2022 C1 Block Test Revision Package Solutions
Chapter 5 Differentiation and its Applications

1(i) **AJC13/C1Mid-year/Q1**

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-(\mathrm{e}^{-\sqrt{x}})^2}}(\mathrm{e}^{-\sqrt{x}})\left(\frac{-1}{2\sqrt{x}}\right)=\frac{-\mathrm{e}^{-\sqrt{x}}}{2\sqrt{x}\sqrt{1-\mathrm{e}^{-2\sqrt{x}}}}$$

> Refer to MF26 for the derivative of $\sin^{-1}x$, and use chain rule.

1(ii)

$$y=\ln\left(\frac{1+\ln x}{x^x}\right)=\ln(1+\ln x)-x\ln x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{1}{x}}{1+\ln x}-\left(x\cdot\frac{1}{x}+\ln x\right)=\frac{1}{x(1+\ln x)}-1-\ln x$$

> Split expression into 2 terms before differentiating individually

2(a) **DHS13/ C1Mid-year/Q8**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^2 3x\right)=(2\sin 3x)(\cos 3x)(3)=3\sin 6x$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left[(\sin x)(\cos^{-1}x)\right]\right)$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(\sin x)\right]+\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(\cos^{-1}x)\right]$$

$$=\frac{\cos x}{\sin x}+\frac{1}{\cos^{-1}x}\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$=\cot x-\frac{1}{(\cos^{-1}x)\sqrt{1-x^2}}$$

(c)

Let $y=x^{\log_3 x}$.

i.e. $\ln y=(\log_3 x)\ln x=\dfrac{(\ln x)^2}{\ln 3}$

$$\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{\ln 3}\frac{\ln x}{x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{\ln 3}\frac{x^{\log_3 x}\ln x}{x}=\frac{2}{\ln 3}x^{(\log_3 x)-1}\ln x$$

> Simplify equation by taking ln on both sides, then apply implicit differentiation

3(a) **MJC13/C1Mid-year/Q7**

$$y=\ln\frac{\mathrm{e}^{x^2y}}{x^2+1}$$

$$y=\ln\left(\mathrm{e}^{x^2y}\right)-\ln\left(x^2+1\right)$$

$$y=x^2y-\ln\left(x^2+1\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=x^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy-\frac{2x}{x^2+1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(1-x^2)=\frac{2xy(x^2+1)-2x}{x^2+1}$$

	$\frac{dy}{dx}=\frac{2x^3y+2xy-2x}{1-x^4}$ or $\frac{2x\left(x^2y+y-1\right)}{1-x^4}$
(b)	$x=3u^2-u$, $y=\tan^{-1}u$ $\frac{dx}{du}=6u-1$, $\frac{dy}{du}=\frac{1}{1+u^2}$ $\frac{dy}{dx}=\frac{\frac{dy}{du}}{\frac{dx}{du}}=\frac{1}{\left(1+u^2\right)\left(6u-1\right)}$ Always use chain rule $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ to differentiate equations in parametric form. There is no need to find Cartesian equation before differentiating. For the curve to be strictly increasing, $\frac{dy}{dx}>0$ $\frac{1}{\left(1+u^2\right)\left(6u-1\right)}>0$ Since $1+u^2>0$ for all $u\in\mathbb{R}$, $\left(6u-1\right)>0\Rightarrow u>\frac{1}{6}$
4(a)	**PJC13/C1Mid-year/Q2** $\frac{d}{dx}(\sec(\ln(3x^2-6)))$ $=\sec(\ln(3x^2-6))\tan(\ln(3x^2-6))(\frac{6x}{(3x^2-6)})$ $=\sec\left(\ln\left(3x^2-6\right)\right)\tan\left(\ln\left(3x^2-6\right)\right)\frac{2x}{x^2-2}$
(b)	$\frac{d}{dx}\cos^{-1}\sqrt{1-x^2}=\frac{d}{dx}[e^{2x}+x+2xy^2]$ $\frac{-\{(\frac{1}{2})(1-x^2)^{\frac{-1}{2}}(-2x)\}}{\sqrt{1-(\sqrt{1-x^2})^2}}=0+1+2y^2+4xy\frac{dy}{dx}$ $\frac{x}{\sqrt{1-x^2}(\sqrt{x^2})}=1+4xy\frac{dy}{dx}+2y^2$ $4xy\frac{dy}{dx}=\frac{x}{\sqrt{x^2(1-x^2)}}-1-2y^2$ $\frac{dy}{dx}=\frac{1}{4xy}\{\frac{x}{\sqrt{x^2(1-x^2)}}-1-2y^2\}=\frac{1}{4xy}\{\frac{x}{\|x\|\sqrt{(1-x^2)}}-1-2y^2\}$ Note: $\sqrt{x^2}=\|x\|$
5(a)	**RI13/C1Mid-year/Q7** $x=2\theta-\sin 2\theta,$ $y=3-2\cos^2\theta$ $\frac{dx}{d\theta}=2-2\cos 2\theta,$ $\frac{dy}{d\theta}=-4\cos\theta(-\sin\theta)=4\sin\theta\cos\theta$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{4\sin\theta\cos\theta}{2-2\cos 2\theta}$$

$$=\frac{4\sin\theta\cos\theta}{2-2(1-2\sin^2\theta)}$$

$$=\frac{4\sin\theta\cos\theta}{4\sin^2\theta}$$

$$=\frac{\cos\theta}{\sin\theta}=\cot\theta$$

(b)

$e^{x^2}y=\pi^x\ln\pi=e^{x\ln\pi}(\ln\pi)$

By Implicit Differentiation, we have

$$2xe^{x^2}y+e^{x^2}\frac{dy}{dx}=(\ln\pi)(\ln\pi)(e^{x\ln\pi})=\pi^x(\ln\pi)^2$$

$$\frac{dy}{dx}=\frac{\pi^x(\ln\pi)^2-2xe^{x^2}y}{e^{x^2}}=\frac{\pi^x(\ln\pi)^2-2x(\pi^x\ln\pi)}{\frac{\pi^x\ln\pi}{y}}\quad(\because\ e^{x^2}y=\pi^x\ln\pi)$$

$$=\frac{\pi^x(\ln\pi)[\ln\pi-2x]}{\frac{\pi^x\ln\pi}{y}}$$

$$=y[\ln\pi-2x]$$

Alternate method

$e^{x^2}y=\pi^x\ln\pi$

$\ln(e^{x^2}y)=\ln(\pi^x\ln\pi)$

$x^2+\ln y=x\ln\pi+\ln(\ln\pi)$

Differentiate w.r.t.x,

$$2x+\frac{1}{y}\frac{dy}{dx}=\ln\pi$$

$$\frac{1}{y}\frac{dy}{dx}=\ln\pi-2x$$

$$\frac{dy}{dx}=y(\ln\pi-2x)\quad\text{(shown)}$$

6(a) **VJC13/C1Mid-Year/Q3**

$y=\sin^{-1}\left(x^2\right),\quad -1<x<1.$

$$\frac{dy}{dx}=\frac{2x}{\sqrt{1-\left(x^2\right)^2}}=\frac{2x}{\sqrt{1-x^4}}$$

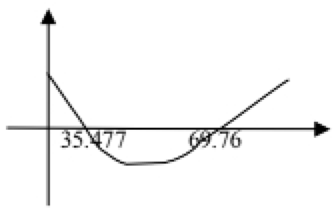
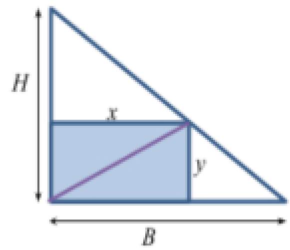
$\frac{dy}{dx}<0 \Rightarrow\ 2x<0$ and $-1<x<1$

For $\sqrt{1-x^4}$ to be valid, we must have $-1<x<1$.

Taking intersection, $-1<x<0$.

Set of values $=\{x\in\mathbb{R}:-1<x<0\}$

(b)	$y=\ln\left[\tan(x+y)\right]$ $e^{y}=\tan(x+y)$ Differentiating wrt x: $e^{y}\frac{dy}{dx}=\sec^{2}(x+y)\left(1+\frac{dy}{dx}\right)$ Since $\sec^{2}(x+y)=1+\tan^{2}(x+y)=1+e^{2y}$, $e^{y}\frac{dy}{dx}=\left(1+e^{2y}\right)\left(1+\frac{dy}{dx}\right)$ (shown)
7(a)	$\ln y=x\ln\left[f(x)\right]$ Differentiate both sides w.r.t. x, $\frac{1}{y}\frac{dy}{dx}=x\left[\frac{f'(x)}{f(x)}\right]+\ln\left[f(x)\right]$ $\Rightarrow \frac{dy}{dx}=xy\left[\frac{f'(x)}{f(x)}\right]+y\ln\left[f(x)\right]$ Let $y=(1+2x)^{x}$. Then we have $\ln y=x\ln(1+2x)$ where $f(x)=1+2x$. Using result from above, $\frac{dy}{dx}=xy\left(\frac{2}{1+2x}\right)+(1+2x)^{x}\ln(1+2x)$
(b) (i)	$y=\tan^{-1}(x+y)$ $\Rightarrow \frac{dy}{dx}=\frac{1}{1+(x+y)^{2}}\cdot\left(1+\frac{dy}{dx}\right)$ $\Rightarrow \left[1+(x+y)^{2}\right]\frac{dy}{dx}=1+\frac{dy}{dx}$ $\Rightarrow (x+y)^{2}\frac{dy}{dx}=1$ ----- (*)
(ii)	Differentiate (*) w.r.t. x, $(x+y)^{2}\frac{d^{2}y}{dx^{2}}+2(x+y)\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}=0$ $\Rightarrow (x+y)^{2}\frac{d^{2}y}{dx^{2}}+2(x+y)\left(\frac{dy}{dx}\right)^{2}+2(x+y)\frac{dy}{dx}=0$ $\Rightarrow (x+y)^{3}\frac{d^{2}y}{dx^{2}}+2(x+y)^{2}\left(\frac{dy}{dx}\right)^{2}+2(x+y)^{2}\frac{dy}{dx}=0$ $\Rightarrow (x+y)^{3}\frac{d^{2}y}{dx^{2}}+2\frac{dy}{dx}+2=0$ ----- (**) At $(1-\frac{\pi}{4},\frac{\pi}{4})$, $\frac{dy}{dx}=1$ using (*) and $\frac{d^{2}y}{dx^{2}}=-4$ using (**)

8(i)	**AJC14/C2Mid-year/Q11** Distance he rowed = BR = $\sqrt{45^2+x^2}$ Distance he walked = RC = $80-x$ $t=\dfrac{\sqrt{2025+x^2}}{2}+\dfrac{80-x}{5}$ [as $\text{time}=\dfrac{\text{distance}}{\text{constan tspeed}}$]
(ii)	$\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{x}{2\sqrt{2025+x^2}}-\dfrac{1}{5}$ When $\dfrac{\mathrm{d}t}{\mathrm{d}x}=0$, $\dfrac{x}{2\sqrt{2025+x^2}}=\dfrac{1}{5}$ $5x=2\sqrt{2025+x^2}$ $\Rightarrow 25x^2=4(2025+x^2)$ $\Rightarrow x^2=\dfrac{4\times 2025}{21}$ $\Rightarrow x=\dfrac{90}{\sqrt{21}}=\dfrac{30\sqrt{21}}{7}$ Exact distance from A to R = $\dfrac{90}{\sqrt{21}}$.
(iii)	Time for John to reach R from B = $\dfrac{\sqrt{2025+x^2}}{2}$. Time for Alex to reach R from A = $\dfrac{x}{1.5}$ $\left\lvert\dfrac{\sqrt{2025+x^2}}{2}-\dfrac{x}{1.5}\right\rvert\le 5\Rightarrow\left\lvert\dfrac{\sqrt{2025+x^2}}{2}-\dfrac{x}{1.5}\right\rvert-5\le 0$ Using GC, sketch graph of $y=\left\lvert\dfrac{\sqrt{2025+x^2}}{2}-\dfrac{x}{1.5}\right\rvert-5$.  For $y\le 0$, we have $35.5\le x\le 69.8$.
9(i)	**MJC14/C2Mid-year/Q3** Let the base length and the height of the rectangle be x and y respectively, and let A denote the area of the inscribed rectangle. $\dfrac{1}{2}BH=\dfrac{1}{2}Hx+\dfrac{1}{2}By$ ← Express y in terms of x using area of triangles. Note that H and B are constants. $\Rightarrow y=\dfrac{BH-Hx}{B}$ 

$\therefore A = xy$

$= x\left(\frac{BH - Hx}{B}\right)$

$= \frac{BHx - Hx^2}{B}$

$= Hx - \frac{H}{B}x^2$

For maximum area ,

$A = Hx - \frac{H}{B}x^2$

$\frac{dA}{dx} = H - \frac{2H}{B}x = 0$

$\therefore x = \frac{B}{2}$

$y = \frac{BH - H\left(\frac{B}{2}\right)}{B} = \frac{H}{2}$ $\frac{dA}{dx} - 0$

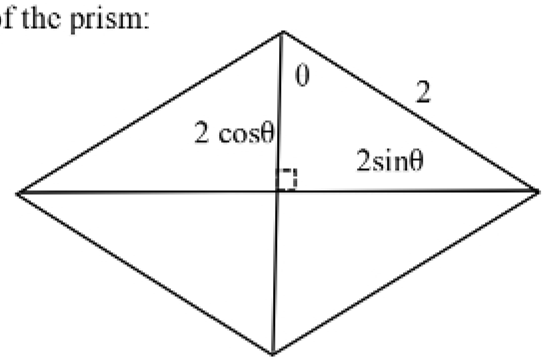
Area is a maximum, since

$\frac{d^2A}{dx^2} = -\frac{2H}{B} < 0 \quad \because H, B$ are lengths > 0

(ii) The two unshaded triangles are congruent. (or the two unshaded triangles have the same area.)

10 **SRJC16/C2 MYE/II/4**

(a) To find the base area of the prism:



Area of the base $= 4\left(\frac{1}{2}2\sin\theta.2\cos\theta\right) = 8\sin\theta\cos\theta = 4\sin 2\theta$

OR Area of the base $= 2\left(\frac{1}{2}(2)(2)\sin 2\theta\right) = 4\sin 2\theta$

Let x be the height of the prism.

Volume = $4\sin 2\theta(x) = 100$ (given)

$\therefore x = \frac{25}{\sin 2\theta}$

Since surface area involves height which is not known, we need to find another equation involving height and θ, so that we can eliminate this unknown in the surface area equation.

Total surface area, $S = 8\sin 2\theta + 8x = 8\sin 2\theta + \frac{200}{\sin 2\theta}$

$$\frac{\mathrm{d}S}{\mathrm{d}\theta}=16\cos 2\theta-\frac{200}{(\sin 2\theta)^2}(2\cos 2\theta)=0 \text{ for turning points}$$

$$16\cos 2\theta\left(1-\frac{25}{(\sin 2\theta)^2}\right)=0$$

$$\cos 2\theta=0 \quad \text{since } 1-\frac{25}{(\sin 2\theta)^2}\neq 0$$

$$2\theta=\frac{\pi}{2},\ \therefore\ \theta=\frac{\pi}{4}$$

θ	$\frac{\pi}{4}^{-}$	$\frac{\pi}{4}$	$\frac{\pi}{4}^{+}$
$\frac{\mathrm{d}S}{\mathrm{d}\theta}$	(+)(–) = – ve	0	(–)(–) = +ve
	\	—	/

Therefore S is minimum when $\theta=\frac{\pi}{4}$

(b)

$$\cos\angle BAC=\frac{1}{2}$$

$$\angle BAC=\frac{\pi}{3}$$

Let $BE=y$.

$$=x^2+25-5x$$

$$y=\sqrt{x^2-5x+25}$$

Express y in terms of x using cosine rule: $BE^2=AE^2+AB^2-2(AE)(AB)\cos BAE$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-5}{2\sqrt{x^2-5x+25}}$$

When $x=5$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2(5)-5}{2\sqrt{(5)^2-5(5)+25}}=\frac{1}{2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{20}\text{cm/s}$$

11 (i) **TJC16/C2 MYE/4**

$$C=\pi(3r)^2k+2\pi(3r)^2(2k)+2\pi rh(2k)$$

$$=45\pi r^2k+4\pi rhk$$

$$\Rightarrow h=\frac{C-45\pi r^2k}{4\pi rk}$$

Note that k, h are constants and r is the variable.

$$V=\frac{2}{3}\pi(3r)^3-\pi r^2h$$

$$= 18\pi r^3 - \pi r^2\left(\frac{C}{4\pi rk} - \frac{45}{4}r\right)$$

$$= \frac{117}{4}\pi r^3 - \frac{Cr}{4k}$$

(ii)

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{117}{4}\pi\left(3r^2\right) - \frac{C}{4k}$$

When V is minimum, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$

Thus $r^2 = \frac{C}{351\pi k}$

Since $\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = \frac{351}{2}\pi r > 0$ since $r > 0$, V is minimum.

Cost of anodizing the flat surfaces $= \$\pi\left(3r\right)^2 k = \$9\pi\left(\frac{C}{351\pi k}\right)k = \$\frac{C}{39}$

12

TJC2017/C1 Mid-year/7

(a)

(i)

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[\tan^{-1}(2x)\right] = \left[\frac{1}{\tan^{-1}(2x)}\right]\left[\frac{2}{1+(2x)^2}\right] = \frac{2}{\left(1+4x^2\right)\left[\tan^{-1}(2x)\right]}$$

(ii)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right) + \cot x\left(-\operatorname{cosec} x \cot x\right)$$

$$= -\operatorname{cosec}^3 x + \cot x\left(-y\right)$$

$$= -\left(\operatorname{cosec}^3 x + y\cot x\right)$$

(b)

$$\mathrm{f}(x) = \frac{x}{2} - 4\ln x + \frac{15}{4}\ln(2x+1)$$

$$\mathrm{f}'(x) = \frac{1}{2} - \frac{4}{x} + \frac{15}{2(2x+1)}$$

$$= \frac{x(2x+1) - 4(2)(2x+1) + 15x}{2x(2x+1)}$$

$$= \frac{x^2-4}{x(2x+1)} \text{ (shown)}$$

At stationary points, $\mathrm{f}'(x) = 0$

$$\Rightarrow \frac{x^2-4}{x(2x+1)} = 0$$

$$\Rightarrow x^2 - 4 = 0$$

$\Rightarrow x = 2$ or -2 (reject, since $x > 0$ for $\ln x$ and $\ln(2x+1)$ to be defined)

Hence the graph has exactly one stationary point at $x = 2$.

Method 1 – 1st Derivative Test

x	2^-	2	2^+
$\mathrm{f}'(x)$	< 0	0	> 0
Shape	\	_	/

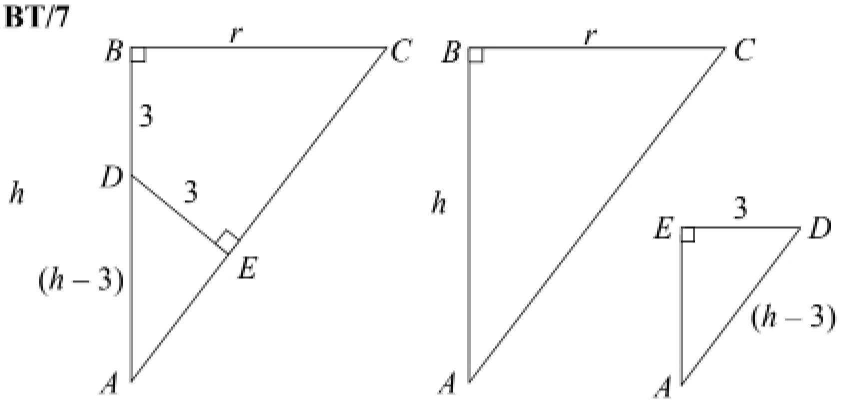
Method 2 – 2nd Derivative Test

$$\mathrm{f}''(x) = \frac{4}{x^2} - \frac{15}{(2x+1)^2}$$

At $x = 2$,

		$f''(x)=\frac{4}{2^2}-\frac{15}{\left[2(2)+1\right]^2}=\frac{2}{5}>0$

Hence the point is a minimum point.

13 **(i)**	**TJC18/C1BT/5** $V=\pi r^2 h \Rightarrow h=\frac{V}{\pi r^2}$ $C=3(2\pi rh)+k(2\pi r^2)$ $=6\pi r\left(\frac{V}{\pi r^2}\right)+2k\pi r^2$ $=\frac{6V}{r}+2k\pi r^2$ (Shown)
(ii)	$\frac{dC}{dr}=-\frac{6V}{r^2}+4k\pi r$ $\frac{dC}{dr}=0 \quad \Rightarrow \quad 6V=4k\pi r^3$ $\Rightarrow \quad 6\pi r^2 h=4k\pi r^3$ or $r^3=\frac{3V}{2k\pi}$ $\Rightarrow \quad \frac{r}{h}=\frac{3}{2k}$ $\frac{d^2C}{dr^2}=\frac{12V}{r^3}+4k\pi>0$ since V, r and $k>0$. Hence C is minimum when $\frac{r}{h}=\frac{3}{2k}$ (shown)
(iii)	Since the costs of producing the curved and flat surfaces remain unchanged, from (ii), the ratio $\frac{r}{h}=\frac{3}{2k}$ is independent of the volume of the can. Hence the worker's suggestion is incorrect.
14	**PJC18/C1BT/7**  $\Delta ABC \sim \Delta AED$ (RHS), ← Tests for similar triangles (AA, SSS, SAS, RHS) are useful in solving maxima/minima questions involving triangles. $\therefore \frac{ED}{BC}=\frac{AE}{AB}$

$$\frac{3}{r}=\frac{\sqrt{(h-3)^2-9}}{h}$$

$$\frac{3}{r}=\frac{\sqrt{h^2-6h}}{h}$$

$$r=\frac{3h}{\sqrt{h^2-6h}} \quad \text{(shown)}$$

Volume of container, $V=\frac{1}{3}\pi r^2 h$

$$=\frac{1}{3}\pi\left(\frac{3h}{\sqrt{h^2-6h}}\right)^2 h$$

$$=\frac{1}{3}\pi\left(\frac{9h^2}{h^2-6h}\right)h$$

$$V=\frac{3\pi h^2}{h-6}$$

For minimum volume, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$,

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{(h-6)(6\pi h)-(3\pi h^2)}{(h-6)^2}=\frac{3\pi h(h-12)}{(h-6)^2}=0$$

$$3\pi h(h-12)=0$$

From $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{3\pi h(h-12)}{(h-6)^2}$, it can be seen that it is easier to do first derivative test. The factor that distinguishes positive or negative $\frac{\mathrm{d}V}{\mathrm{d}h}$ for 12^+ and 12^- is (h –12).

$\therefore\ h=0$ (reject since h > 0) or $h=12$

Using first derivative test,

h	12^-	12	12^+
$\frac{\mathrm{d}V}{\mathrm{d}h}$	–	0	+
	\	–	/

Hence, V is minimum when h = 12.

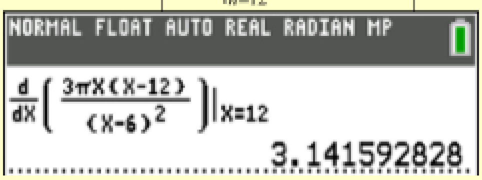
Alternatively,
Using second derivative test,

$$\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{(h-6)^2\left[3\pi h+3\pi(h-12)\right]-3\pi h(h-12)2(h-6)}{(h-6)^4}$$

When h = 12,

$$\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{(12-6)^2\left[3\pi(12)\right]}{(12-6)^4}=\pi>0$$

OR: Using GC to find second derivatives, $\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right|_{h=12}=3.14>0$



Hence, V is minimum when h = 12.
Minimum volume of container,

$$V=\frac{3\pi(12)^2}{12-6}$$
$$=226.19 \quad \text{or} \quad 72\pi$$
$$=226\ \text{(3s.f.)} \quad \text{or} \quad 72\pi$$

15 VJC14/C2Mid-year/Q9 (modified)

(i)

$x = t^2 - t, \quad y = t^2 + 4t + 8, \quad \text{for } t \in \mathbb{R}$

$\frac{dx}{dt} = 2t - 1; \ \frac{dy}{dt} = 2t + 4$

Thus, $\frac{dy}{dx} = \frac{2t+4}{2t-1}$

For min. pt., $\frac{dy}{dx} = 0 \Rightarrow t = -2$

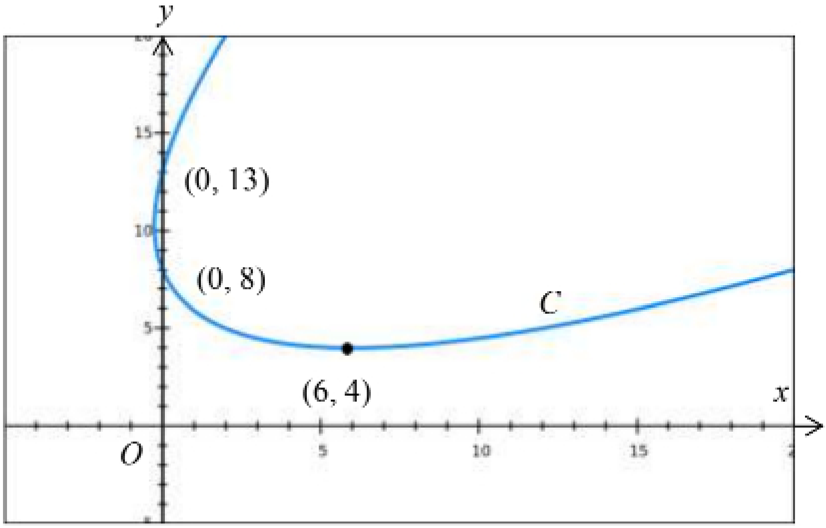
Hence coordinates of the min. pt. is $(6, \ 4)$.

For tangent to be vertical line, $\frac{dy}{dx}$ is undefined.

$2t - 1 = 0 \Rightarrow t = \frac{1}{2} \quad \therefore x = t^2 - t = -\frac{1}{4}$.

Distance of this tangent from the y- axis is $\frac{1}{4}$ unit.

(ii)



(iii)

The equation of tangent at P is

$y - \left(p^2 + 4p + 8\right) = \frac{2p+4}{2p-1}\left(x - p^2 + p\right)$

$(2p-1)y - (2p-1)\left(p^2 + 4p + 8\right) = (2p+4)\left(x - p^2 + p\right)$

(iv)

Since the tangent passes through $(0,0)$,

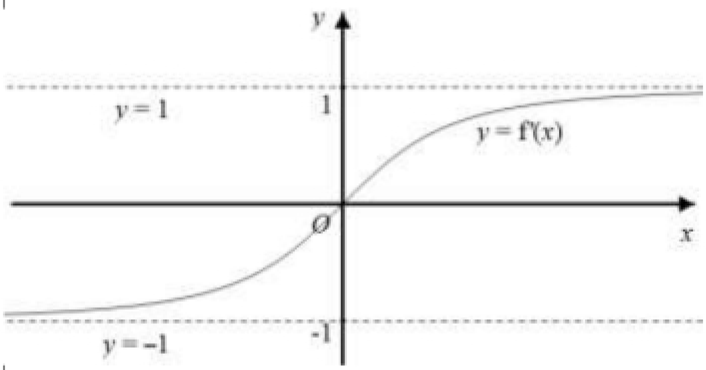
$-(2p^3 + 7p^2 + 12p - 8) = (2p+4)\left(-p^2 + p\right)$

$5p^2 + 16p - 8 = 0$

$p = -3.6396 \text{ or } 0.43961$

Hence the possible coordinates of P are $(-0.246, 9.952)$ and $(16.886, 6.688)$

16 (a) **CJC16/Prelim/II/3 (modified)**



(b)(i)

Given $x = \tan\theta$, $y = \sec\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \sec\theta\tan\theta,$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\Big/\frac{\mathrm{d}x}{\mathrm{d}\theta}$$

$$= \frac{\sec\theta\tan\theta}{\sec^2\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$$

At point P, gradient of normal $= -\frac{1}{\sin\theta}$.

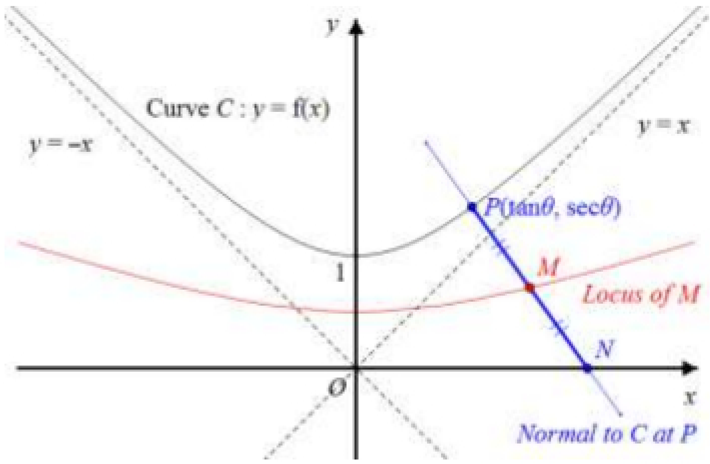
Equation of the normal to the curve at P:

$$y - \sec\theta = -\frac{1}{\sin\theta}(x - \tan\theta),$$

$$y - \frac{1}{\cos\theta} = -x\frac{1}{\sin\theta} + \frac{1}{\cos\theta},$$

$$y = -x\ \mathrm{cosec}\,\theta + 2\sec\theta \quad \text{(shown)}$$

(b)(ii)



x-intercept of the normal at P:

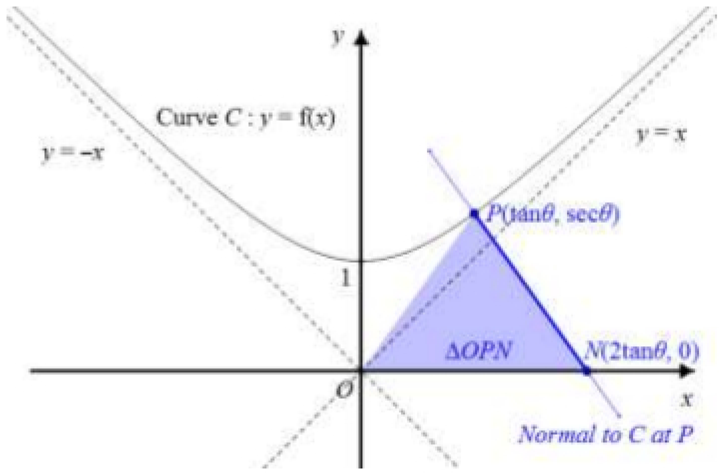
$$0 = -x\ \mathrm{cosec}\,\theta + 2\sec\theta,$$

$$x = 2\frac{\sec\theta}{\mathrm{cosec}\,\theta} = 2\tan\theta.$$

$\therefore$ Point N is $(2\tan\theta,\ 0)$.

Mid-point of PN is $M\left(\frac{x_P + x_N}{2},\ \frac{y_P + y_N}{2}\right) = \left(\frac{3}{2}\tan\theta,\ \frac{1}{2}\sec\theta\right)$.

> When asked to find coordinates of midpoint of PN in terms of θ, we are expressing the midpoint of PN parametrically, as θ varies. The locus of M is the curve where M moves as θ varies.

(b) (iii)	 A, area of $\Delta OPN = \frac{1}{2}(ON)\left(\substack{\text{Height of } P\\ \text{w.r.t. } x\text{-axis}}\right)$ $= \frac{1}{2}(2\tan\theta)\ (\sec\theta) = \tan\theta\sec\theta$ ($ON = 2\tan\theta$, from (b)(ii) assuming $\theta > 0$.)
	$\frac{dA}{d\theta} = \left(\sec^2\theta\right)\sec\theta + \tan\theta\left(\sec\theta\tan\theta\right)$ $= \sec^3\theta + \sec\theta\tan^2\theta$ Rate of change of area of ΔOPN, $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$ $= (\sec^3\theta + \sec\theta\tan^2\theta) \times \cos\theta$ $= \sec^2\theta + \tan^2\theta$ Alternatively, Differentiating $A = \tan\theta\sec\theta$ implicitly with respect to time t, $\frac{dA}{dt} = \left[\left(\sec^2\theta\right)\sec\theta + \tan\theta\left(\sec\theta\tan\theta\right)\right] \times \frac{d\theta}{dt}$ $= (\sec^3\theta + \sec\theta\tan^2\theta) \times \cos\theta$ $= \sec^2\theta + \tan^2\theta$
	When $\theta = \frac{\pi}{6}$, $\sec\theta = \frac{1}{\cos\theta} = \frac{2}{\sqrt{3}}$, $\tan\theta = \frac{1}{\sqrt{3}}$ $\therefore$ $\frac{dA}{dt} = \sec^2\theta + \tan^2\theta = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$, rate of change of the area of ΔOPN when $\theta = \frac{\pi}{6}$.
17 (i)	**HCI 16/Prelim/I/7** $\frac{dy}{dx} = \frac{(kx^2 + x - 2) - (x-2)(2kx+1)}{(kx^2 + x - 2)^2} = \frac{-kx^2 + 4kx}{(kx^2 + x - 2)^2}$ When $x = 0$, $\frac{dy}{dx} = \frac{0}{(-2)^2} = 0$ and $y = \frac{-2}{-2} = 1$ Hence required equation of tangent is $y = 1$.

(ii) For axial intercepts, when $y=0$, $x=2$.
when $x=0$, $y=1$.

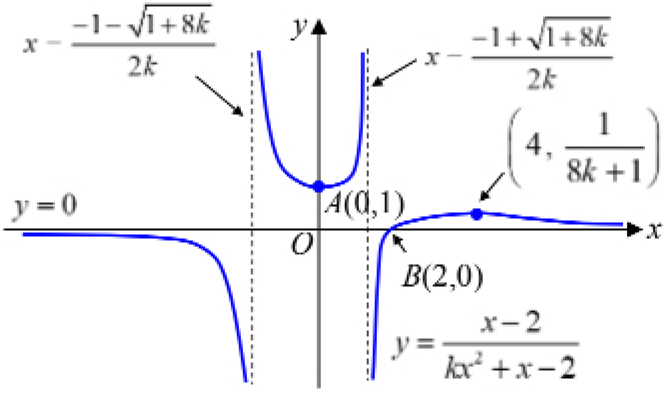
For vertical asymptotes, $kx^2+x-2=0$

$$\therefore\ x=\frac{-1\pm\sqrt{1+8k}}{2k}$$

For turning points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$$-kx^2+4kx=0$$
$$-kx(x-4)=0$$

$\therefore\ x=0$ or $x=4$



To get rough shape of graph, substitute any $k>1$ into equation and plot graph in GC.
Eg. $y=\frac{x-2}{2x^2+x-2}$.
Remember to express features in terms of k.

(iii) At $x=2$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-4k+8k}{(4k)^2}=\frac{4k}{16k^2}=\frac{1}{4k}$

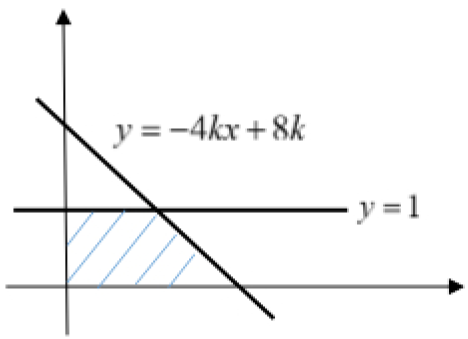
$\therefore$ gradient of normal $=-4k$

Hence required equation of normal is $y-0=-4k(x-2)$

(iv) When $y=1$, $1=-4kx+8k\ \Rightarrow\ x=\frac{8k-1}{4k}$

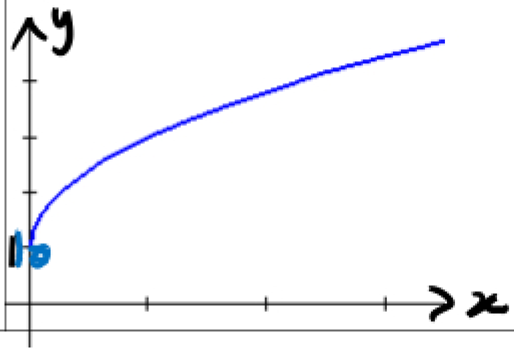
$\therefore$ required area

$$=\frac{1}{2}\left(\frac{8k-1}{4k}+2\right)(1)$$
$$=\frac{16k-1}{8k}$$
$$=2-\frac{1}{8k}\ >\ 2-\frac{1}{8}=\frac{15}{8}\quad(\text{since } k>1)$$



18 **RI 16/Prelim/I/9**

(i)



(ii)

$x=t^2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=2t \qquad y=1+2t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}=2$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}/\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{2}{2t}=\frac{1}{t}$

Equation of tangent at $P(p^2,1+2p)$,

$y-(1+2p)=\frac{1}{p}(x-p^2)$

$y-1-2p=\frac{1}{p}x-p$

$y=\frac{1}{p}x+1+p$

Equation of normal at $P(p^2,1+2p)$:

$y-(1+2p)=-p(x-p^2)$

$y-1-2p=-px+p^3$

$y=-px+p^3+1+2p$

Subs. $x=0$ into $y=\frac{1}{p}x+1+p$, we have $y=1+p$

$\therefore T$ is $(0,1+p)$

Subs. $x=0$ into $y=-px+p^3+1+2p$, $y=p^3+1+2p$

$\therefore N$ is $(0,\ p^3+1+2p)$

Given that P is $(p^2,1+2p)$

$PT^2=(p^2-0)^2+(1+2p-1-p)^2$

$\quad =p^4+p^2$

$TN=\left|(p^3+1+2p)-(1+p)\right|$

$\quad =\left|p^3+p\right|$

$\quad =p^3+p \quad (p\geq 0)$

$\frac{PT^2}{TN}=\frac{p^4+p^2}{p^3+p}=\frac{p^2(p^2+1)}{p(p^2+1)}=p$ (shown)

---

19 (i) **VJC18/C1 Mid-year/8**

$x=\theta-\sin\theta \quad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta}=1-\cos\theta$

$y=1-\cos\theta \quad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta}=\sin\theta$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\sin\theta}{1-\cos\theta}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-\left(1-2\sin^2\frac{\theta}{2}\right)}=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\cot\frac{\theta}{2}$ (shown)

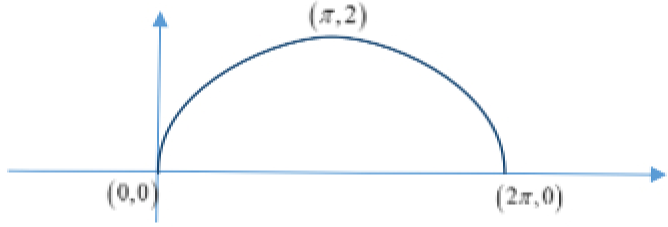
At $\theta=\pi$, $x=\pi$, $y=2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\cot\frac{\pi}{2}=0$. Hence, the equation of the tangent at $\theta=\pi$ is

$y=2$

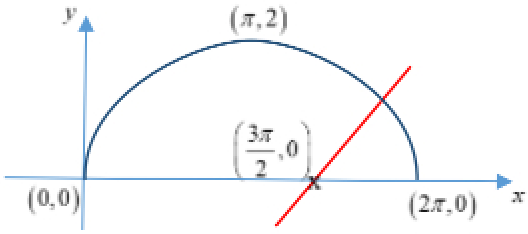
$\frac{dy}{dx} = \cot\frac{\theta}{2}$. As $\theta \to 0$ and $\theta \to 2\pi$, the tangent lines become steeper.

Additionally, the coordinates that correspond to $\theta = 0$ and $\theta = 2\pi$ are $(0,0)$ and $(2\pi,0)$ respectively. Hence, the tangent line at $(0,0)$ and $(2\pi,0)$ are both vertical lines that are parallel to the y-axis.

(ii)



(iii)



The gradient of the line perpendicular to curve C is $-\tan\frac{1}{2}\theta$.

This line also passes through $\left(\frac{3\pi}{2}, 0\right)$ and $(\theta - \sin\theta, 1-\cos\theta)$. Thus,

$$\frac{(1-\cos\theta)-0}{\left((\theta-\sin\theta)-\frac{3\pi}{2}\right)} = -\tan\tfrac{1}{2}\theta.$$

To solve the above equation for θ, we can consider **two** methods.

Method A (Algebra):

$$1-\cos\theta = -\tan\tfrac{1}{2}\theta\left(\theta - \sin\theta - \tfrac{3\pi}{2}\right)$$

$$2\sin^2\tfrac{1}{2}\theta = \frac{-\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\left(\theta - \tfrac{3\pi}{2} - 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\right)$$

Since the interception happened in mid-air, $\theta \neq 0$ and $\theta \neq 2\pi$, thus $\sin\frac{1}{2}\theta \neq 0$. Therefore,

$$2\sin\tfrac{1}{2}\theta = \frac{-1}{\cos\frac{1}{2}\theta}\left(\theta - \tfrac{3\pi}{2} - 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\right)$$

$$2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = -\left(\theta - \tfrac{3\pi}{2} - \sin\theta\right)$$

$$\sin\theta = -\theta + \tfrac{3\pi}{2} + \sin\theta$$

$$\theta = \tfrac{3\pi}{2}$$

Coordinate of interception: $\left(\frac{3\pi}{2} - \sin\frac{3\pi}{2}, 1-\cos\frac{3\pi}{2}\right) = \left(\frac{3\pi}{2}+1, 1\right)$

Method B (Using GC)

Solve $\frac{(1-\cos\theta)-0}{\left((\theta-\sin\theta)-\frac{3\pi}{2}\right)} = -\tan\frac{1}{2}\theta$. Using a GC, we have

$\theta = 4.7124$, $\theta = 0$ or $\theta = 2\pi$. Since interception is in mid-air, $\theta = 4.7124$.

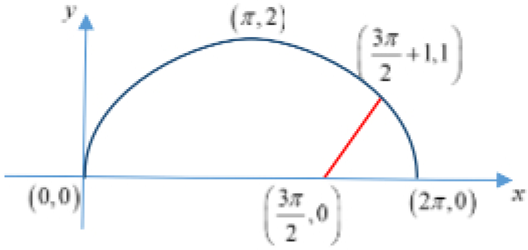
Coordinate of interception:

$(4.7124 - \sin 4.7124, 1-\cos 4.7124) = (5.71, 1.00)$ (3 s.f.)

(iv) Equation of the path of the missile:

$$y=-\tan\left(\frac{\frac{3\pi}{2}}{2}\right)\left(x-\frac{3\pi}{2}\right)$$

$$y=x-\frac{3\pi}{2} \quad (\text{or } y=1.00x-4.71)$$



Range of values of x:

$$\left[\frac{3\pi}{2}\le x\le\frac{3\pi}{2}+1\right] \text{ or }$$

$$\frac{3\pi}{2}\le x\le 5.71$$

20 (a) (i) **DHS18/BT/9**

$$\frac{dx}{dt}=75\cos(\tfrac{1}{15}\pi t)(\tfrac{1}{15}\pi)$$

$$\frac{dy}{dt}=-75(-\sin(\tfrac{1}{15}\pi t))(\tfrac{1}{15}\pi)=75\sin(\tfrac{1}{15}\pi t)(\tfrac{1}{15}\pi)$$

$$\therefore\ \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{75\sin(\frac{1}{15}\pi t)(\frac{1}{15}\pi)}{75\cos(\frac{1}{15}\pi t)(\frac{1}{15}\pi)}=\tan(\tfrac{1}{15}\pi t)$$

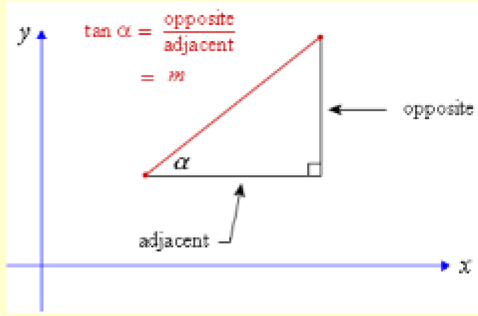
At point P, $t=p$:

$x=75\sin(\tfrac{1}{15}\pi p)+120$,

$y=-75\cos(\tfrac{1}{15}\pi p)+165$,

Gradient of tangent $=\frac{dy}{dx}=\tan(\tfrac{1}{15}\pi p)$

Note:



Thus $\frac{dy}{dx}=\tan\theta$

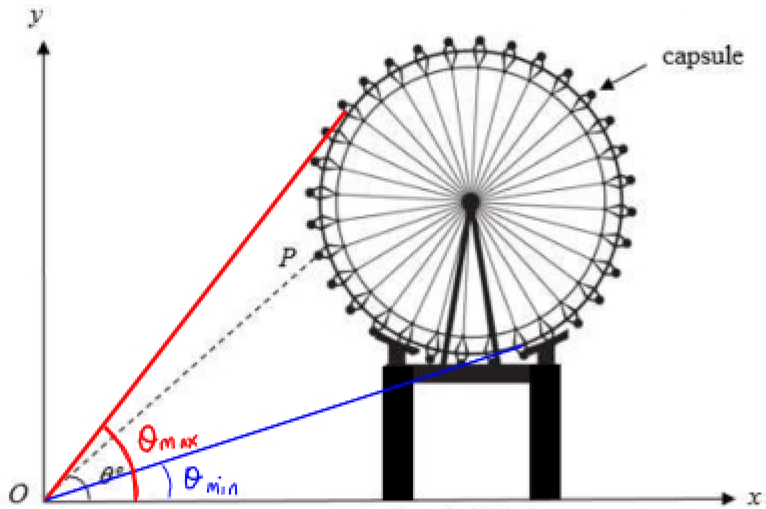
Equation of tangent at P:

$$y-(-75\cos(\tfrac{1}{15}\pi p)+165)=\tan(\tfrac{1}{15}\pi p)(x-(75\sin(\tfrac{1}{15}\pi p)+120))$$

$$y-y_1=m(x-x_1)$$

$$\therefore\ y=\tan(\tfrac{1}{15}\pi p)(x-(75\sin(\tfrac{1}{15}\pi p)+120))-75\cos(\tfrac{1}{15}\pi p)+165$$

(ii) **<u>Method 1: Using equation of tangent from part (i)</u>**



Ferris Wheel

To find largest angle of elevation, consider tangents that pass through origin.

Substitute $(0,0)$ into equation of tangent:

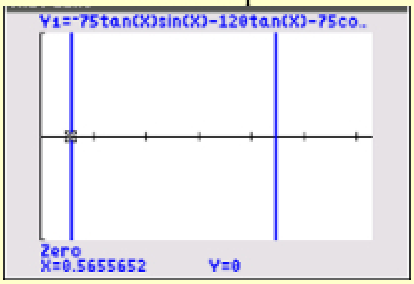
$$0=\tan(\tfrac{1}{15}\pi p)(0-(75\sin(\tfrac{1}{15}\pi p)+120))-75\cos(\tfrac{1}{15}\pi p)+165$$

$\therefore\ -75\tan(\frac{1}{15}\pi p)\sin(\frac{1}{15}\pi p)-120\tan(\frac{1}{15}\pi p)-75\cos(\frac{1}{15}\pi p)+165=0$

Using GC,

$\therefore\ \frac{1}{15}\pi p=0.5656$ or $\frac{1}{15}\pi p=4.460$ (since $0\le p\le 30$, $\therefore 0\le \frac{1}{15}\pi p\le 2\pi$)

> There are 2 possible tangents to curve that pass through origin (see diagram), hence two values of $\frac{1}{15}\pi p$.

> Using GC to solve for k:
> ```
> Plot1  Plot2  Plot3
> \Y1=-75tan(X)sin(X)-120ta▸
> \Y2=
> ```
> Adjust WINDOW to see x-intercepts:
> ```
> WINDOW
>  Xmin=0
>  Xmax=6.283185307
>  Xscl=1
>  Ymin=-1
>  Ymax=1
> ```
> Find first x-intercept:
>
> 
>

When $\frac{1}{15}\pi p=0.5656$,

Gradient of tangent at that point

$=\tan(\frac{1}{15}\pi p)$

$=\tan(0.5656)$

$\therefore\ \theta$ at that point

$=\tan^{-1}(\tan(0.5656))$

$=0.5656$ rad

$=32.4^\circ$

> 32.4° is smallest value θ can take (see diagram above).

When $\frac{1}{15}\pi p=4.460$,

Gradient of tangent at that point

$=\tan(\frac{1}{15}\pi p)$

$=\tan(4.460)$

$\therefore\ \theta$ at that point

$=\tan^{-1}(\tan(4.460))$

$=1.318$ rad

$=75.5^\circ$

> Note that θ is acute, thus basic angle in radian, ie. 1.318 rad is found.

Hence largest $\theta=75.5^\circ$

<u>Method 2: Consider solution type between line $y=kx$ and cartesian equation of circle</u>

$y=kx$ -----(1)

$(x-120)^2+(y-165)^2=75^2=5625$ ----(2)

> See part (b)(i) below on how to obtain cartesian equation of C

Substitute (1) into (2):

$(x-120)^2+(kx-165)^2=5625$

$x^2-240x+14400+k^2x^2-330kx+27225=5625$

$(1+k^2)x^2-(240+330k)x+36000=0$

For $y=kx$ to intersect C at one point, quadratic equation above should have only one solution:

$\therefore$ Discriminant $=(240+330k)^2-4(1+k^2)(36000)=0$

$57600+158400k+108900k^2-144000-144000k^2=0$

$35100k^2-158400k+86400=0$

$k=0.6347$ or $k=3.8781$

For the two lines with above k-values, angles that they make with positive x-axis are:

$\tan^{-1}(0.6347)=32.4°$

$\tan^{-1}(3.8781)=75.5°$

$\therefore$ Largest $\theta=75.5°$

> Note that k is also gradient of $y=kx$ and that $\frac{dy}{dx}=\tan\theta$.

(b) (i) **<u>Method 1:</u>**

For C,

$x = 75\sin(\frac{1}{15}\pi t) + 120$,

$y = -75\cos(\frac{1}{15}\pi t) + 165$.

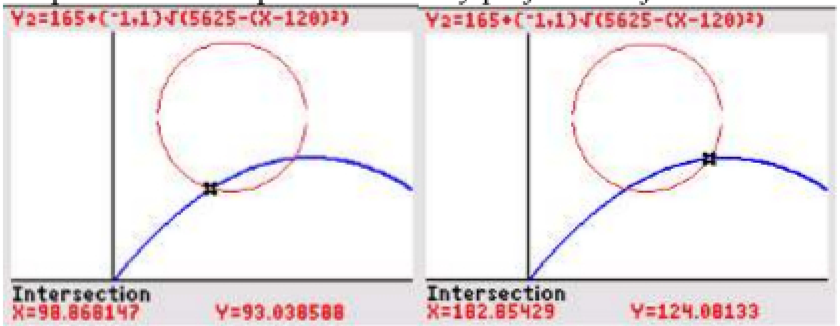
$\therefore$ cartesian eqn is:

$(x-120)^2 + (y-165)^2 = 75^2 \sin^2(\frac{1}{15}\pi t) + 75^2 \cos^2(\frac{1}{15}\pi t)$

$(x-120)^2 + (y-165)^2 = 75^2$

$\therefore\ (x-120)^2 + (y-165)^2 = 5625$

Graphs of C and of path travelled by projected object:



Using GC, intersection points are $(98.9,\ 93.0)$ and $(183,\ 124)$.

<u>Method 2: Solving simultaneous equations using parametric form of circle</u>

$x = 75\sin(\frac{1}{15}\pi t) + 120,\ y = -75\cos(\frac{1}{15}\pi t) + 165 \quad ...(1)$

$y = -\frac{1}{320}(x-200)^2 + 125 \quad ...(2)$

Substitute (1) into (2):

$-75\cos(\frac{1}{15}\pi t) + 165 = -\frac{1}{320}((75\sin(\frac{1}{15}\pi t) + 120) - 200)^2 + 125$

$-75\cos(\frac{1}{15}\pi t) + 40 = -\frac{1}{320}(75\sin(\frac{1}{15}\pi t) - 80)^2$

$-75\cos(\frac{1}{15}\pi t) + 40 + \frac{1}{320}(75\sin(\frac{1}{15}\pi t) - 80)^2 = 0$

Since $0 \le t \le 30$, solutions are:

$t = 4.745$ or $t = 28.636$ ←

These two values of t indicate the times at which P reaches the intersection points between the circle and the parabolic path of projected object.

Using (1),

When $t = 4.745$,

$x = 75\sin(\frac{1}{15}\pi t) + 120 = 182.86$ (5 s.f.)

$y = -75\cos(\frac{1}{15}\pi t) + 165 = 124.09$ (5 s.f.)

When $t = 28.636$,

$x = 75\sin(\frac{1}{15}\pi t) + 120 = 98.865$ (5 s.f.)

$y = -75\cos(\frac{1}{15}\pi t) + 165 = 93.040$ (5 s.f.)

Intersection points are $(98.9,\ 93.0)$ and $(183,\ 124)$.

(ii) **<u>Method 1 (if Method 1 is used in part (i))</u>**

For projected object, since $x = 8t$, the time taken to reach these points $(98.9,\ 93.0)$ and $(183,\ 124)$ are:

$t_1 = \frac{98.865}{8} = 12.358$ (5s.f.) and $t_2 = \frac{182.86}{8} = 22.858$ (5s.f.) respectively.

The horizontal displacements of P at these times t are:

$x_1 = 75\sin(\frac{1}{15}\pi(12.358)) + 120 = 159.41 \ne 98.865$ and

$x_2 = 75\sin(\frac{1}{15}\pi(22.858)) + 120 = 45.211 \ne 182.86$

Thus, the object will not hit P.

Method 2 (if Method 2 is used in part (i))

Finding the horizontal displacements of projected object at the two times t:

When $t = 4.745$,

$$8t = 37.96 \neq 182.86$$

When $t = 28.636$,

$$8t = 229.088 \neq 98.865$$

Since the projected object is at different positions when P reaches the two intersection points, the object will not hit P.

21

(i) **ACJC14/C2Mid-year/Q3**

$$\left(\frac{h}{2}\right)^2 + r^2 = 26^2$$

$$4r^2 + h^2 = 2704$$

Method 1:

$$\Rightarrow \frac{dh}{dt} = \frac{dh}{dr} \times \frac{dr}{dt} = \left(-\frac{4r}{h}\right)\left(-\frac{1}{2}\right) = \frac{2(24)}{20} = 2.4\text{cms}^{-1}$$

Method 2:

Implicit diff,

$$8r\frac{dr}{dt} + 2h\frac{dh}{dt} = 0$$

$$8(24)\left(-\frac{1}{2}\right) + 2(20)\frac{dh}{dt} = 0$$

$$\frac{dh}{dt} = 2.4\text{cms}^{-1}$$

(ii)

$$A = 2\pi rh = 2\pi r\left(2704 - 4r^2\right)^{\frac{1}{2}}$$

$$\frac{dA}{dr} = 2\pi\left(2704 - 4r^2\right)^{\frac{1}{2}} + \frac{2\pi r\left(\frac{1}{2}\right)(-8r)}{\left(2704 - 4r^2\right)^{\frac{1}{2}}} = 0$$

$$r = \sqrt{338} = 18.4\text{cm}$$

22. **NYJC 16/Prelim/I/5**

Let $\sin\theta = \frac{y}{r}$

Differentiate wrt y : $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}y} = \frac{1}{r} \Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}y} = \frac{1}{r\cos\theta}$

$\therefore\ m = \tan\theta \Rightarrow \frac{\mathrm{d}m}{\mathrm{d}y} = \sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}y} = \frac{1}{r\cos^3\theta}$

ie, $$\frac{\mathrm{d}m}{\mathrm{d}y} = \frac{1}{r\left(\frac{r^2-y^2}{r^2}\right)^{\frac{3}{2}}} = \frac{r^2}{(r^2-y^2)^{\frac{3}{2}}}$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{r^2}{(r^2-y^2)^{\frac{3}{2}}} \times \frac{r}{1000} = \frac{r^3}{10^3(\sqrt{r^2-y^2})^3} = \left(\frac{r}{10\sqrt{r^2-y^2}}\right)^3 \quad \text{(shown)}$$

$\frac{\mathrm{d}m}{\mathrm{d}t}$ is the rate of change of the gradient of the line OP.

23 **TJC 16/Prelim/I/5**

At time t, $AB = 3t$, $AP = 500 - 4t$

$$\tan\theta = \frac{AB}{AP} = \frac{3t}{500-4t}$$

$$\theta = \tan^{-1}\left(\frac{3t}{500-4t}\right) \quad \text{(shown)}$$

(i)

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{3t}{500-4t}\right)^2} \times \frac{(500-4t)(3)-3t(-4)}{(500-4t)^2}$$

$$= \frac{(500-4t)^2}{(500-4t)^2+(3t)^2} \times \frac{1500}{(500-4t)^2} = \frac{1500}{9t^2+(500-4t)^2}$$

$$\left(= \frac{1500}{25t^2-4000t+250000} = \frac{60}{t^2-160t+10000}\right)$$

(ii)

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \frac{-1500\left(18t+2(500-4t)(-4)\right)}{\left(9t^2+(500-4t)^2\right)^2} = \frac{-1500(50t-4000)}{\left(9t^2+(500-4t)^2\right)^2}$$

$$\frac{d^2\theta}{dt^2} = 0 \Rightarrow -1500(50t-4000) = 0 \Rightarrow t = 80$$

t	80^-	80	80^+
$\frac{d^2\theta}{dt^2}$	+ve	0	−ve
slope	/	—	\

Using first derivative test, rate of change of θ is maximum at $t = 80$

24 (a)

Let h be the depth of the water in the conical tank and r be the radius of the surface of the water.

$$\tan\frac{\pi}{6} = \frac{r}{h} \Rightarrow h = \sqrt{3}\,r$$

Volume of water in the tank, $V = \frac{1}{3}\pi r^2 h = \frac{1}{\sqrt{3}}\pi r^3$

$$\frac{dV}{dr} = \sqrt{3}\pi r^2$$

Volume of water in the tank after 25 minutes

$$= 5000\pi - (0.9\pi \times 25 \times 60)$$

$$= 3650\pi$$

$$3650\pi = \frac{1}{\sqrt{3}}\pi r^3$$

$r = \sqrt[3]{3650\sqrt{3}} = 18.4906$ after 25 minutes

After 25 minutes,

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{\sqrt{3}\pi(\sqrt[3]{3650\sqrt{3}})^2} \times -0.9\pi$$

$$= -0.001520 \text{ cm s}^{-1}$$

← $\frac{dV}{dt} < 0$ since volume is decreasing.

The radius of the water surface is decreasing at the rate of 0.001520 cm s^{-1}.

(b) **(i)**	$V = 2x^2y + \frac{\pi}{2}\left(\frac{x}{2}\right)^2(2x)$ $y = \frac{V - \frac{\pi x^3}{4}}{2x^2} = \frac{V}{2x^2} - \frac{\pi x}{8}$ Area of the walls $= 2xy + 2(2xy) = 6xy$ Surface area of the roof $= \pi\left(\frac{x}{2}\right)^2 + \frac{1}{2}\pi x(2x) = \frac{5}{4}\pi x^2$ $C = k(6xy) + 4k\left(\frac{5}{4}\pi x^2\right) + 0.5k(2x^2)$ $= 6kx\left(\frac{V}{2x^2} - \frac{\pi x}{8}\right) + 5k\pi x^2 + kx^2$ $= \frac{3kV}{x} + \frac{17}{4}k\pi x^2 + kx^2$ (shown)
(ii)	$\frac{dC}{dx} = -\frac{3kV}{x^2} + \frac{17}{2}k\pi x + 2kx$ $\frac{dC}{dx} = 0$ $\left(\frac{17\pi}{2} + 2\right)x^3 = 3V$ $x = \sqrt[3]{\frac{6V}{17\pi + 4}}$ $\frac{d^2C}{dx^2} = \frac{6kV}{x^3} + \frac{17}{2}k\pi + 2k$ When $x = \sqrt[3]{\frac{6V}{17\pi + 4}}$, $\frac{d^2C}{dx^2} > 0$