NUMERICAL METHODS [FM]
EULER'S METHOD (SOLUTIONS)

1 A solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + \sqrt{xy}$ has $y = 1$ when $x = 1$.

(i) Use two iterations of Euler method of step size 0.5 to estimate the value of y when $x = 2$. [3]

(ii) Explain whether you would expect this value to be an under-estimate or an over-estimate of the true value. [2]

(iii) Explain why it is usually better to improve accuracy by using the improved Euler method rather than by simply using smaller step sizes in Euler method. [1]

[TJC/FM/2018/P1//Q2]

[Solution]

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + \sqrt{xy}$

Given $x_1 = 1$, $y_1 = 1$ and $h = 0.5$

When $x_2 = 1.5$,

$$y_2 = y_1 + h \times \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_1} = 1 + 0.5 \times \left(2 + \sqrt{(1)(1)}\right) = 2.5$$

When $x_3 = 2$,

$$y_3 = y_2 + h \times \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_2} = 2.5 + 0.5 \times \left(2 + \sqrt{(1.5)(2.5)}\right) = 4.47 \quad \text{(correct to 3 s.f.)}$$

(ii)

x_n	$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{x=x_n}$
1	3
1.5	3.936
2	$2 + \sqrt{2(4.46825)} = 4.989$

Since $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing over [1, 2], the value is an under-estimate.

Alternatively, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sqrt{x} \times \frac{1}{2\sqrt{y}} \frac{\mathrm{d}y}{\mathrm{d}x} + \sqrt{y} \times \frac{1}{2\sqrt{x}}$

Since $x > 0$, $y > 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$, we have $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$

Thus the curve is concave upwards

(iii) It is usually more efficient to use the improved Euler method as the convergence is faster and so requires fewer computations or lesser computing time.

2 The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 \tan x = 1,$$

where $y = 1$ when $x = 1,$ is to be solved numerically.

(i) Carry out two steps of Euler's method with step length 0.1 to estimate the value of y when $x = 1.2$, giving your answer to 4 decimal places. [3]

(ii) The method in part **(i)** is now replaced by the improved Euler method. The estimate obtained is 2.0156, given to 4 decimal places. State, with a reason, whether this estimate and the one found in part **(i)** are likely to be overestimates or underestimates of the actual value of y when $x = 1.2$. [2]

(iii) Explain why it would be inappropriate to continue the process in part **(i)** to estimate the value of y when $x = 1.6$. [2]

[VJC/FM/2019/P2/Q3]

[Solution]

1 **(i)** $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2 \tan x, \quad x_0 = 1, y_0 = 1, \ \mathrm{f}(x, y) = 1 + y^2 \tan x$,

By Euler's Method with h = 0.1:

$$\begin{aligned} y_1 &= 1 + 0.1\mathrm{f}(1,1) \\ &= 1 + 0.1(1 + \tan 1) \\ &\approx 1.255741 \end{aligned}$$

$$\begin{aligned} y_2 &= 1.25574 + 0.1\mathrm{f}(1.1, 1.255741) \\ &\approx 1.665560 \end{aligned}$$

$\therefore$ the value of y when x = 1.2 is 1.6656 (4d.p) .

(ii) The improved Euler method usually gives a better estimate than the Euler method. Since 2.0156 > 1.6656, it is likely that the actual value of y is bigger than these estimates. Hence both estimates are likely to underestimate the actual value.

(iii) $\mathrm{f}(x, y) = 1 + y^2 \tan x$ is discontinuous at $x = \frac{\pi}{2} \approx 1.57$ **.**

3 Determine the maximum number of iterations, using the *Euler Method*, with step size 0.01 on the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y, \text{ where } y = 0 \text{ when } x = 1,$$

such that the error between the actual value of y and the approximated value of y is less than 0.001. [6]

[NYJC/FM/2018/P1//Q2]

[Solution]

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+y}{x^2}$$

$$\int \frac{1}{1+y} dy = \int \frac{1}{x^2} dx$$

$$\ln|1+y| = -\frac{1}{x} + c$$

$$1 + y = A\mathrm{e}^{-\frac{1}{x}}$$

$$y = A\mathrm{e}^{-\frac{1}{x}} - 1$$

Given $y = 0, x = 1$,

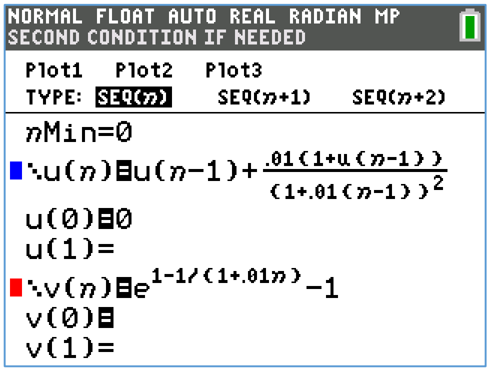
$$0 = A\mathrm{e}^{1} - 1$$

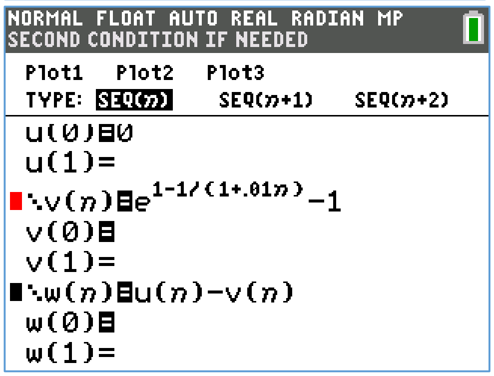
$$A = \frac{1}{\mathrm{e}^{-1}} = \mathrm{e}$$

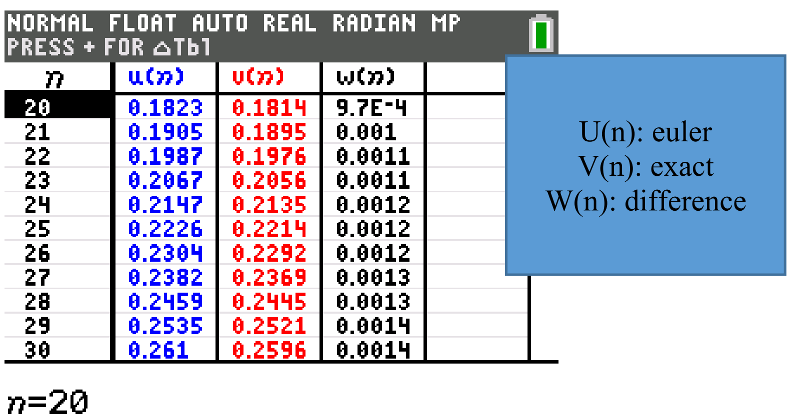
$$y = \mathrm{e}\mathrm{e}^{-\frac{1}{x}} - 1 = \mathrm{e}^{1-\frac{1}{x}} - 1$$

Using Euler Method,

$$y_n = y_{n-1} + 0.01\frac{1 + y_{n-1}}{(1 + 0.01(n-1))^2}, n \geq 1 \text{ , } y_0 = 0$$







n	y_n	y_{actual}	Difference
20	0.182334	0.181360	0.000974
21	0.190545	0.189525	0.00102

From GC, max no of iterations = 20

4. The variables x and y are related by the differential equation $\dfrac{dy}{dx} = f(x, y)$.

(i) Taking the initial value as $y(x_0) = y_0$, explain with the aid of a diagram, how the Euler method can be applied once on the differential equation to approximate the solution at $x = x_0 + h$. [3]

Given that $f(x, y) = xy - \dfrac{y}{x}$ and $(x_0, y_0) = (1.5, 2)$.

(ii) Apply the following methods with a step size 0.5 to estimate y at $x = 2.5$.

(a) Euler method [2]

(b) Improved Euler method [3]

(iii) Comment on the accuracy of the estimates found in part (ii). [2]

(iv) State one advantage and one disadvantage in using the improved Euler method compared to the Euler method. [2]

[RVHS/FM/2018/P1/Q9]

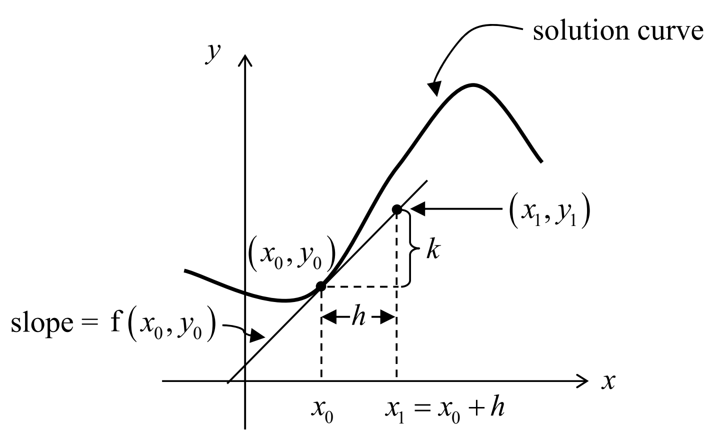
[Solution]

(i) $\dfrac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

Let $x_1 = x_0 + h$

At the initial point (x_0, y_0), the tangent line has slope $\left.\dfrac{dy}{dx}\right|_{x=x_0} = f(x_0, y_0)$.

$\Rightarrow k = hf(x_0, y_0)$



Hence the solution at $x = x_0 + h$ is approximated by

$y_1 = y_0 + hf(x_0, y_0)$

(ii)(a) $h = 0.5$

x	y	$\Delta y = h\mathrm{f}(x, y)$
1.5	2	0.833333
2	2.83333	2.12500
2.5	4.95833	

$\therefore y(2.5) \approx 4.96$

(ii)(b) $h = 0.5$

x	y	$\mathrm{f}(x, y)$	u	$\frac{\Delta y}{\Delta x}$
1.5	2	1.66667	2.83333	2.95833
2	3.47917	5.21875	6.08854	9.00234
2.5	7.98034			

$\therefore y(2.5) \approx 7.98$

(iii) As the gradient of the solution curve is increasing, the estimate found by the improved Euler method is more accurate than that found by the Euler method because gradient used in improved Euler method is adjusted by taking mean of gradients at x_i and $x_i + h$, as the iteration goes.

However, noticing that the gradient is increasing very rapidly from $x = 2$ to $x = 2.5$, the accuracy of both method is likely to be poor (as seen also from the difference in the two estimates).

(iv)
Advantage:
- Estimate is more accurate
- Estimate is numerically stable

Disadvantage:
- Require large computation time
- Error estimation is not easily done

5 A particle moves along a straight line which passes through a fixed point O. It is acted on by two resistive forces, one of which is proportional to its displacement x from O while the other is proportional to its speed v. As a result, the motion of the particle is governed by the differential equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -7x - 24v.$$

Given that $v = 121$ when $x = 0$, estimate the value of v when $x = 1$ using

(i) one iteration of Euler's Formula, [2]

(ii) one iteration of the improved Euler formula. [2]

Hence, explain why v is approximately a linear function of x for $0 \le x \le 1$. [2]

By considering the values of $\frac{x}{v}$ for $0 \le x \le 1$, use the given differential equation to find an expression for this linear function. [2]

[VJC/FM/2018/P1//Q5]

[Solution]

$\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{7x}{v} - 24 = \mathrm{f}(x, v)$

(a) $x_0 = 0, v_0 = 121$: $h = 1$

One iteration of Euler formula: $v_1 = v_0 + (1)\,\mathrm{f}(x_0, v_0) = 121 + \left(-\frac{7(0)}{121} - 24\right) = 97$

(b) One iteration of improved Euler formula: $u_1 = 97$, $h = 1$

$$\begin{aligned} v_1 &= v_0 + \frac{1}{2}\left[\mathrm{f}(x_0, v_0) + \mathrm{f}(x_1, u_1)\right] \\ &= 121 + \frac{1}{2}\left[-24 + \left(-\frac{7(1)}{97} - 24\right)\right] \\ &= 96.9639... \\ &\approx 97.0 \end{aligned}$$

The two estimates are approximately equal. This means that the tangent at (x_0, v_0) used in the Euler formula has a gradient that is almost equal to the average of the gradients at (x_0, v_0) and $(1, u_1)$. In other words, tangents at (x_0, v_0) and $(1, u_1)$ have about the same gradient. This suggests that v is approximately a linear function of x for $0 \le x \le 1$.

Since $\frac{x}{v}$ is small for $0 \le x \le 1, \frac{\mathrm{d}v}{\mathrm{d}x} \approx -24.$

$\therefore v \approx -24x + C.$

Given $x = 0, v = 121,$ we have $C = 121.$

$\therefore v \approx 121 - 24x.$

6 **(i)** Show that the substitution $z = y^2$ transforms the differential equation

$$\left(1+x^2\right) y \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^2 = 3, \quad y \geq 0 \quad ---(1)$$

to

$$\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{4xz}{1+x^2} = \frac{6}{1+x^2}.$$ [2]

(ii) Hence find y in terms of x and sketch three members of the family of solution curves. [6]

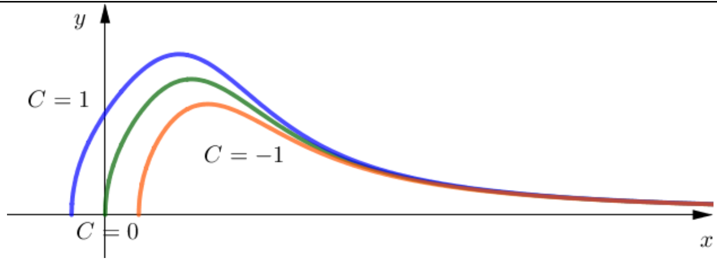
(iii) A curve that has a y–intercept at 1 is defined by the differential equation in (1). Use Euler's method with step length 0.5 to estimate the value of y when $x = 2$, giving your answer to 3 decimal places. [3]

(iv) An estimation is considered to be a good estimation if the error is less than 1% of the step length. Determine if the estimation in **(iii)** is good. [2]

[NJC/FM/2019/MYE/P1/Q10]

[Solution]

(i)	$z = y^2 \quad \Rightarrow \quad \frac{\mathrm{d}z}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(1+x^2\right) y \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^2 = 3$ $2y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4x}{1+x^2}y^2 = \frac{6}{1+x^2}$ $\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{4xz}{1+x^2} = \frac{6}{1+x^2}$
(ii)	$\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{4xz}{1+x^2} = \frac{6}{1+x^2}$ $\frac{\mathrm{d}z}{\mathrm{d}x} + \left(\frac{4x}{1+x^2}\right)z = \frac{6}{1+x^2}$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(z\mathrm{e}^{\int \frac{4x}{1+x^2}\mathrm{d}x}\right) = \frac{6}{1+x^2}\mathrm{e}^{\int \frac{4x}{1+x^2}\mathrm{d}x}$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(z\mathrm{e}^{2\ln\left(1+x^2\right)}\right) = \frac{6}{1+x^2}\mathrm{e}^{2\ln\left(1+x^2\right)}$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(z\left(1+x^2\right)^2\right) = \frac{6}{1+x^2}\left(1+x^2\right)^2$ $z\left(1+x^2\right)^2 = \int 6\left(1+x^2\right)\mathrm{d}x$ $z\left(1+x^2\right)^2 = 6x + 2x^3 + C$ $z = \frac{6x+2x^3+C}{\left(1+x^2\right)^2}$ $y = \frac{\sqrt{6x+2x^3+C}}{1+x^2}$

	
(iii)	$\left(1+x^2\right)y\frac{\mathrm{d}y}{\mathrm{d}x}+2xy^2=3, \quad y\geq 0$
(iv)	When $x=2$, $y=1.07703$ (5 d.p.) Hence the error is 0.003. Thus good estimation because $0.003<0.01\times 0.5=0.005$.

x	y	$\frac{\mathrm{d}y}{\mathrm{d}x}$	$h\frac{\mathrm{d}y}{\mathrm{d}x}$	$y+h\frac{\mathrm{d}y}{\mathrm{d}x}$
0	1	3	1.5	2.5
0.5	2.5	-1.04	-0.52	1.98
1	1.98	-1.22242	-0.61121	1.36878
1.5	1.36978	-0.58912	-0.29456	1.07422
2	1.07422			

7 The function $y = y(x)$ satisfies $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{5}\left(\tan x + x^3 y\right)$. The value of $y(h)$ is to be found, where h is a small positive number, and $y(0) = 0$.

(i) Use one step of the improved Euler formula to find an alternative approximation to $y(h)$ in terms of h. [2]

(ii) It can be shown that $y = y(x)$ satisfies $y(h) = \mathrm{e}^{0.05h^4}\int_0^h \frac{\tan x}{5}\mathrm{e}^{-0.05x^4}\,\mathrm{d}x$. Assume that h is small and hence find another approximation to $y(h)$ in terms of h. [2]

(iii) Discuss the relative merits of these two methods employed to obtain these approximations. [2]

[EJC/FM/2018/P1/Q1]

[Solution]

7 (i) $y = y(x)$, $\mathrm{f}(x, y) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{5}\left(\tan x + x^3 y\right)$.

Let $x_1 = 0$, $y_1 = y(x_1) = 0$,

Step size is h.

$u_2 = y_1 + h\mathrm{f}(x_1, y_1) = 0$

$y_2 = y_1 + \frac{h}{2}\left[\mathrm{f}(x_1, y_1) + \mathrm{f}(x_2, u_2)\right] = 0 + \frac{h}{2}\left[\mathrm{f}(0,0) + \mathrm{f}(h,0)\right] = \frac{h}{10}\tan(h)$

(ii) h is small, we may use the following power series approximations to estimate the integral. For $0 < x < h$,

$\tan x \approx x$

$\mathrm{e}^{\pm 0.05x^4} \approx 1 \pm \frac{1}{20}x^4$

$\mathrm{e}^{-0.05x^4}\tan x \approx \left(1 - \frac{1}{20}x^4\right)(x) \approx x - \frac{1}{20}x^5$

$\int_0^h \frac{\tan x}{5}\mathrm{e}^{-0.05x^2}\mathrm{d}x \approx \frac{1}{5}\int_0^h x - \frac{1}{20}x^5\mathrm{d}x = \frac{1}{5}\left[\frac{h^2}{2} - \frac{h^6}{120}\right]$

$y(h) \approx \mathrm{e}^{0.05h^4}\,\frac{1}{5}\left[\frac{h^2}{2} - \frac{h^6}{120}\right]$ or $y(h) \approx \left(1 + \frac{1}{20}h^4\right)\cdot\frac{1}{5}\left[\frac{h^2}{2} - \frac{h^6}{120}\right]$

<u>Remark (not required in solution):</u>

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{5}\left(\tan x + x^3 y\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{-x^3}{5}\right)y = \frac{\tan x}{5}$

Integrating factor $= \mathrm{e}^{\int \frac{-x^3}{5}\mathrm{d}x} = \mathrm{e}^{-0.05x^4}$. Multiply the integrating factor on both sides of the D.E., we can re-write it as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{-0.05x^4}\right)=\frac{\tan x}{5}\mathrm{e}^{-0.05x^4}$$
$$\int_0^h\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{-0.05x^4}\right)\mathrm{d}x=\int_0^h\frac{\tan x}{5}\mathrm{e}^{-0.05x^4}\mathrm{d}x$$
$$\mathrm{e}^{-0.05h^4}y\left(h\right)-\mathrm{e}^{-0.05(0)^4}y\left(0\right)=\int_0^h\frac{\tan x}{5}\mathrm{e}^{-0.05x^4}\mathrm{d}x$$
$$\mathrm{e}^{-0.05h^4}y\left(h\right)=\int_0^h\frac{\tan x}{5}\mathrm{e}^{-0.05x^4}\mathrm{d}x$$

(iii) As for the method employed in (ii), depending on how small h is and how high the order of approximations used for the power series, the approximation from (ii) may be potentially much more accurate than those from (i) and (ii), possibly at the expense of using an arbitrary large degree polynomial from truncating the power series of the integrand.

8 A differential equation is given by $(x+4)\frac{\mathrm{d}y}{\mathrm{d}x}+(x+5)y+2\mathrm{e}^{x}=0$, where $x \neq -4$.

(i) By using the substitution $u=(x+4)y$, show that the differential equation can be reduced to $\frac{\mathrm{d}u}{\mathrm{d}x}+u=-2\mathrm{e}^{x}$. Hence, find y in terms of x, given that $y=0$ when $x=0$. Hence obtain the value of y when $x=0.2$. [6]

Consider the differential equation $(x+4)\frac{\mathrm{d}y}{\mathrm{d}x}+(x+5)y+2+2x+x^{2}=0$, where $y=0$ when $x=0$.

(ii) Use the Euler method with step length 0.1 to estimate the value of y when $x=0.2$. [3]

(iii) Use the improved Euler method with step length 0.1 to estimate the value of y when $x=0.2$. [3]

(iv) Comment on your numerical answers for the values of y from parts **(i)**, **(ii)** and **(iii)**. [2]

[HCI/FM/2019/MCT//Q7]

[Solution]

7(i)	Given $u=(x+4)y$, differentiate u wrt x, $\frac{\mathrm{d}u}{\mathrm{d}x}=(x+4)\frac{\mathrm{d}y}{\mathrm{d}x}+y$ $\Rightarrow (x+4)\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}-y$ Therefore, $\left(\frac{\mathrm{d}u}{\mathrm{d}x}-y\right)+(x+5)y+2\mathrm{e}^{x}=0$ $\frac{\mathrm{d}u}{\mathrm{d}x}+u=-2\mathrm{e}^{x}$ The integrating factor $=\mathrm{e}^{\int \mathrm{d}x}=\mathrm{e}^{x}$ $\Rightarrow \mathrm{e}^{x}\frac{\mathrm{d}u}{\mathrm{d}x}+u\mathrm{e}^{x}=-2\mathrm{e}^{x}\cdot\mathrm{e}^{x}$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(u\mathrm{e}^{x}\right)=-2\mathrm{e}^{2x}$ $u\mathrm{e}^{x}=\int -2\mathrm{e}^{2x}\,\mathrm{d}x$ $u\mathrm{e}^{x}=-\mathrm{e}^{2x}+C$ for any arbitrary constant C. $\Rightarrow (x+4)y\mathrm{e}^{x}=-\mathrm{e}^{2x}+C$ When $x=0$ and $y=0$, $C=1$.

$(x+4)ye^{x} = -e^{2x} + 1$

$y = \dfrac{e^{-x} - e^{x}}{x+4}$ is the particular solution.

When $x = 0.2$, $y = -0.095874 = -0.0959$ (3 s.f.) .

(ii)

$h = 0.1$ and $y_{i+1} = y_i + 0.1\,\mathrm{f}(x_i, y_i)$, where $\mathrm{f}(x, y) = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-(x+5)y - (2+2x+x^2)}{x+4}$

i	x_i	y_i	$\mathrm{f}(x_i, y_i)$
0	0	0	– 0.5
1	0.1	– 0.05	– 0.476829
2	0.2	– 0.097683	

$\therefore y(0.02) \approx y_2 = -0.0977\,(3\text{sf})$

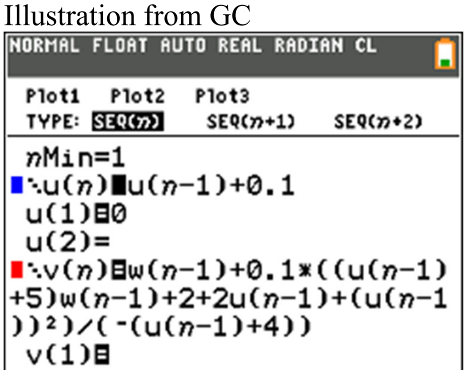
(iii) Using improved Euler's method, $h = 0.1$,

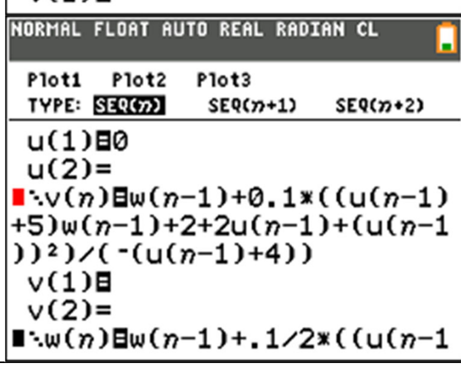
$u_{i+1} = y_i + 0.1\,\mathrm{f}(x_i, y_i)$ and $y_{i+1} = y_i + \dfrac{0.1}{2}\left[\mathrm{f}(x_i, y_i) + \mathrm{f}(x_{i+1}, u_{i+1})\right]$

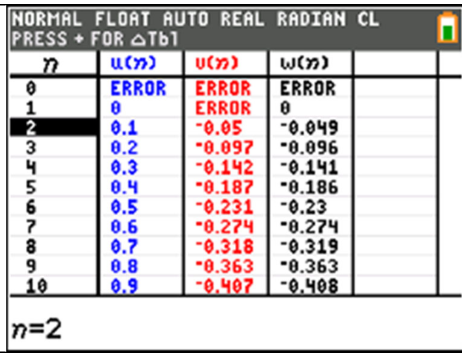
i	x_i	u_i	$\mathrm{f}(x_i, u_i)$	y_i	$\mathrm{f}(x_i, y_i)$
0	0			0	– 0.5
1	0.1	– 0.05	– 0.476829	– 0.048841	– 0.478270
2	0.2	– 0.096669	– 0.461268	– 0.095818	

$\therefore y(0.02) \approx y_2 = -0.0958\,(3\text{sf})$

Illustration from GC







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n	u(n)	v(n)	w(n)
0	ERROR	ERROR	ERROR
1	0	ERROR	0
2	0.1	-0.05	-0.049
3	0.2	-0.097	-0.096
4	0.3	-0.142	-0.141
5	0.4	-0.187	-0.186
6	0.5	-0.231	-0.23
7	0.6	-0.274	-0.274
8	0.7	-0.318	-0.319
9	0.8	-0.363	-0.363
10	0.9	-0.407	-0.408

n=2

(iv) The values are all close to one another. This is because the expansion of $2e^x$ is $2\left(1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\right)$ and can be approximated by $2\left(1+x+\dfrac{x^2}{2}\right)=2+2x+x^2$.
However the value obtained by improved Euler's method is closer than Euler's method as the improved Euler's method tends to produce result of better accuracy.

Answers

1 (i) 4.47 (correct to 3 s.f.)

2 (i) 1.6656 (4d.p)

3 max no of iterations = 20

4 (ii) a. $y(2.5)\approx 4.96$ b. $y(2.5)\approx 7.98$

5 (i) 97 (ii) 97.0; $v\approx 121-24x$.

6 (ii) $y=\dfrac{\sqrt{6x+2x^3+C}}{1+x^2}$ (iii) 1.07422

7 (i) $\dfrac{h}{10}\tan(h)$ (ii) $y(h)\approx e^{0.05h^4}\dfrac{1}{5}\left[\dfrac{h^2}{2}-\dfrac{h^6}{120}\right]$ or $y(h)\approx\left(1+\dfrac{1}{20}h^4\right)\cdot\dfrac{1}{5}\left[\dfrac{h^2}{2}-\dfrac{h^6}{120}\right]$

8 (i) $y=\dfrac{e^{-x}-e^{x}}{x+4}$; -0.0959 (3 s.f.) (ii) $-0.0977(3\text{sf})$ (iii) $-0.0958(3\text{sf})$