HCI 2024 H2 Mathematics Preliminary Examinations Paper 1

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Marks	4	5	5	7	7	11	9	12	14	14	12

Question 1 [4]

The curve $y = f(x)$ cuts the axes at $(0, \frac{1-p}{p})$ and $(1 - p, 0)$, where p is a constant such that $0 < p < 1$. It is given that f^{-1} exists.

State, if possible to do so, the coordinates of the points where the following curves cut the axes.

(a) $y = f(x) + 1$

(b) $y = f(x - p)$

(c) $y = f(3x - p)$

(d) $y = f^{-1}(x)$

Question 2 [5]

Part (a) [1]

It is given that $f(x)$ and $g(x)$ are non-zero polynomials.

When solving an inequality $\frac{f(x)}{g(x)} \geq 1$, a student writes $f(x) > g(x)$.

Comment on the student's working. **[1]**

Part (b) [4]

Find the exact set of values of x for which $\frac{2x^2-x-9}{x^2-x-6} \geq 1$. **[4]**

Question 3 [5]

The region bounded by the curve with equation $x = \frac{y}{\sqrt{2y - y^2}}$, the lines $y = 1$, $y = 1.6$ and the y-axis is rotated through 2π radians about the y-axis to form a solid ornament.

Part (a) [4]

Find the exact volume V_1 of the ornament, giving your answer in terms of π.

Part (b) [1]

An ornament designer designs a different ornament by rotating the region bounded by another curve with equation $x = \frac{by}{\sqrt{2y - y^2}}$, where $b > 0$, the lines $y = 1$, $y = 1.6$ and the y-axis. The region is now rotated through 2π radians about the y-axis. The volume generated is now V_2. State the ratio of V_1 to V_2.

Question 4 [7]

Part (a) [4]

The 11th, 15th and 23rd terms of an arithmetic progression are three distinct consecutive terms of a geometric progression. Find the common ratio of the geometric progression.

Part (b) [3]

The sum, S_n, of the first n terms of a sequence v_1, v_2, v_3 is given by

$$S_n = \frac{3^{n+2} - (-2)^{n+2} - 5}{6}.$$

Find an expression for v_n in terms of n, simplifying your answer.

Question 5 [7]

Part (a) [4]

By using the substitution $x = sec\,\theta$, where $0 \leq \theta \leq \frac{\pi}{2}$, show that

$$\int_{\sqrt{2}}^{2} \frac{1}{\sqrt{x^2-1}}\,dx = \int_{\theta_1}^{\theta_2} g(\theta)\,d\theta,$$

Where θ_1 and θ_2 are exact constants to be stated,
and g is a single trigonometric function to be determined.

Part (b) [3]

Hence, find the exact value of $\int_{\sqrt{2}}^{2} \frac{1}{\sqrt{x^2-1}}\,dx$.

Question 6 [11]

The functions f and g are defined by

$$f : x \mapsto ln\,[(x+4)^2 - 9], \qquad \text{for } x \in R,\ x > k,$$

$$g : x \mapsto \frac{3-2x}{1+2x}, \qquad \text{for } x \in R,\ x > \frac{1}{2}.$$

Part (a) [2]

Find the least value of k for which the function f^{-1} exists.

Use the value of k found in part (a) for the rest of this question.

Part (b) [2]

Without finding f^{-1}, find the exact value of α if $g(\frac{3}{2}) = f^{-1}(\alpha)$.

The function h is defined by

$$h: x \mapsto \frac{1}{\sqrt[4]{x(a-x)}}, \text{ for } 0 < x < a, \text{ where } a \text{ is a constant.}$$

Part (c) [3]

Sketch the graph of $y = h(x)$, stating the coordinates of the stationary point and the equations of any asymptotes.

Part (d) [2]

Given that the composite function gh exists, find the range of values of a.

Part (e) [2]

By considering $y = \frac{1}{h(x)}$ and its stationary point, or otherwise, find the value of a for which $[h(x)]^2 = 1$ only has one real root.

Question 7 [9]

It is given that $f(x) = ax^5 + bx^3 + cx$, where a, b, and c are non-zero real constants.

Part (a) [1]

Show that $f(-x) = -f(x)$.

Part (b) [3]

It is given that $f(x) = 0$ has only one real root and one of the non-real roots is $p + qi$, where p and q are non-zero real constants. Find, in terms of p and q, all the other non-real roots of $f(x) = 0$, justifying your answers.

Part (c) [2]

Given that $\int_{0}^{3} f(x)\, dx = -5$, state the values of $\int_{-3}^{3} f(x)\, dx$ and $\int_{-3}^{3} f(|x|)\, dx$.

Let $a = 1$ and $b = 3$.

Part (d) [3]

By considering $f'(x)$, find the range of values of c such that the curve with equation $y = f(x)$ has 2 stationary points, showing your working clearly.

Question 8 [12]

A curve C has parametric equations

$$x = t^2, \qquad y = lnt, \qquad \text{for } t > 0.$$

Part (a) [3]

Find the equation of the tangent to C at the point with parameter t.

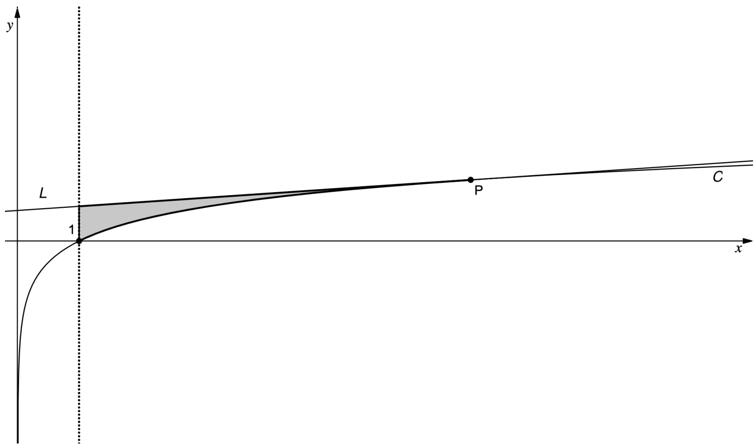
Part (b) [3]

The line L is the tangent to C at the point $P(p^2, ln\,p)$, where p is a positive constant.

Part (c) [2]

Find $\int ln\, x\, dx$.

The diagram below shows the parts of C and L for which $x > 0$.

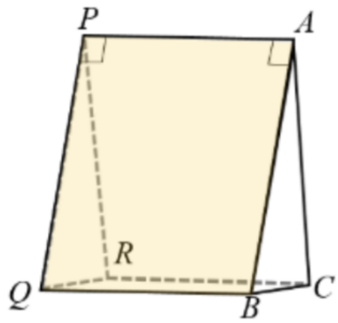


Part (d) [4]

Find the cartesian equation of C in the form of $y = f(x)$. **By using the results in parts (b) and (c)**, find the exact area of the shaded region bounded by C, L and the line $x = 1$.

Question 9 [14]

A right triangular prism has its 2 triangular faces ABC and PQR adjoined by 3 rectangles as shown in the diagram below.



The coordinates of points A, B and C are $(-5,-4,1)$, $(-3,6,2)$ and $(-3,-4,2)$ respectively.

Part (a) [3]

Find the area of triangle ABC.

Part (b) [2]

Find the cartesian equation of the plane which contains A, B and C.

Part (c) [3]

It is given that the plane which contains P, Q and R has equation

$$\underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -2$$

Find the volume of the right triangular prism.

Part (d) [3]

Find the coordinates of P.

Part (e) [3]

A circle with centre at the origin O passes through A and another point D with coordinates $(1, 5, -4)$. Find the length of the minor arc AD, **giving your answer correct to 3 decimal places**.

(arc length = $r\theta$ where θ is in radians)

Question 10 [14]

In a particular chemical reaction, every 2 grams of compound Y and every 3 grams of compound Z react to form 1 gram of compound X. Let x, y and z denote the masses (in grams) of compounds X, Y and Z respectively at any time t (in minutes) after the start of the reaction. 24 grams of compound Y and 24 grams of compound Z are used at the start of the reaction, and there is none of compound X present initially.

Part (a) [1]

Express y as $\alpha + \beta x$, where α and β are constants to be determined.

Part (b) [2]

At any time t, the rate of change of x with respect to t is directly proportional to the product of y and z. Show that

$$\frac{dx}{dt} = k(x-12)(x-8), \text{ where } k \text{ is a positive constant.}$$

Part (c) [6]

By solving the differential equation in part **(b)**, obtain an expression for x in terms of t and k.

Part (d) [1]

State the theoretical mass of compound X formed in the long run.

Part (e) [2]

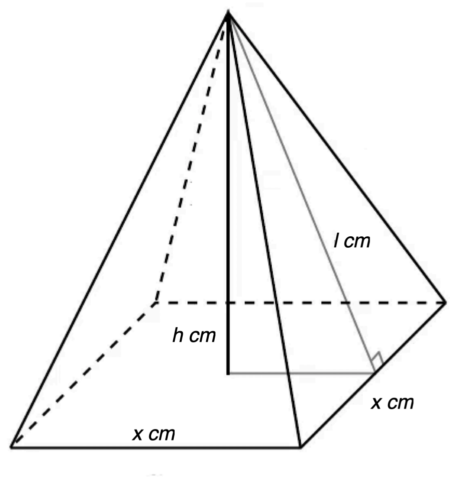
It is observed that there are 4 grams of compound X formed 5 minutes after the start of the reaction. Determine the exact value of k.

Part (f) [2]

Sketch the graph of x against t with the value of k found in part **(e)**.

Question 11 [12]

[The volume of a right square-based pyramid is $\frac{1}{3}$ × base area × height.]



Jane designs a model in the shape of a right square-based pyramid. The square base has sides x cm. Each of the four lateral faces is a triangle with base x cm and perpendicular height l cm. The four lateral faces converge at the top of the pyramid to form an apex directly above the centre of the square base. The vertical height of the pyramid is h cm. The model is assumed to be made of material of negligible thickness.

Part (a) [1]

Form an equation involving x, l and h.

In the design of the model, Jane hopes to fix the total surface area, A cm^2 of the model but maximise the volume, V cm^3 of the model.

Part (b) [1]

Using the result in part (a), show that

$$A = x^2 + 2x\sqrt{h^2 + \frac{x^2}{4}}.$$

Part (c) [2]

Hence show that

$$V^2 = \frac{Ax^2(A-2x^2)}{36}.$$

Part (d) [5]

Use differentiation to show that the maximum V occurs when $x = \frac{\sqrt{A}}{2}$ and find a simplified expression for the maximum V in terms of A.

Part (e) [3]

Given that V is a maximum, find the angle made by a lateral face and the base of the model, **giving your answer to the nearest degree**.