EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2021

General Certificate of Education Advanced Level

Higher 2

1 The *Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art) is a Chinese mathematical manuscript written somewhere in the middle of the 3rd century. It demonstrated that the Chinese had an understanding of negative numbers way before the Europeans did. The following problem is translated and modified from the original text found in the *Jiuzhang Suanshu*:

> "Sell 2 cows and 5 sheep to buy 13 pigs: there is a surplus of 1000 coins.
>
> Sell 3 cows and 3 pigs to buy 9 sheep: there are exactly enough coins.
>
> Sell 6 sheep and 8 pigs, then buy 5 cows: there is a deficit of 600 coins.
>
> Tell me: what is the surplus or deficit from buying 3 cows and selling 4 pigs and 3 sheep?"

Solve this problem, giving your answer in context. [4]

2 Using the substitution $x = \dfrac{\cos^2\theta}{k}$, find the exact value of $\displaystyle\int_0^{\frac{1}{2k}} \sqrt{\frac{kx}{1-kx}}\,\mathrm{d}x$ in terms of k, where k is a positive constant. [5]

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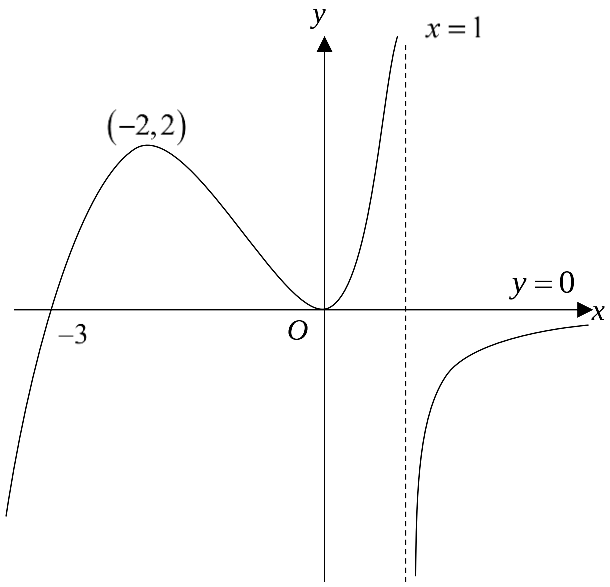
3 **(i)** Let k be a positive constant. Sketch the curve with equation $y = \dfrac{kx+2}{1-x}$, stating the equations of the asymptotes. On the same diagram, sketch the line with equation $y = -x + 2$. [3]

(ii) Hence, solve, in terms of k, the inequality $\dfrac{kx+2}{1-x} < -x + 2$. [2]

(iii) Solve, in terms of k, $\dfrac{k\mathrm{e}^x+2}{1-\mathrm{e}^x} < -\mathrm{e}^x + 2$. [2]

4 **(a)** The graph of $y = \mathrm{f}(x)$ is given below. It has one vertical asymptote at $x = 1$ and one horizontal asymptote at $y = 0$. The graph cuts the x-axis at $x = -3$ and has turning points at $(-2, 2)$ and the origin.



Sketch the graph of $y = \mathrm{f}'(x)$, stating the equations of any asymptotes, and indicating any axial intercepts. [3]

(b) It is given that $\mathrm{g}(x) = x^3 - 3x + p$, where p is an unknown constant.

(i) Given that the graph of $y = \dfrac{1}{\mathrm{g}(x)}$ has a vertical asymptote at $x = 2$, find the value of p. [1]

(ii) Given instead that the graph of $y = \dfrac{1}{\mathrm{g}(x)}$ has a minimum point at $y = \dfrac{1}{5}$, find the value of p. [3]

5 **(a)** **(i)** It is given that $y\dfrac{\mathrm{d}y}{\mathrm{d}x} + x = \sqrt{x^2 + y^2}$. Using the substitution $w = x^2 + y^2$, show that the differential equation can be transformed to $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \mathrm{f}(w)$, where the function $\mathrm{f}(w)$ is to be found. [2]

(ii) Hence, given that $y = 4$ when $x = 3$, solve the differential equation $y\dfrac{\mathrm{d}y}{\mathrm{d}x} + x = \sqrt{x^2 + y^2}$. [3]

(b) Solve the differential equation $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{1}{x}$, where $x > 0$. [4]

6 It is given that $\ln(y+1)=1-\tan x$.

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-(y+1)\sec^2 x$. [1]

(ii) By further differentiation of the above result, find the Maclaurin series for y, up to and including the term in x^2. [4]

(iii) Verify the correctness of your result in part **(ii)** by using small angle approximation and standard series from the List of Formulae (MF26). [3]

(iv) Use your series from part **(ii)** to estimate $\int_{-0.1}^{0} \mathrm{e}^{1-\tan x}\,\mathrm{d}x$, correct to 5 decimal places. [2]

7 **(i)** Show that $\dfrac{4r-1}{3^{r+2}}$ can be expressed as $\dfrac{Ar}{3^r}-\dfrac{B(r+2)}{3^{r+2}}$, where A and B are constants to be determined. [2]

(ii) Hence, find an expression for $\displaystyle\sum_{r=1}^{n}\frac{4r-1}{3^{r+2}}$ in terms of n. You need not express your answer as a single fraction. [3]

(iii) Give a reason why the series $\displaystyle\sum_{r=1}^{n}\frac{4r-1}{3^{r+2}}$ converges and write down the value of the sum to infinity. [2]

(iv) Hence, by finding $\displaystyle\sum_{r=1}^{\infty}\frac{1}{3^{r+2}}$, find the value of $\displaystyle\sum_{r=1}^{\infty}\frac{r}{3^r}$. [3]

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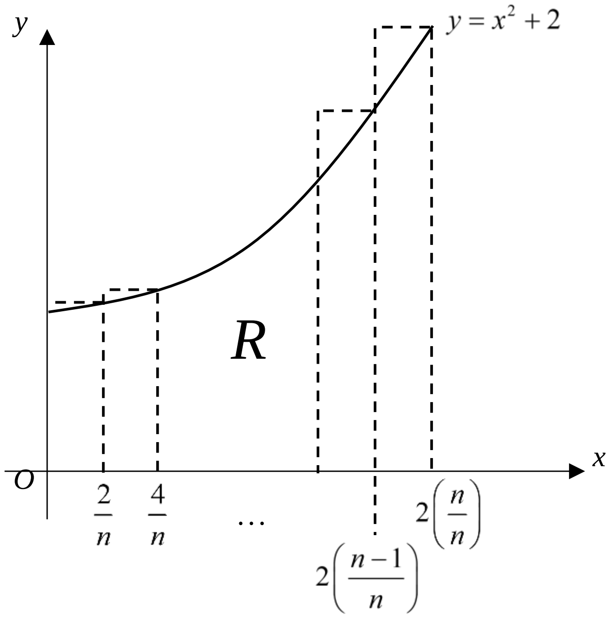


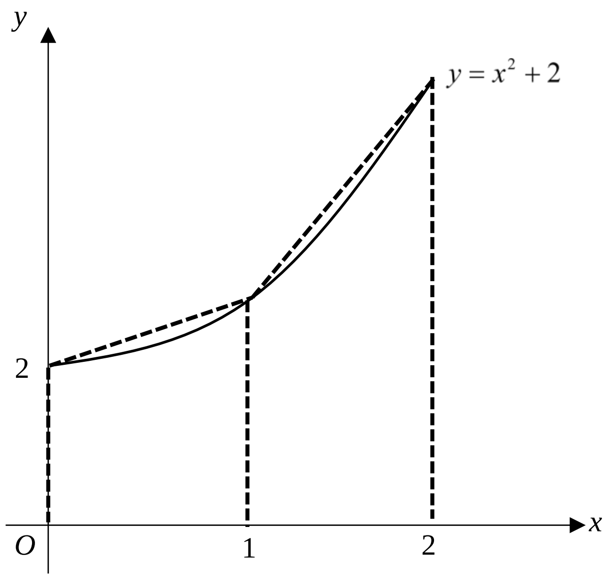
The graph of $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=x^2+2, 0\le x\le 2$, is shown. Let R be the region bounded by the x-axis, the curve $y=x^2+2$, and the lines $x=0$ and $x=2$. Let n rectangles of equal width $\frac{2}{n}$ be used to estimate the area of R, as shown in the diagram above.

(i) Given that $\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$, show that the total area of n rectangles is $\frac{4(n+1)(2n+1)}{3n^2}+4$. [4]

(ii) Hence find the least value of n such that the total area of n rectangles differs from the area of R by less than 0.1. [2]

Instead, n trapeziums of equal width are used to estimate the area of R.

For example, when $n=2$, the trapeziums look like this:



(iii) Find the difference between the total area of the trapeziums and the area of R when 4 trapeziums are used. [3]

(iv) Using your answers to part **(ii)** and part **(iii)**, comment on the use of trapeziums over rectangles to estimate the value of R. [1]

9 A curve C has parametric equations $x=t+\mathrm{e}^{t}$, $y=3t+\mathrm{e}^{-t}$.

(i) Find the exact coordinates of the stationary point of curve C. [4]

(ii) Find the coordinates of the point P on curve C for which both x and y have the same rate of change with respect to t. Hence, find the equation of the normal to the curve C at this point P. [4]

(iii) The normal at P meets the curve C again at point Q. Find the coordinates of point Q. [2]

(iv) Determine, with justification, if triangle OPQ is a right-angled triangle, where O is the origin. [2]

10 Kai is prescribed a drug called 'Aixocra' to manage a chronic long-term illness. Kai's body is able to remove half the amount of Aixocra remaining in his bloodstream every 8 hours. One pill of Aixocra contains 120mg, and one pill is to be taken every 24 hours.

(i) Show that the amount of Aixocra in Kai's bloodstream immediately after he takes the second pill is 135mg. [1]

(ii) Find, in terms of n, the amount of Aixocra in Kai's bloodstream immediately after he has taken the n^{th} pill. [4]

(iii) What is the long-term amount of Aixocra in Kai's bloodstream? [1]

Nya is also prescribed Aixocra for the same illness. However, unknown to her, her body is only able to remove 40% of Aixocra remaining in her bloodstream every 24 hours.

(iv) Studies have shown that patients with this illness will experience heart palpitations if there is more than 280mg of Aixocra in their bloodstream. Find the least number of pills it will take for Nya to experience heart palpitations. [3]

(v) On the day she experiences heart palpitations, Nya stops her intake of Aixocra. She is considered to have completely cleared Aixocra from her system when she has less than 1mg of Aixocra remaining in her bloodstream. Find the number of complete days for Aixocra to be completely cleared from her bloodstream. [2]

(vi) Nya's doctor starts her on an alternative drug called "Nefurb". She is to take 3mg of Nefurb on the first day, and increase her dosage by 2mg every day until she reaches 25mg. She is to then take 25mg every day subsequently. The doctor stops this treatment after a total of 28 days on Nefurb. What is the total amount of Nefurb taken by Nya at the end of treatment? [3]

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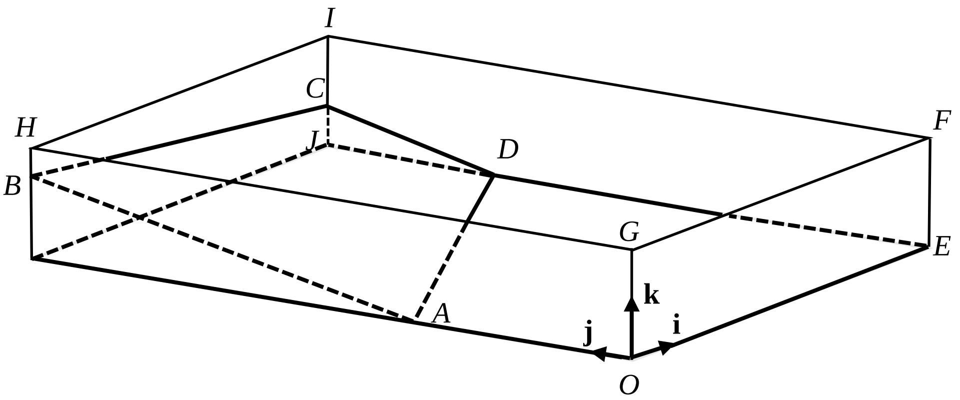


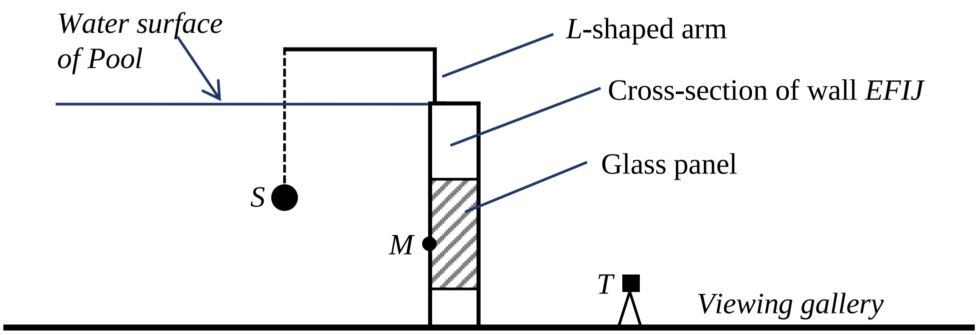
A scuba diving training pool is modelled by a rectangular cuboid (see diagram above). The floor of the pool is made up of the horizontal surface *OADE* and the inclined surface *ABCD*. The surface of the water in the pool is the horizontal plane *FGHI*. The point *O* is the origin and vectors **i**, **j, k** each of length 1 m, are taken along *OE*, *OA* and *OG* respectively.

OADE is part of the plane p_1 and *ABCD* is part of the plane p_2 where p_2 has equation $-3x+6y-20z=90$.

(i) Find the acute angle between p_1 and p_2. [2]

(ii) Find a vector equation for the line passing through A and D. [2]

One important aspect of scuba diving is neutral buoyancy. Trainees learn to descend to a certain depth indicated by a spherical weight that is suspended in the water. The spherical weight hangs at the end of a chain attached to a L-shaped arm such that the centre of the spherical weight is at point $S(18, 2.5, 2)$ (see diagram below).



(iii) The diagram shows a glass panel built into the wall *EFIJ*. A video camera is located at point T in the viewing gallery behind the glass panel and it captures the underwater activity surrounding S through the glass. It is given that $M(20, 2, 1.5)$ is a point on the glass panel such that M is the mid-point of the line segment ST. Find the coordinates of T. [2]

(iv) When trainees are being assessed for their neutral buoyancy skills, a diving instructor stands along the line AD to monitor the safety of the trainees diving around the spherical weight. The instructor will proceed to swim towards S if he senses any irregularity. To ensure that the instructor is standing at a spot along AD that is nearest to S, the location where he should stand is marked with an "X". Determine the coordinates of the point marked with "X". [3]

The L-shaped arm is subsequently mounted along a different edge of the pool such that the spherical weight is now suspended in the water at point $Q(2, 15, k)$. Two objects are secured onto the floor of the pool, one object on p_1 and another on p_2. Based on a signal given by the instructor, trainees will swim from Q to either p_1 or p_2 to retrieve the object on the specified plane.

(v) Find the value of k such that Q is equidistant from p_1 and p_2. [3]