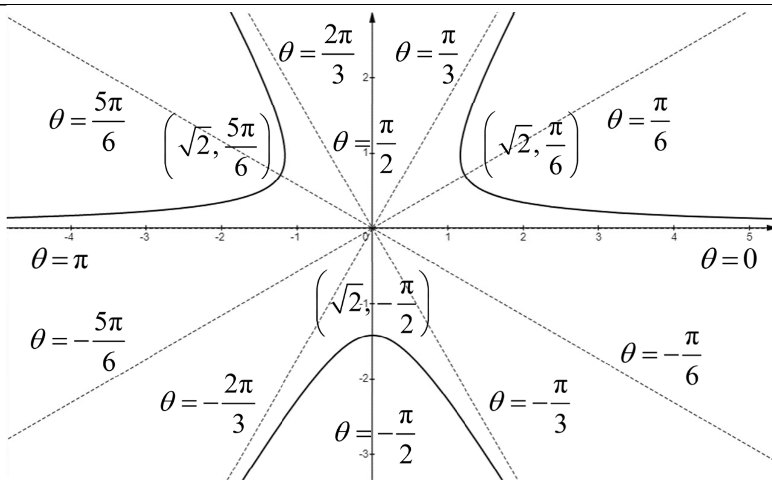
1	**Solution**
	Let P_n be the proposition $2^{n+1} > n^2$ where $n \in \mathbb{Z}^+, n \geq 3$. When $n = 3$: $2^{3+1} = 16$ $3^2 = 9$ $\therefore P_3$ is true as $16 > 9$. Assume P_k is true for some $k \in \mathbb{Z}^+, k \geq 3$: $2^{k+1} > k^2$ To prove P_{k+1} is true: $2^{k+2} = 2\left(2^{k+1}\right)$ $> 2\left(k^2\right)$ $2k^2 - \left(k+1\right)^2 = k^2 - 2k - 1$ $= \left[k - \left(1 - \sqrt{2}\right)\right]\left[k - \left(1 + \sqrt{2}\right)\right]$ $> 0 \quad \text{as } k \geq 3$ $\therefore 2^{k+2} > \left(k+1\right)^2$, as $2k^2 > (k+1)^2$ Since P_3 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by Mathematical Induction, P_n is true for all positive integers $n \geq 3$.

2	**Solution**
(a)	$r=\sqrt{\frac{9}{\cos^2\theta+(\sin\theta-\cos\theta)^2}}\Rightarrow r^2=\frac{9}{\cos^2\theta+(\sin\theta-\cos\theta)^2}$ $\Rightarrow(r\cos\theta)^2+(r\sin\theta-r\cos\theta)^2=9$ Using $x=r\cos\theta,\ y=r\sin\theta$, $\therefore x^2+(y-x)^2=9$ $\Rightarrow 2x^2-2xy+y^2=9$
(b)	$r=\frac{1}{\sin\theta-\cos\theta}\Rightarrow r\sin\theta-r\cos\theta=1\Rightarrow y=x+1$ $\therefore 2x^2-2x(x+1)+(x+1)^2=9$ $\Rightarrow 2x^2-2x^2-2x+x^2+2x+1=9$ $\Rightarrow x^2=8$ $\Rightarrow x=\pm2\sqrt{2}$ Hence the points of intersections are $\left(2\sqrt{2},1+2\sqrt{2}\right)$ and $\left(-2\sqrt{2},1-2\sqrt{2}\right)$
	<u>**Otherwise:**</u> Using $(r\cos\theta)^2+(r\sin\theta-r\cos\theta)^2=9$ and $r\sin\theta-r\cos\theta=1$, $(r\cos\theta)^2+1^2=9\Rightarrow r\cos\theta=\pm2\sqrt{2}=x$ Hence the points of intersections are $\left(2\sqrt{2},1+2\sqrt{2}\right)$ and $\left(-2\sqrt{2},1-2\sqrt{2}\right)$

3	**Solution**
(a)	$\mathrm{f}(x)=\dfrac{2x-1}{x}$ $\qquad \mathrm{f}^2(x)=\dfrac{3x-2}{2x-1}$ $\mathrm{f}^3(x)=\dfrac{4x-3}{3x-2}$ $\qquad \mathrm{f}^4(x)=\dfrac{5x-4}{4x-3}$ Conjecture: $\mathrm{f}^n(x)=\dfrac{(n+1)x-n}{nx-(n-1)}$
(b)	Let P_n be the proposition $\mathrm{f}^n(x)=\dfrac{(n+1)x-n}{nx-(n-1)}$ where $n\in\mathbb{Z}^+$. When n = 1: $\mathrm{LHS}=\mathrm{f}(x)$ $\mathrm{RHS}=\dfrac{2x-1}{x}=2-\dfrac{1}{x}$ $\therefore P_1$ is true. Assume P_k is true for some $k\in\mathbb{Z}^+$: $\mathrm{f}^k(x)=\dfrac{(k+1)x-k}{kx-(k-1)}$ To prove P_{k+1} is true: P_{k+1} : $\mathrm{f}^{k+1}(x)=\dfrac{(k+2)x-(k+1)}{(k+1)x-k}$ $\mathrm{f}^{k+1}(x)=\mathrm{f}\left[\dfrac{(k+1)x-k}{kx-(k-1)}\right]$ $=\dfrac{2(k+1)x-2k-kx+(k-1)}{(k+1)x-k}$ $=\dfrac{(k+2)x-(k+1)}{(k+1)x-k}$ Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+$.
(c)	For $\mathrm{f}^n(x)$ to be defined, $x\neq\dfrac{n-1}{n}$. $\therefore$ Largest subset of the reals for $\mathrm{f}^n(x)$ to be defined for all $n\in\mathbb{Z}^+$ is $\left\{x\in\mathbb{R}:\ x\neq\dfrac{n-1}{n},\ n\in\mathbb{Z}^+\right\}$.

4	**Solution**
(a)	$r=\sqrt{\dfrac{2}{\sin 3\theta}}$ is undefined for $\sin 3\theta \le 0$ Then the range of values are $-\pi<\theta\le-\dfrac{2\pi}{3}$ or $-\dfrac{\pi}{3}\le\theta\le 0$ or $\dfrac{\pi}{3}\le\theta\le\dfrac{2\pi}{3}$ or π. Hence the equation of the asymptotes are $\theta=0,\pm\dfrac{\pi}{3},\pm\dfrac{2\pi}{3}$, and π.
(b)	

5	**Solution**
(a)	Since the coefficients of x^2 and y^2 must have different sign, $A<0$.
(b)	Since the equation of H is $Ax^2+2y^2-4\sqrt{3}y+4=0$, the centre is $\left(0,\sqrt{3}\right)$ and by symmetry, the other focus, O' is $\left(0,2\sqrt{3}\right)$. Comparing $\left(y-\sqrt{3}\right)^2-\frac{1}{-2/A}x^2=1$ to a standard equation of a hyperbola, $\frac{\left(y-h\right)^2}{a^2}-\frac{x^2}{b^2}=1$, we have $e^2=1+\frac{b^2}{a^2} \Rightarrow 3=1+\frac{\left(-2/A\right)}{1} \Rightarrow A=-1$
(c)	$2y^2-4\sqrt{3}y-x^2+4=0 \Rightarrow \left(y-\sqrt{3}\right)^2-\frac{x^2}{2}=1$ For curve H, the derivative will be $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x_0}{2\left(y_0-\sqrt{3}\right)}$ The angle that T makes with the horizontal, $\theta=\tan^{-1}\left(\frac{x_0}{2\left(y_0-\sqrt{3}\right)}\right)$ The angle that OP makes with the horizontal, $\alpha=\tan^{-1}\left(\frac{y_0}{x_0}\right)$ The angle that $O'P$ makes with the horizontal, $\beta=\tan^{-1}\left(\frac{y_0-2\sqrt{3}}{x_0}\right)$ To show $\alpha-\theta=\theta-\beta$, we can show $2\theta=\alpha+\beta$. $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=2\left(\frac{x_0}{2\left(y_0-\sqrt{3}\right)}\right)\div\left[1-\left(\frac{x_0}{2\left(y_0-\sqrt{3}\right)}\right)^2\right]$ $=\left(\frac{x_0}{y_0-\sqrt{3}}\right)\frac{4\left(y_0-\sqrt{3}\right)^2}{4\left(y_0-\sqrt{3}\right)^2-x_0^{\,2}}$ $=\left(\frac{x_0}{y_0-\sqrt{3}}\right)\frac{4\left(y_0-\sqrt{3}\right)^2}{4+2x_0^{\,2}-x_0^{\,2}}$ $=\frac{4x_0\left(y_0-\sqrt{3}\right)}{x_0^{\,2}+4}$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\left(\frac{y_0}{x_0}+\frac{y_0-2\sqrt{3}}{x_0}\right)\div\left[1-\left(\frac{y_0}{x_0}\right)\left(\frac{y_0-2\sqrt{3}}{x_0}\right)\right]$$

$$=\left(\frac{2y_0-2\sqrt{3}}{x_0}\right)\frac{x_0^{\ 2}}{x_0^{\ 2}-\left(y_0^{\ 2}-2\sqrt{3}y_0\right)}$$

$$=\left(\frac{2x_0\left(y_0-\sqrt{3}\right)}{x_0^{\ 2}-\left(\frac{x_0^{\ 2}}{2}-2\right)}\right)$$

$$=\frac{4x_0\left(y_0-\sqrt{3}\right)}{x_0^{\ 2}+4}$$

Since $\tan(2\theta)=\tan(\alpha+\beta)\Leftrightarrow 2\theta=\alpha+\beta$ over the interval $(0,\pi)$, we have $2\theta=\alpha+\beta\Rightarrow\alpha-\theta=\theta-\beta$.

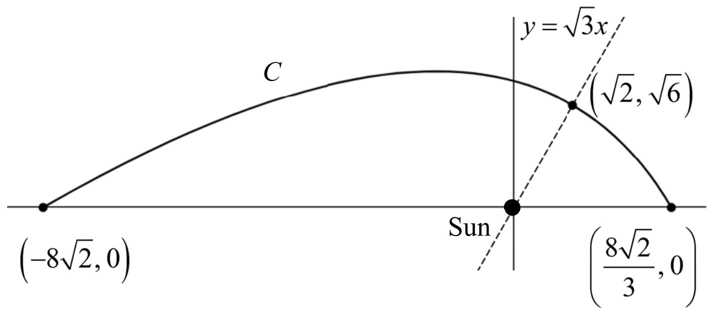
Hence the angles made by the line segments OP and $O'P$ with the tangent T is the same.

6	**Solution**
(a)	$2^n X_n = \frac{2^{n-1}}{2} X_{n-1} + 2$ Substitute $v_n = 2^n X_n$: $v_n = \frac{1}{2} v_{n-1} + 2$ Let $v_n = A\left(\frac{1}{2}\right)^n + B$, where *A*, *B* are constants Substituting v_o, v_1 : $v_0 = 6 = A + B$ $v_1 = 5 = \frac{1}{2}A + B$ Solving, $A = 2,\ B = 4$. $\therefore v_n = 2^{1-n} + 4$ $X_n = \frac{1}{2^n} v_n$ $X_n = 2^{1-2n} + 2^{2-n}$
(b) (i)	As $u_n \to l$ as $n \to \infty$, $l = \frac{l^2 + 5}{2l + 4}$ $l - \frac{l^2 + 5}{2l + 4} = 0$ $\frac{2l^2 + 4l - l^2 - 5}{2l + 4} = 0$ $\frac{(l + 5)(l - 1)}{2l + 4} = 0$ $\therefore l = -5$ OR $l = 1$.
(b) (ii)	Using GC, When $a = -2.01$, the sequence increases towards the limit $l = -5$. When $a = -1.99$, the sequence decreases towards the limit $l = 1$.
(b) (iii)	$u_{n+1} + 2 = \frac{u_n{}^2 + 4u_n + 13}{2(u_n + 2)} = \frac{(u_n + 2)^2 + 9}{2(u_n + 2)}$ Note $(u_n + 2)^2 + 9 > 0$ If $u_n > -2$, $u_n + 2 > 0,\ \therefore u_{n+1} + 2 > 0 \Rightarrow u_{n+1} > -2$

	If $u_n < -2$, $u_n + 2 < 0$, $\therefore u_{n+1} + 2 < 0 \Rightarrow u_{n+1} < -2$ When $a = -2.01 < -2$, all terms in the sequence will be less than –2, hence the sequence converges to $\alpha = -5 < -2$. When $a = -1.99 > -2$, all terms in the sequence will be greater than than –2, hence the sequence converges to $\alpha = 1 > -2$.

7	**Solution**
(a)	$x_n - x_{n-1} = k\left(x_{n-1} - x_{n-2}\right)$ $x_n = (k+1)x_{n-1} - kx_{n-2},\ n \in \mathbb{Z},\ n \geq 3$
(b)	$x_n - (k+1)x_{n-1} + kx_{n-2} = 0$ Auxiliary equation: $\lambda^2 - (k+1)\lambda + k = 0$ $\lambda = \dfrac{k+1 \pm \sqrt{(k+1)^2 - 4k}}{2}$ $= \dfrac{k+1 \pm \sqrt{(k-1)^2}}{2}$ $\lambda = k$ OR $\lambda = 1$ General solution: $x_n = A + B(k^n)$, where A, B are constants When $n = 1$, $x_1 = 8 = A + Bk$ ----- (1) When $n = 2$, $x_2 = 23 = A + Bk^2$ ----- (2) (2) − (1): $B = \dfrac{15}{k(k-1)}$ $A = \dfrac{8k-23}{k-1} = 8 - \dfrac{15}{k-1}$ $\therefore \quad x_n = 8 + \dfrac{15}{k-1}\left(k^{n-1} - 1\right),\ n \in \mathbb{Z}^+$
(c)	When $k > 1$, $k^{n-1} \to \infty$ as $n \to \infty \Rightarrow x_n \to \infty$ as $n \to \infty$ When $0 < k < 1$, $k^{n-1} \to 0$ as $n \to \infty \Rightarrow x_n \to 8 - \dfrac{15}{k-1}$ as $n \to \infty$ If $k > 1$, we would expect the spread of the virus to escalate and affect the whole population. If $0 < k < 1$, we would expect the spread of the virus to stop, and the total number of Omega variant cases in Singapore to approach approximately $8 - \dfrac{15}{k-1}$.
(d)	**Any of the following or equivalent:** • k is unlikely to be constant for an extended period of time in a virus outbreak • k may decrease if a lot of the population has gotten the virus (eg. if there is herd immunity)

	• x_n cannot grow indefinitely due to population size limits • If k is not an integer, x_n takes non-integer values which is not feasible in real life.
(e)	Consider $x_n - x_{n-1}$ for some $n \in \mathbb{Z},\ n \geq 2$ $k=1$: $x_m - x_{m-1} = x_{m-1} - x_{m-2}$ for all $m \in \mathbb{Z},\ m \geq 3$ For $n=2$: $x_2 - x_1 = 15$ For $n \geq 3$: $x_n - x_{n-1} = x_{n-1} - x_{n-2}$ $= x_{n-2} - x_{n-3}$ $= \ldots$ $= x_2 - x_1 = 15$ $\therefore$ for all $n \in \mathbb{Z},\ n \geq 2,\ x_n - x_{n-1} = 15$ (constant indep of n) Therefore x_n is an arithmetic progression with a common difference of 15.
(f)	When $n \leq 5$, $k=4$: $x_n = 3 + 5(4^{n-1}),\ n \leq 5$ $x_4 = 323$ $x_5 = 1283$ When $n \geq 6$, $k=1$ and x_n increases as an arithmetic progression with common difference $1283 - 323 = 960$. $\therefore x_n = 1283 + (n-5)960,\ n \geq 5$ $1283 + (n-5)960 \geq 10000$ $n - 5 \geq 9.0802$ $n \geq 14.0802$ Alert Orange will be triggered on the 15th week of the outbreak.

8	**Solution**
(a)	$r+r\cos(\theta-\alpha)=d$ At $\left(\frac{8\sqrt{2}}{3},0\right)$, $\frac{8\sqrt{2}}{3}+\frac{8\sqrt{2}}{3}\cos(-\alpha)=d$ $\Rightarrow\ d=\frac{8\sqrt{2}}{3}+\frac{8\sqrt{2}}{3}\cos\alpha$ At $\left(2\sqrt{2},\frac{\pi}{3}\right)$, $2\sqrt{2}+2\sqrt{2}\cos\left(\frac{\pi}{3}-\alpha\right)=d$ $\Rightarrow\ d=2\sqrt{2}+\sqrt{2}\cos\alpha+\sqrt{6}\sin\alpha$ Then $d=\frac{8\sqrt{2}}{3}+\frac{8\sqrt{2}}{3}\cos\alpha=2\sqrt{2}+\sqrt{2}\cos\alpha+\sqrt{6}\sin\alpha$ $\Rightarrow\ 8+8\cos\alpha=6+3\cos\alpha+3\sqrt{3}\sin\alpha$ $\Rightarrow\ 2+5\cos\alpha=3\sqrt{3}\sin\alpha$ (shown)
(b)	$\Rightarrow\ (2+5\cos\alpha)^2=\left(3\sqrt{3}\sin\alpha\right)^2$ $\Rightarrow\ 4+20\cos\alpha+25\cos^2\alpha=27\sin^2\alpha=27-27\cos^2\alpha$ $\Rightarrow\ 52\cos^2\alpha+20\cos\alpha-23=0$ $\Rightarrow\ \cos\alpha=\frac{-20+\sqrt{400+4784}}{104}$ $(\because \alpha \text{ is acute})$ $\Rightarrow\ \cos\alpha=\frac{1}{2}\ \Rightarrow\ \alpha=\frac{\pi}{3}$
(c)	$r=\frac{d}{1+\cos\left(\theta-\frac{\pi}{3}\right)}$. At $\left(2\sqrt{2},\frac{\pi}{3}\right)$, $2\sqrt{2}=\frac{d}{1+\cos(0)}\ \Rightarrow\ d=4\sqrt{2}$
(d)	
(e)	At the closest point to the Sun, which is the vertex of the parabola, the tangent is perpendicular to the line of symmetry. Hence the gradient of the path $=-\frac{1}{\sqrt{3}}$ $\therefore y-\sqrt{6}=-\frac{1}{\sqrt{3}}\left(x-\sqrt{2}\right)\ \Rightarrow\ y=-\frac{1}{\sqrt{3}}x+\frac{4\sqrt{6}}{3}$
(f)	At the y-axis, the distance is $\frac{4\sqrt{6}}{3}$ A.U.s

	Hence the distance in kilometres $= \dfrac{4\sqrt{6}}{3} \times 150 \times 10^6$ km $= 489.898 \times 10^6$ km $= 4.90 \times 10^8$ km (3 s.f.)