + 2a + \frac{4a^2}{x - 2a}, \ x \in \mathbb{R}, \ x \neq 2a,$$

where a is a positive real constant.
It is known that one of the turning points is $(4a, 8a)$.

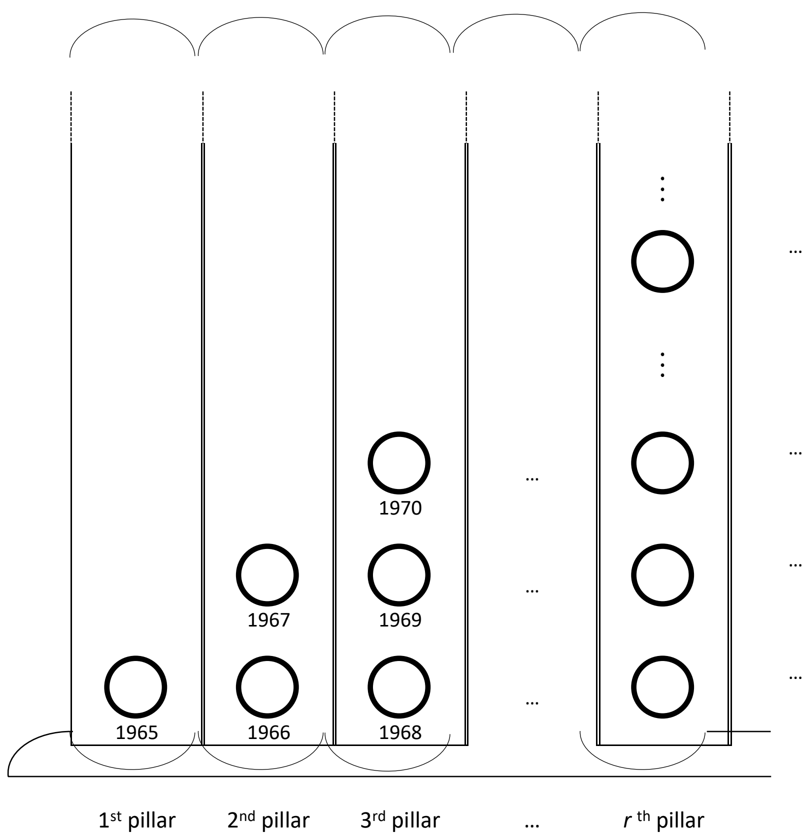
(i) Sketch the curve $y = \mathrm{f}(x)$, showing clearly the coordinates of the turning points and the equations of any asymptotes. [3]

(ii) By adding a suitable curve to your sketch in part **(i)**, solve, in terms of a, the inequality

$$\mathrm{f}(x) - |x| > 2a .$$ [2]

(iii) Solve $|\mathrm{f}(x)| - |x| = 2a$, leaving your answer in exact form in terms of a. [3]

11 An artist was commissioned to create an installation art piece for the 2022 National Day celebrations. Inspired by the traditional Malay board game *Congkak* from his childhood, which is played on a wooden boat-shaped block with circular holes, he built a row of identical rectangular pillars. He drilled 1 hole in the first pillar, 2 holes in the second pillar, 3 holes in the third pillar, and so on, with each hole commemorating a year starting from 1965, the year of Singapore's independence. Part of the front view of the pillars is shown in the diagram.



To represent the national colour white which signifies pervading and everlasting purity and virtue, the artist put white sand into each hole and sealed the white sand at the lower half of the hole. In the first hole commemorating the year 1965, he put in 1000g of white sand. In the second hole commemorating the year 1966, he poured 1% less white sand than the amount he put in the first hole. In the third hole commemorating the year 1967, he put in 1% less white sand than the amount he put in the second hole, and so on.

(i) By considering the number of holes drilled on the r^{th} pillar, show that the total number of holes drilled in the first r pillars is given by

$$\frac{r(r+1)}{2}.$$ [2]

(ii) Find M, the minimum amount of white sand in grams needed for all the holes up to and including the hole commemorating the year 2022. Give your answer correct to the nearest integer. [2]

(iii) It is intended that the art piece be installed with additional holes and pillars every National Day after 2022. In the years to come, the first hole to contain less than 250 g of white sand occurs in the m^{th} pillar. Find the value of m. [3]

To represent the national colour red which symbolises universal brotherhood and the equality of man, the artist placed red *Saga* (*Adenanthera pavonina*, an exotic tree species which has naturalised in Singapore) seeds on top of the white sand in each hole. In the first hole commemorating the year 1965, 2 seeds were placed. In the second hole commemorating 1966, 4 seeds were placed. In the third hole commemorating 1967, 6 seeds were placed, and so on.

(iv) Find, in terms of r,

(a) the number of *Saga* seeds in the top-most hole of the r^{th} pillar. [1]

(b) the number of *Saga* seeds in the bottom-most hole of the r^{th} pillar; [2]

(c) the total number of *Saga* seeds that can be placed in all the holes of the r^{th} pillar. [2]

12 [It is given that volume of a trapezoid = cross-sectional area $\times$ length.]

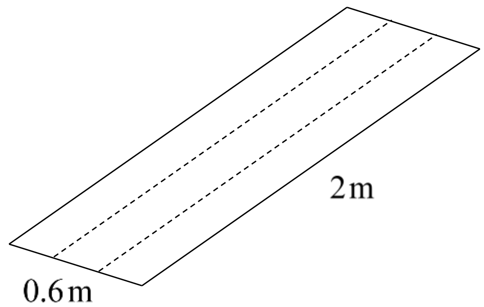


Figure 1

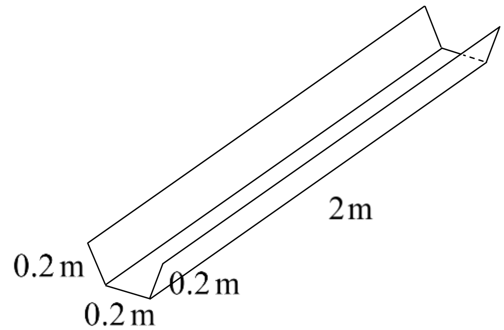


Figure 2

In Figure 1, a polyvinyl chloride (PVC) sheet of length $2\,\text{m}$ and breadth $0.6\,\text{m}$ with negligible thickness is used to construct a rain gutter in the shape of a trapezoid. The PVC sheet is bent along its length such that its breadth is divided into 3 equal parts as shown in Figure 2. The two walls of the rain gutter are such that they are at an angle θ with the horizontal. The cross-section of the rain gutter is shown in Figure 3.

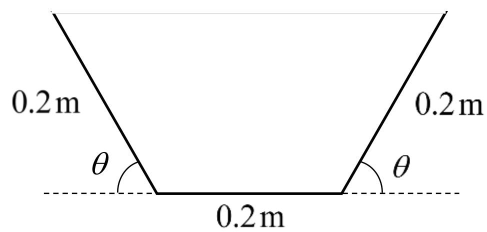


Figure 3

(i) Show that the value of the total volume of the rain gutter is given by

$$0.04\left(2\sin\theta+\sin 2\theta\right).$$ [2]

(ii) Use differentiation to determine the exact value of θ for the rain gutter to hold the maximum amount of rainwater. [4]

For the rest of the question, let $\theta = \frac{\pi}{4}$.

On a particular day with heavy downpour, it is found that a constant amount of $0.002\,\text{m}^3$ of rainwater is collected by the rain gutter every second. The cross-section of the rain gutter on a day with heavy downpour is shown in Figure 4.

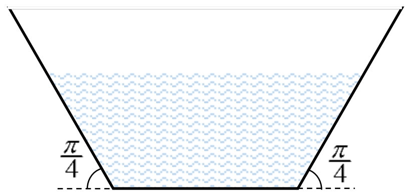


Figure 4

(iii) By taking *p* as the depth of rainwater in the rain gutter at time *t*, show that, when the depth of rainwater in the rain gutter is $0.1\,\text{m}$, the rate of change of depth of rainwater in the rain gutter is given by $0.0025\,\text{ms}^{-1}$. [4]

(iv) Hence find the rate of change of the total surface area of the rain gutter in contact with the rainwater at the instant when the depth of rainwater in the rain gutter is $0.1\,\text{m}$. [3]