2015 H2 MATH (9740/02) JC 2 PRELIM EXAMINATION – MARKING SCHEME

Qn	Solution
1	**Mathematical Induction**
1(i)	$u_1 = 3 = \dfrac{2^2 - 1}{2 - 1}$. $u_2 = 3 - \dfrac{2}{3} = \dfrac{7}{3} = \dfrac{2^3 - 1}{2^2 - 1}$. $u_3 = 3 - \dfrac{2}{\left(\frac{7}{3}\right)} = \dfrac{15}{7} = \dfrac{2^4 - 1}{2^3 - 1}$. Hence $a = 2$.
1(ii)	Let P_n be the statement that $u_n = \dfrac{2^{n+1} - 1}{2^n - 1}$ for $n \geq 1$. When $n = 1$, LHS = $u_1 = 3$ RHS $= \dfrac{2^2 - 1}{2 - 1} = \dfrac{3}{1} = 3 =$ LHS (shown). $\therefore \mathrm{P}_1$ is true. Assume P_k is true some $k \in \mathbb{Z}^+$, i.e., $u_k = \dfrac{2^{k+1} - 1}{2^k - 1}$. To prove P_{k+1} is also true, i.e. $u_{k+1} = \dfrac{2^{k+2} - 1}{2^{k+1} - 1}$. LHS = u_{k+1} $= 3 - \dfrac{2}{u_k}$ $= 3 - \dfrac{2}{\left(\dfrac{2^{k+1} - 1}{2^k - 1}\right)}$ $= 3 - \dfrac{2\left(2^k - 1\right)}{2^{k+1} - 1}$ $= \dfrac{3.2^{k+1} - 3 - 2^{k+1} + 2}{2^{k+1} - 1}$ $= \dfrac{2.2^{k+1} - 1}{2^{k+1} - 1}$ $= \dfrac{2^{k+2} - 1}{2^{k+1} - 1}$ = RHS $\therefore \mathrm{P}_k$ is true $\Rightarrow \mathrm{P}_{k+1}$ is true. Since P_1 is true, and P_k is true $\Rightarrow \mathrm{P}_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

1(iii)	$u_1u_2u_3...u_n=\left(\frac{2^2-1}{2-1}\right)\left(\frac{2^3-1}{2^2-1}\right)\left(\frac{2^4-1}{2^3-1}\right)...\left(\frac{2^{n+1}-1}{2^n-1}\right)$ $=\left(\frac{2^{n+1}-1}{2-1}\right)$ $=2^{n+1}-1\to\infty$ as $n\to\infty$. Hence, the limit does not exist.

Qn	Solution
2	**Differentiation (Maxima and Minima)**
2(i)	Base of isosceles triangle $=10-2x$ $y+2h=20$ By Pythagoras Theorem, $x^2-h^2=(5-x)^2$ $x^2-h^2=25-10x+x^2$ $x=\dfrac{25+h^2}{10}$ $V=\dfrac{1}{2}(10-2x)hy$ $=\dfrac{1}{2}\left[10-2\left(\dfrac{25+h^2}{10}\right)\right]h(20-2h)$ $=\dfrac{1}{5}h(10-h)(25-h^2)$ $=\dfrac{1}{5}h(h^3-10h^2-25h+250)$ $=\dfrac{1}{5}(h^4-10h^3-25h^2+250h)$ (shown)
(ii)	Differentiate wrt *x*, $\dfrac{dV}{dh}=\dfrac{1}{5}(4h^3-30h^2-50h+250)$ For maximum *V*, $\dfrac{dV}{dh}=0$. $\therefore\dfrac{dV}{dh}=\dfrac{1}{5}(4h^3-30h^2-50h+250)=0$ $4h^3-30h^2-50h+250=0$ Using GC, $h=8.0902$ cm (rejected $\because 2x=18.090>10$), $h=-3.0902$ cm (rejected $\because h>0$) or $h=2.5$ cm. Differentiate wrt *x*, $\dfrac{d^2V}{dh^2}=\dfrac{1}{5}(12h^2-60h-50)$ For $h=2.5$ cm, $\dfrac{d^2V}{dh^2}=\dfrac{1}{5}\left(12(2.5)^2-60(2.5)-50\right)$ $=-25<0$ $\therefore$ V is maximum. **<u>Alternatively</u>** Using GC, $\left.\dfrac{d^2V}{dh^2}\right\|_{h=2.5}=-25<0$ $\therefore$ V is maximum.

Alternative (By 1st derivative test)

h	2.5^-	2.5	2.5^+
$\dfrac{\mathrm{d}V}{\mathrm{d}h}$	/	—	\

$\therefore$ V is maximum.

$$\therefore \text{ maximum } V = \frac{1}{5}\left(\left(2.5\right)^4 - 10\left(2.5\right)^3 - 25\left(2.5\right)^2 + 250\left(2.5\right)\right)$$
$$= 70.3125 \text{ cm}^3$$

Qn	Solution
3	**Complex 1 & 2**
(ai)	**Method 1:** Since $1+2\text{i}$ is a root to $x^3+ax^2+bx-5=0$, $(1+2\text{i})^3+a(1+2\text{i})^2+b(1+2\text{i})-5=0$ $(-11-2\text{i})+a(-3+4\text{i})+b(1+2\text{i})-5=0$ Comparing real part, $-11-3a+b-5=0$ $-3a+b=16---(1)$ Comparing imaginary part, $-2+4a+2b=0$ $4a+2b=2---(2)$ Solving (1) and (2), $a=-3, b=7$. $x^3-3x^2+7x-5=0$ Using GC, the other roots are $1-2\text{i}$ and 1. **Method 2:** Since the coefficients of the polynomial are all real, $1+2\text{i}$ is a root implies $1-2\text{i}$ is also a root. $(x-(1+2\text{i}))(x-(1-2\text{i}))=\left((x-1)^2-(2\text{i})^2\right)$ $=x^2-2x+5$ By comparing coefficients, $x^3+ax^2+bx-5=\left(x^2-2x+5\right)(x-1)$ By comparing coefficient of x^2, $a=-1-2=-3$ By comparing coefficient of x, $b=2+5=7$ Therefore, the other roots are $1-2\text{i}$ and 1.
(bi)	$z^4=2\text{e}^{\text{i}\pi}$ $z=2^{\frac{1}{4}}\text{e}^{\text{i}\left(\frac{\pi+2k\pi}{4}\right)}$ $=2^{\frac{1}{4}}\text{e}^{\text{i}\left(\frac{1}{4}+\frac{k}{2}\right)\pi}$ $\quad k=-2,-1,0,1$ $=2^{\frac{1}{4}}\text{e}^{\text{i}\left(-\frac{3}{4}\right)\pi}, 2^{\frac{1}{4}}\text{e}^{\text{i}\left(-\frac{1}{4}\right)\pi}, 2^{\frac{1}{4}}\text{e}^{\text{i}\left(\frac{1}{4}\right)\pi}, 2^{\frac{1}{4}}\text{e}^{\text{i}\left(\frac{3}{4}\right)\pi}$
(ii)	$w=\text{e}^{\text{i}\frac{\pi}{2}}$ or $w=\text{i}$

(iii)	**Hence method:** $\lvert wz_3 - w^* z_3 \rvert^2 = \lvert z_4 - z_2 \rvert^2$ $= \left(2\left(2^{\frac{1}{4}}\right)\right)^2$ $= 4\sqrt{2}$ **Otherwise method:** $\lvert wz_3 - w^* z_3 \rvert^2 = \lvert z_3 \left(w - w^*\right) \rvert^2$ $= \lvert z_3 \rvert^2 \lvert 2\mathrm{i} \rvert^2$ $= 4\left(2^{\frac{1}{4}}\right)^2$ $= 4\left(2^{\frac{1}{2}}\right) = 4\sqrt{2}$	B1 B1 B1 B1

Qn	Solution
4	**Vectors 2 & 3**
(i)	$\cos\theta = \dfrac{\left\lvert \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right\rvert}{\sqrt{3}\sqrt{2}} = \dfrac{2}{\sqrt{6}}$ $\theta = 0.615\text{rad} \text{ or } 35.3^\circ$
(ii)	$x+y+z=1$ $x \quad +z=2$ By GC, $l:\mathbf{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\0\\1\end{pmatrix},\ \lambda\in\mathbb{R}$
(iii)	$p_k:\ x+\left(\frac{1}{2}\right)^{k-1}y+z=2\left[1-\left(\frac{1}{2}\right)^{k}\right]=2-\left(\frac{1}{2}\right)^{k-1}$ Since p_k tends to q as $k\to\infty$ and p is p_1, the limit of the acute angle is 0.615rad.
(iv)	$\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\\left(\frac{1}{2}\right)^{k-1}\\1\end{pmatrix}=-1+1=0$ Thus p_k is parallel to l for any $k\in\mathbb{Z}^+$. $\begin{pmatrix}2\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\\left(\frac{1}{2}\right)^{k-1}\\1\end{pmatrix}=2-\left(\frac{1}{2}\right)^{k-1}$ Thus the point $(2,-1,0)$ lies in p_k for any $k\in\mathbb{Z}^+$. l lies in all planes in P.
(v)	Since $a\neq c$ and $\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}=-a+c\neq 0$, π is not parallel to l, π and any two planes in P must **intersect at only 1 point** which lies on l.
(vi)	$\pi:\ \mathbf{r}\cdot\begin{pmatrix}1\\\frac{1}{2}\\1\end{pmatrix}=1$ Since π is parallel to the p_2 in P, and $(2,-1,0)$ lies in p_2, the perpendicular distance is also the shortest distance between π and p_2. required distance $=\dfrac{\frac{3}{2}-1}{\sqrt{1+1+\frac{1}{4}}}=\dfrac{1}{3}$

	Alternatively Note that (1,0,0) lies in π . $$\text{required distance} = \frac{\left|\left[\begin{pmatrix}2\\-1\\0\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix}\right]\cdot\begin{pmatrix}1\\\frac{1}{2}\\1\end{pmatrix}\right|}{\sqrt{1+1+\frac{1}{4}}} = \frac{1}{3}$$

Qn	Solution
5	**Permutation and Combination**
5(a)	Number of arrangements = $5!\times 4!=2880$
(b)(i)	Number of arrangements = $^{9}C_{8}\times\dfrac{8!}{8}=45360$
(ii)	Number of arrangements = $^{7}C_{6}\times\dfrac{6!}{6}\times 6\times 5=25200$ **Alternatively:** Number of arrangements = $^{7}C_{6}\times\dfrac{6!}{6}\times{}^{6}C_{2}\times 2!=25200$

Qn	Solution
6	**Binomial Distribution**
(a)	Let X be the number of students, out of 10, who score distinction $X \sim \text{B}(10, 0.01p)$ $\text{P}(X \geq 2) = 0.95$ $1 - \text{P}(X = 0) - \text{P}(X = 1) = 0.95$ $1 - \binom{10}{0}(0.01p)^0(1-0.01p)^{10} - \binom{10}{1}(0.01p)^1(1-0.01p)^9 = 0.95$ $1 - (1-0.01p)^{10} - 10(0.01p)(1-0.01p)^9 = 0.95$ $(1-0.01p)^{10} + 10(0.01p)(1-0.01p)^9 = 0.05$ Using GC, $p = 39.416 \approx 39.4$
(b)	Let W be the number of students, out of 10, who score distinction $W \sim \text{B}(10, 0.4)$ Since n = 50 is large, by Central limit theorem, $\overline{W} \sim \text{N}\left(4, \frac{2.4}{50}\right)$ approximately $\text{P}\left(\overline{W} < 3.5\right) = 0.011239 \approx 0.0112$ **Alternatively: (not for students)** Let Y be the number of students, out of 500 , who score distinction $Y \sim \text{B}(500, 0.4)$ $\text{P}(Y < 3.5 \times 50) = \text{P}(Y < 175)$ $= \text{P}(Y \leq 174)$ $= 0.0095558 \approx 0.00956 \text{ (3 s.f)}$

Qn 7	Suggested Solutions
7	**Poisson and Normal Distribution**
(i)	Let X be the number of people arriving at a particular bus stop in a randomly chosen period of 5 min. $X \sim \text{Po}(2)$ Let Y be the number of people arriving at a particular bus stop in a period of t min. $Y \sim \text{Po}\left(\frac{2}{5}t\right)$ $\text{P}(Y > 9) < 0.759$ $1 - \text{P}(Y \le 9) < 0.759$ Using table method, When $t = 30$, $1 - \text{P}(Y \le 9) = 0.75761 < 0.759$ When $t = 31$, $1 - \text{P}(Y \le 9) = 0.79081 > 0.759$ Hence greatest possible integer for t = 30 (shown) **Alternatively,** Using GC, $t < 30.039917$ Hence greatest possible integer for t = 30 (shown)
(ii)	Let A be the number of people arriving at a particular bus stop in a randomly chosen period of 60 min. $A \sim \text{Po}\left(\frac{2}{5} \times 60\right) \Rightarrow A \sim \text{Po}(24)$ Since $\lambda = 24 > 10$, $A \sim \text{N}(24, 24)$ approximately. Let L be the number of people leaving the particular bus stop in a randomly chosen period of 60 min. $L \sim \text{Po}\left(\frac{2.5}{5} \times 60\right) \Rightarrow L \sim \text{Po}(30)$ Since $\lambda = 30 > 10$, $L \sim \text{N}(30, 30)$ approximately. Since there were 5 people at the bus stop at 0800 and more than 15 people at the bus stop at 0900, $5 + A - L > 15 \Rightarrow A - L > 10$ $A - L \sim \text{N}(24 - 30, 24 + 30)$ approximately $A - L \sim \text{N}(-6, 54)$ approximately $\text{P}(A - L > 10)$ $= \text{P}(A - L > 10.5)$ after continuity correction $= 0.0124$ (3sf)
(iii)	People arriving and leaving the bus stop may not occur at a constant average rate as the outcomes are influenced by peak hours and lull periods. **Alternatively:** The mean number of people arriving and leaving the bus stop is not constant throughout each time interval as the outcomes are influenced by peak hours and lull periods.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>8</td><td>Hypothesis Testing</td></tr>
<tr><td>(i)</td><td>Let $u = x - 16$
Unbiased estimate of population mean, $\bar{x} = 16 + \bar{u} = 16 + \frac{-23.8}{14} = 14.3$
Unbiased estimate of population variance,
$s_x{}^2 = s_u{}^2 = \frac{1}{13}\left(149.127 - \frac{(-23.8)^2}{14}\right) = 8.359$
Let μ denote the population mean time taken by the students (in hrs).
Given $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$ $\therefore \overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$
H_0: $\mu = 16$
H_1: $\mu \neq 16$
Test statistic: $T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$
Level of Significance: 5%
Reject H_0 if p-value < 0.05
OR Reject H_0 if $|t\text{-value}| > 2.1603$

Under H_0, using G.C., p-value = 0.046492
OR t-value = –2.2001

Since p-value = 0.0465 < 0.05 (OR t-value = –2.2001<–2.1603), we reject H_0 and conclude that there is sufficient evidence at 5% level, that there has been a change in the mean time taken to complete the homework given in a week.</td></tr>
<tr><td>(ii)</td><td>H_0: $\mu = 16$
H_1: $\mu \neq 16$
Test statistic: $Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$
Level of Significance: 5%
Reject H_0 if z-value > 1.9600 or z-value < –1.9600
Since we do not reject H_0, $-1.9600 < \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < 1.9600$
$-1.9600 < \frac{\bar{x} - 16}{3/\sqrt{14}} < 1.9600$
$14.429 < \bar{x} < 17.571$

$\therefore \left\{\bar{x} \in \mathbb{R} : 14.4 < \bar{x} < 17.6\right\}$</td></tr>
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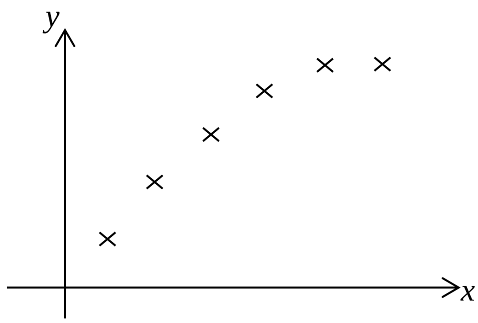
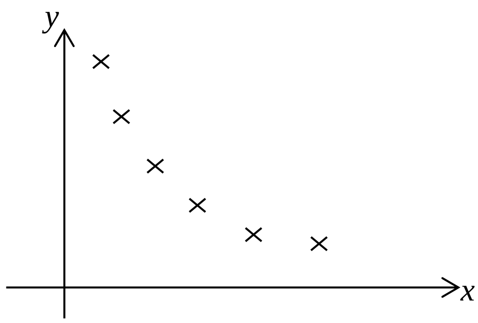
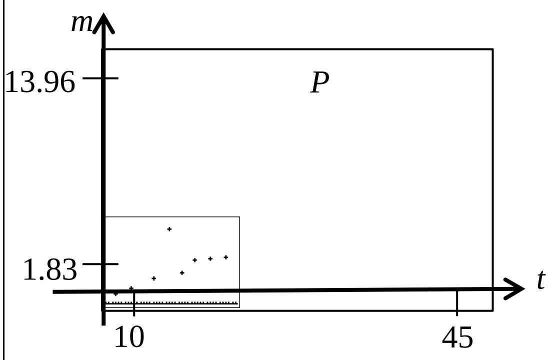
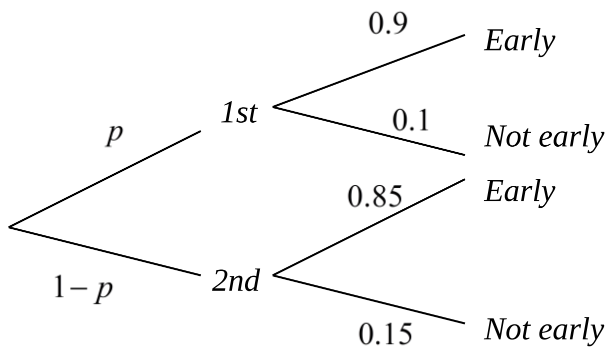
Qn	Solution
9	**Correlation and Regression**
(a)	
(aii)	
bi)	
(bii)	With the removal of the point P, **<u>as t increases, m increases at an approximately constant rate</u>** which is consistent with a linear model. With the removal of the point P, **<u>the remaining points lie close to a straight line</u>** which is consistent with a linear model.
(biii)	$m = 0.204t - 0.0323$ (3 s.f)
(biv)	$m = 0.16934t + 0.72035$ $\dfrac{40.21+k}{8} = 0.16934(31) + 0.72035$ $\therefore k = 7.55$ (2 d.p)

Qn	Solution
10	**Normal and Sampling Distributions**
(i)	Let X be the mass of a randomly chosen chocolate bar (in grams). Let Y be the mass of a randomly chosen candy bar (in grams). $X \sim \mathrm{N}\left(25, 0.3^2\right)$ $Y \sim \mathrm{N}\left(32, 0.5^2\right)$ Let $W = \dfrac{X_1 + X_2 + Y_1 + Y_2 + Y_3}{5} \sim \mathrm{N}\left(\dfrac{25\times 2 + 32\times 3}{5}, \dfrac{0.3^2\times 2 + 0.5^2\times 3}{25}\right)$ i.e. $W \sim \mathrm{N}\left(29.2, 0.0372\right)$ $\mathrm{P}\left(W < 29\right) = 0.14988 = 0.150 \ \left(3 \text{ s.f.}\right)$
(ii)	$X_1 - X_2 \sim \mathrm{N}\left(25 - 25, 0.3^2 + 0.3^2\right)$ i.e. $X_1 - X_2 \sim \mathrm{N}\left(0, 0.18\right)$ $\mathrm{P}\left(\left\|X_1 - X_2\right\| < k\right) = 0.95$ $\mathrm{P}\left(-k < X_1 - X_2 < k\right) = 0.95$ Using G.C., $k = 0.83154 = 0.832 \ \left(3 \text{ s.f.}\right)$
(iii)	$0.08X + 0.03Y \sim \mathrm{N}\left(25\times 0.08 + 32\times 0.03, 0.3^2\times 0.08^2 + 0.5^2\times 0.03^2\right)$ i.e. $0.08X + 0.03Y \sim \mathrm{N}\left(2.96, 0.000801\right)$ $\mathrm{P}\left(0.08X + 0.03Y > 3\right) = 0.078779 = 0.0788 \ \left(3 \text{ s.f.}\right)$

Qn	Solution
11	**Probability**
(a)(i)	$\mathrm{P}(R=3)=\dfrac{\binom{15}{3}\binom{10}{5}}{\binom{25}{8}}=0.10601=0.106$ (3 s.f) (Shown) **Alternatively: Probability method** $\mathrm{P}(R=3)=\left(\frac{15}{25}\right)\left(\frac{14}{24}\right)\left(\frac{13}{23}\right)\left(\frac{10}{22}\right)\left(\frac{9}{21}\right)\left(\frac{8}{20}\right)\left(\frac{7}{19}\right)\left(\frac{6}{18}\right)\left(\frac{8!}{3!5!}\right)$ $=0.10601$ $=0.106$ (3 s.f) (Shown)
(ii)	$\mathrm{P}(R=r)>\mathrm{P}(R=r+1)$ $\dfrac{\binom{15}{r}\binom{10}{8-r}}{\binom{25}{8}}>\dfrac{\binom{15}{r+1}\binom{10}{7-r}}{\binom{25}{8}}$ $\binom{15}{r}\binom{10}{8-r}>\binom{15}{r+1}\binom{10}{7-r}$ $\dfrac{15!}{r!(15-r)!}\dfrac{10!}{(8-r)!(2+r)!}>\dfrac{15!}{(r+1)!(14-r)!}\dfrac{10!}{(7-r)!(3+r)!}$ $(r+1)!(14-r)!(7-r)!(3+r)!>r!(15-r)!(8-r)!(2+r)!$ (shown) $\dfrac{(r+1)!(14-r)!(7-r)!(3+r)!}{r!(15-r)!(8-r)!(2+r)!}>1$ $\dfrac{(r+1)(r+3)}{(15-r)(8-r)}>1$ Using GC, when $r=4, \dfrac{(r+1)(r+3)}{(15-r)(8-r)}=0.795<1$ when $r=5, \dfrac{(r+1)(r+3)}{(15-r)(8-r)}=1.6>1$ $r=5$
(b)	 $\mathrm{f}(p)=\mathrm{P}(\text{chooses first route} \mid \text{early for school})$

	$=\dfrac{0.9p}{0.9p+0.85(1-p)}$ $=\dfrac{0.9p}{0.05p+0.85}$ $=\dfrac{18p}{p+17}$ (shown) $\mathrm{f}(p)=\dfrac{18p}{p+17}$ $\mathrm{f}'(p)=\dfrac{(p+17)18-18p(1)}{(p+17)^2}$ $=\dfrac{306}{(p+17)^2}>0 \Rightarrow \mathrm{f}(p)$ is increasing $\mathrm{f}''(p)=\dfrac{-612}{(p+17)^3}<0 \Rightarrow \mathrm{f}(p)$ is concave downwards Therefore, when $0\le p\le 1$, $\mathrm{f}(p)$ is increasing at a decreasing rate.