Name:____________________________________ Class:_____________

JURONG PIONEER JUNIOR COLLEGE

JC1 Year End Examination 2024

MATHEMATICS
Higher 2

9758/01

27 Sept 2024

Paper 1

3 hours

Additional materials: Printed Answer Booklet
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.

Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given by [ ] at the end of each question or part question.

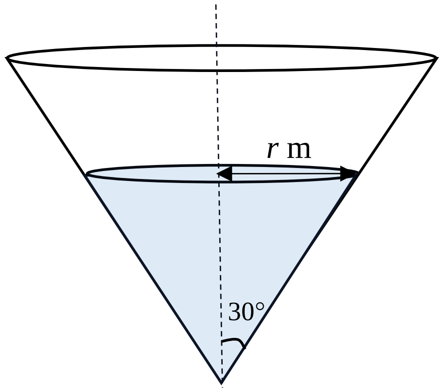
1 Differentiate each of the following expressions with respect to *x*.

(a) $\cos^{-1}\left(3x^2\right)$, [2]

(b) $\ln\left(\dfrac{\sqrt{2x+1}}{x^3}\right)$. [3]

2 Water is poured at a rate of 0.1 m^3 per minute into a container in the form of an open cone. The semi-vertical angle of the cone is 30°. At time t minutes after the start, the radius of the water surface is r m (see diagram). Find the rate of increase of the depth of water when the volume of the water is 3 m^3. [5]

[The volume of a cone of base radius r and height h is given by $V = \dfrac{1}{3}\pi r^2 h$]



3 A curve C has equation $\mathrm{f}(x) = \dfrac{a}{x^2} + bx + c$, where a, b and c are constants. It is given that C passes through the points $(-1, 4)$ and $(2, -11)$ and the curve of $y = \dfrac{1}{\mathrm{f}(x)}$ has a vertical asymptote with equation $x = 1$. Find the equation of C. [4]

4 By writing $x = A(8x-8) + B$, where A and B are constants, find $\displaystyle\int_0^1 \frac{x}{4x^2 - 8x + 5}\,\mathrm{d}x$, giving your answer in the form $p\ln q + r\tan^{-1} s$, where p, q, r and s are values to be determined. [5]

5 The curve C has equation $y = \dfrac{x^2 + ax + b}{x+1}$, where a and b are constants. It is given that C has a minimum point at $(2, 0)$.

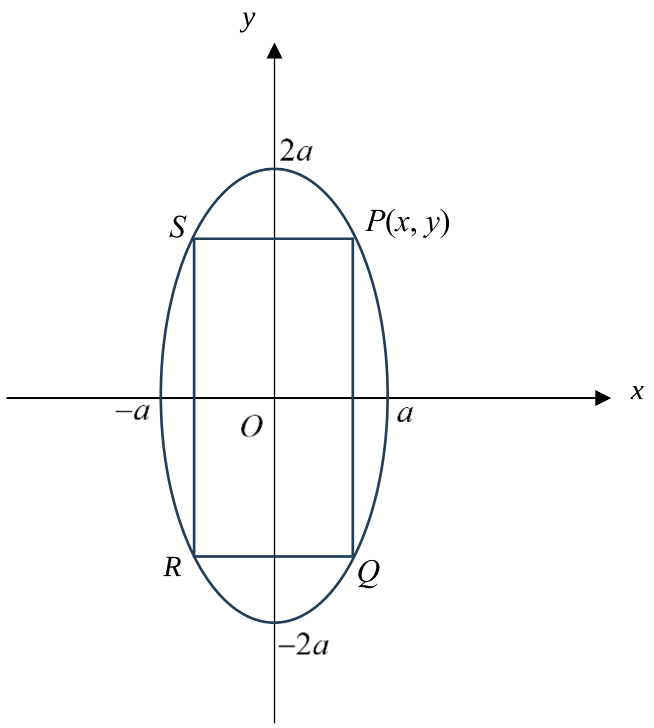
(i) Show that $a = -4$ and $b = 4$. [3]

(ii) Sketch the graph of C, indicating clearly the equations of any asymptotes and the coordinates of any turning points and axial intercepts. [3]

(iii) By adding an appropriate graph to your sketch in **(ii)**, deduce the number of real roots of the equation

$$x^3 + ax^2 + (b+1)x + 1 = 0.$$ [2]

6 The diagram shows an ellipse, centred at the origin with semi-minor axis a and semi-major axis $2a$, where $a > 0$. The rectangle $PQRS$ is inscribed in the ellipse such that its four vertices P, Q, R and S lie on the ellipse. The coordinates of P is (x, y), where x and y are positive.



(i) Write down the cartesian equation of the ellipse in terms of a. [1]

(ii) By considering the coordinates of P, show that A, the area of rectangle $PQRS$, is $8x\sqrt{a^2 - x^2}$. [2]

(iii) Using differentiation, find, in terms of a, the value of x when A is a maximum. You do not need to prove that A is a maximum. [3]

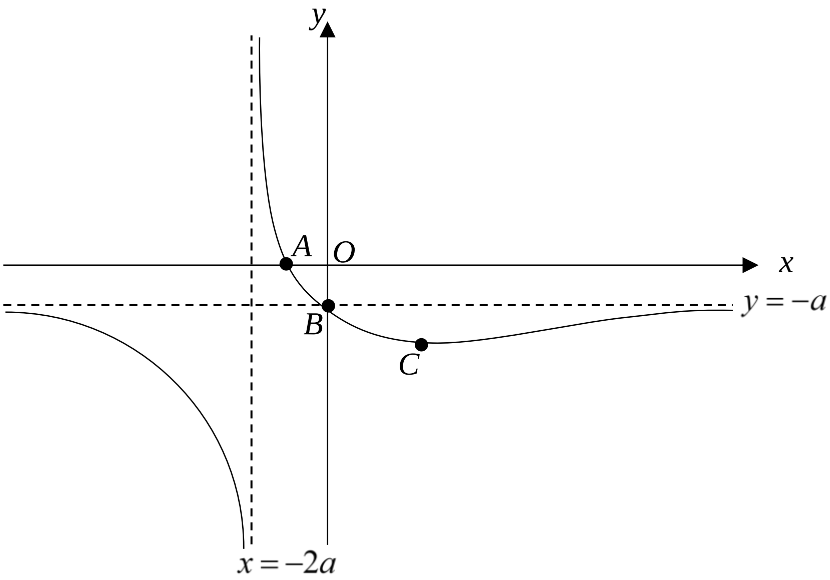
(iv) Find the value of a if the maximum value of A is 100. [2]

[Turn over

7 **(i)** Without using a calculator, solve the inequality $\dfrac{5-3x}{x^2+x-2} \geq -2$. [4]

(ii) Deduce the range of values of x for which $\dfrac{-5-3x}{x^2-x-2} \geq 2$. [3]

8 **(a)** The diagram below shows the graph of $y = \mathrm{f}(x)$. The curve passes through the points $A(-a, 0)$ and $B(0, -a)$ and has a minimum point at $C(2a, -2a)$, where $a > 0$. The lines $x = -2a$ and $y = -a$ are asymptotes of the curve.



On separate diagrams, sketch the graphs of

(i) $y = \mathrm{f}(x-a) + a$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

indicating clearly in each case, where appropriate, the equations of any asymptote(s), axial intercepts and coordinates of the points corresponding to *A*, *B* and *C*.

(b) The curve whose equation is $y = \sin x$, undergoes in succession, the following transformations:

A: A translation by π units in the negative x-direction.
B: A scaling parallel to the x-axis by a factor of 3.
C: A translation by 2 units in the positive y-direction.

Find the equation of the resulting curve. [3]

9 **(a)** Use the substitution $x = \sin\theta$ to find $\displaystyle\int \frac{2x^2}{\sqrt{1-x^2}}\,\mathrm{d}x$. [4]

(b) **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin(\ln x)$. [1]

(ii) Hence, using integration by parts, find $\int \sin(\ln x)\,\mathrm{d}x$. [3]

10 A curve C has parametric equations

$$x = t^2 + t, \qquad y = t^2 - t, \qquad t \in \mathbb{R}.$$

(i) Sketch the graph of C, indicating the coordinates of the points where the curve crosses either axis. [2]

(ii) Find the coordinates of the point on C where the tangent to C is parallel to the line $5y = 4x - 20$. [3]

(iii) Find the equation of the tangent to C at the point $(0, 2)$ and show algebraically that this tangent does not cut C again. [5]

11 Referred to the origin O, points A and B have position vectors **a** and **b** respectively, where **a** and **b** are non-zero and non-parallel vectors. Point C is the midpoint of OA and point D lies on BC such that BD: DC = 2: 3.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of **a** and **b**. [2]

(ii) Find the area of triangle OCD in the form of $k|\mathbf{a}\times\mathbf{b}|$, where k is a constant to be found. [3]

(iii) Given that $|\mathbf{a}| = 2$, $|\mathbf{b}| = \sqrt{2}$ and angle AOB is $\dfrac{\pi}{4}$ radian, using scalar product, show that CD is perpendicular to OA. [3]

(iv) Given instead that **a** is a unit vector, give a geometrical interpretation of $|\mathbf{a}\cdot\mathbf{b}|$. [1]

12 The line l contains the point P with coordinates $(1,-5,-5)$ and is parallel to $3\mathbf{i}+\mathbf{j}-6\mathbf{k}$. The plane π_1 has cartesian equation $2x-5y-3z=4$.

(i) Find the acute angle between l and π_1, giving your answer to the nearest degree. [3]

(ii) Find the coordinates of Q, the point of intersection between l and π_1. [3]

The plane π_2 contains l and is perpendicular to π_1.

(iii) Find a cartesian equation of π_2. [3]

(iv) Find a vector equation of the line where π_1 meets π_2. [2]

13 The function f is defined by

$$\mathrm{f}: x \mapsto 1-\frac{4}{x^2}, \quad \text{for } x \in \mathbb{R},\ x<-1.$$

(i) Sketch the graph of $y=\mathrm{f}(x)$, stating the equation of the asymptote and the coordinates of the point where the curve crosses the axes. [2]

(ii) Show that f has an inverse. Hence, find $\mathrm{f}^{-1}(x)$ and state its domain. [4]

(iii) By considering the domains of $y=\mathrm{ff}^{-1}(x)$ and $y=\mathrm{f}^{-1}\mathrm{f}(x)$, find the range of values of x such that $\mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)$. [2]

Another function g is defined by

$$\mathrm{g}: x \mapsto -\mathrm{e}^{2x}-2, \quad \text{for } x \in \mathbb{R}.$$

(iv) Explain why fg exists. Find fg and its range. [3]

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