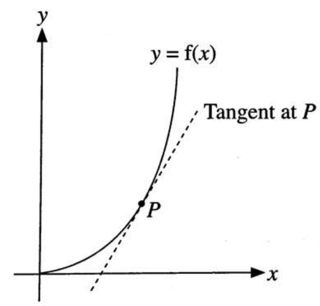
Revision Notes

Learning Objectives

- ✓ *Apply the rules of differentiation to differentiate algebraic expressions*
- ✓ *Find the derivative of f(x)* as the gradient of the tangent to a curve *y = f(x)* at a particular point
- ✓ *Find higher derivatives of functions*
- ✓ *Apply standard derivative notations such as* $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}\left[=\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$, $f'(x)\left[=\frac{d}{dx}f(x)\right]$, $f''(x)\left[=\frac{d}{dx}f'(x)\right]$
- ✓ *Apply* $\frac{dy}{dx}$ *to increasing and decreasing functions*

1. Gradient of a curve

For the curve $y = f(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ represents the gradient of the tangent to the curve at a point P. $\frac{\mathrm{d}y}{\mathrm{d}x}$ measures the **rate of change of *y* with respect to *x***.



2. Formulae

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(ax^n) = anx^{n-1}$

(c) In general, $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]^n = n[f(x)]^{n-1}f'(x)$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}(k) = 0$

3. Derivative rules:

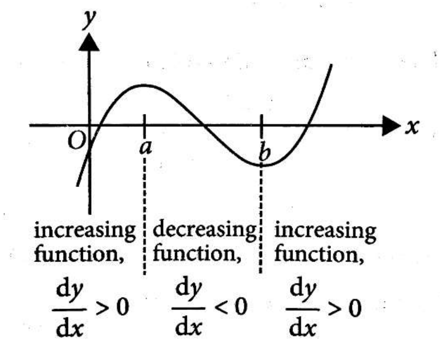
(a) Addition/Subtraction Rule: $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x) \pm \mathrm{g}(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)] \pm \frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{g}(x)]$

(b) Chain Rule: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$

(c) Product Rule: $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

(d) Quotient Rule: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

4.



If y is an increasing function (y increases as x increases), the gradient is positive: $\frac{dy}{dx} > 0$.

If y is an decreasing function (y decreases as x increases), the gradient is negative: $\frac{dy}{dx} < 0$.

5. Higher Derivatives

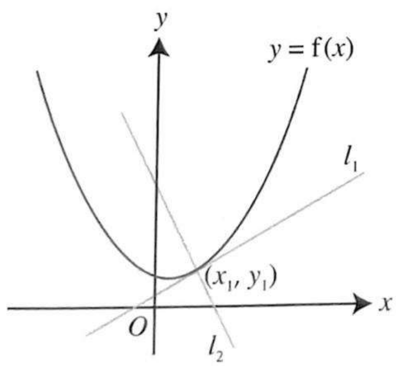
Function in x	**y**	**$f(x)$**
First derivative	$\frac{dy}{dx}$	$f'(x)$
Second derivative	$\frac{d^2y}{dx^2}$	$f''(x)$
Third derivative	$\frac{d^3y}{dx^3}$	$f'''(x)$

Applications of Differentiation

Learning Objectives

- ✓ *Find the equations of the tangents and the normal to a curve at a particular point*
- ✓ *Apply differentiation to problems on rate of change*
- ✓ *Find the nature of a stationary point on a curve and determine whether it is a maximum point, a minimum point or a stationary point of inflexion*
- ✓ *Apply differentiation to solve problems involving maximum and minimum values*

1.



If $y = f(x)$ and $\frac{dy}{dx}$ gives the gradient, m, of the tangent,

- Gradient of the normal = $-\frac{1}{m}$
- the equation of the tangent at the point (x_1, y_1), i.e. l_1, is $y - y_1 = m(x - x_1)$,
- the equation of the normal at the point (x_1, y_1), i.e. l_2, is $y - y_1 = -\frac{1}{m}(x - x_1)$.

2. If the variables x and y both vary with another variable, say t(time), then the rates of change of x and y with respect to t, i.e. $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are related by $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

3. If a point (x_0, y_0) is a stationary point of the curve $y = f(x)$, then $\frac{dy}{dx} = 0$ when $x = x_0$, i.e. the gradient of the tangent at $x = x_0$ is 0.

4. **First derivative test:** to test for nature of stationary point

Substitute values of x^-, x_0, x^+ into $\frac{dy}{dx}$.

(a) **Maximum point:** value of $\frac{dy}{dx}$ changes from positive to negative before and after the stationary point

(b) **Minimum point:** value of $\frac{dy}{dx}$ changes from negative to positive before and after the stationary point

(c) **Point of inflexion:** sign of $\frac{dy}{dx}$ does not change before and after the stationary point

5. **Second derivative test:** used to discriminate between maxima and minima

(a) $\frac{d^2y}{dx^2} < 0$: **Maximum point**

(b) $\frac{d^2y}{dx^2} > 0$: **Minimum point**

(c) $\frac{d^2y}{dx^2} = 0$: Maximum point, minimum point or point of inflexion.

Second derivative test cannot determine point of inflexion.

6. To solve a problem on maximum or minimum values:

Step 1: Find a relationship between the quantity to be maximised or minimised and the variable(s) involved.

Step 2: If there is more than one variable involved, use substitution to reduce it to one independent variable only.

Step 3: Find $\frac{dy}{dx}$ of the expression obtained above.

Step 4: Equate $\frac{dy}{dx}$ to 0 to obtain the value(s) of the variable.

Step 5: Check the nature of the stationary point (first derivative test)

Step 6: Find the required maximum or minimum value of the quantity.

Differentiation (Trigonometric, Logarithmic & Exponential)

Learning Objectives

- ✓ *Differentiate trigonometric functions*
- ✓ *Differentiate logarithmic functions*
- ✓ *Differentiate exponential functions*
- ✓ *Solve problems in the applications of differentiation involving trigonometric, logarithmic and exponential functions*

1. Derivatives of Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sin^n x) = n\sin^{n-1}x\cos x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\cos^n x) = -n\cos^{n-1}x\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\tan^n x) = n\tan^{n-1}x\sec^2 x$
$\frac{d}{dx}(\sec x) = \sec x\tan x$	$\frac{d}{dx}[\sin^n(ax+b)] = an\sin^{n-1}(ax+b)\cos(ax+b)$
$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x$	$\frac{d}{dx}[\cos^n(ax+b)] = -an\cos^{n-1}(ax+b)\sin(ax+b)$
$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	$\frac{d}{dx}[\tan^n(ax+b)] = an\tan^{n-1}(ax+b)\sec^2(ax+b)$
$\frac{d}{dx}[\sin(ax+b)] = a\cos(ax+b)$	$\frac{d}{dx}[\sin^n f(x)] = n\sin^{n-1}f(x) \times \frac{d}{dx}[\sin f(x)]$
$\frac{d}{dx}[\cos(ax+b)] = -a\sin(ax+b)$	$\frac{d}{dx}[\cos^n f(x)] = n\cos^{n-1}f(x) \times \frac{d}{dx}[\cos f(x)]$
$\frac{d}{dx}[\tan(ax+b)] = a\sec^2(ax+b)$	$\frac{d}{dx}[\tan^n f(x)] = n\tan^{n-1}f(x) \times \frac{d}{dx}[\tan f(x)]$

2. Derivatives of Logarithmic Functions

$\frac{d}{dx}(\ln x) = \frac{1}{x}$
$\frac{d}{dx}[\ln(ax+b)] = \frac{a}{ax+b}$
$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$, where $f'(x) = \frac{d}{dx}[f(x)]$

As far as possible, make use of the laws of logarithms to simplify logarithmic expressions before finding the derivatives.

3. Derivatives of Exponential Functions

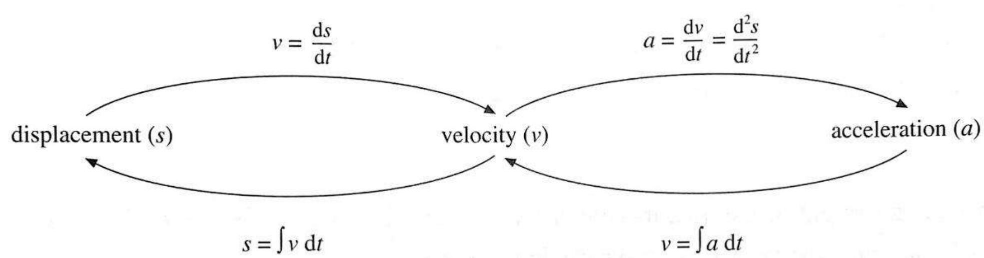
$\frac{d}{dx}\left[e^x\right]=e^x$
$\frac{d}{dx}\left[e^{ax+b}\right]=ae^{ax+b}$
$\frac{d}{dx}\left[e^{f(x)}\right]=f'(x)\,e^{f(x)}$, where $f'(x)=\frac{d}{dx}[f(x)]$

Kinematics

Learning Objectives

✓ *Apply differentiation and integration to problems involving displacement, velocity and acceleration of a particle moving in a straight line*

1. Relationship between displacement(s), velocity(v) and acceleration(a):



2. Common terms used in Kinematics:

(a) Initial: $t = 0$

(b) At rest/Stationary: $v = 0$

(c) Particle is at the fixed point: $s = 0$

(d) Maximum/minimum displacement: $v = 0$

(e) Maximum/minimum velocity: $a = 0$

3. Average speed $= \dfrac{\text{total distance travelled}}{\text{total time taken}}$

4. To find the distance travelled in the first n seconds:

Step 1: Let $v = 0$ to find the value(s) of t.

Step 2: Find s for each of the values of t found in Step 1.

Step 3: Find s for $t = 0$ and for $t = n$.

Step 4: Draw the path of the particle on a displacement-time graph.