RAFFLES INSTITUTION
H2 Mathematics (9758)
2024 Year 6

2024 Year 6 Timed Practice Revision Practice Paper 3 Solution
Source: 2021 Year 6 Term 3 Common Test

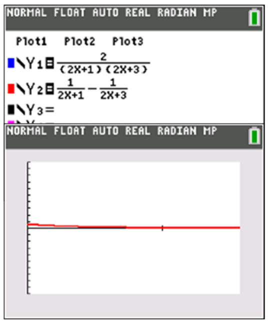
Section A: PURE MATHS (52 Marks)

1	An arithmetic series has first term a and common difference d, where a and d are non-zero. If the first, fifth and eleventh terms of the arithmetic series are equal to the first, second and third terms of a geometric series respectively, find the exact sum of the first ten terms of the geometric series, in the form ka, where k is a simplified rational number. [4]

Solution		**Comments**
[4]	The first 3 terms of the GP are $a, a+4d, a+10d$ $\frac{a+4d}{a}=\frac{a+10d}{a+4d}$ $(a+4d)^2=a(a+10d)$ $a^2+8ad+16d^2=a^2+10ad$ $16d^2-2ad=0$ $d=\frac{a}{8}$ (Since $d\neq 0$) Let the common ratio of the GP be r. $r=\frac{a+4d}{a}=\frac{a+4\left(\frac{a}{8}\right)}{a}=\frac{3}{2}$. Sum of the first 10 terms $=\frac{a(r^{10}-1)}{r-1}$ $=\frac{a\left[\left(\frac{3}{2}\right)^{10}-1\right]}{\frac{3}{2}-1}$ $=2a\left[\left(\frac{3}{2}\right)^{10}-1\right]$ $=\left(\frac{58025}{512}\right)a$	The question asked for a simplified rational number, which is a number of the form $\frac{p}{q}$, where p and q are integers, ie, express k as a fraction in lowest terms.

	Alternatively, to find the required sum of the geometric series, we just need to find r. We can do so by eliminating d from the equations: $a+4d=ar \quad \cdots(1)$ $a+10d=ar^2 \quad \cdots(2)$ $(1) \Rightarrow d=\dfrac{a(r-1)}{4}$ and thus substituting into (2): we have $a+10\left(\dfrac{a(r-1)}{4}\right)=ar^2$ $\Rightarrow 4+10(r-1)=4r^2$ since $a \neq 0$ $\Rightarrow 2r^2-5r+3=0$ $\Rightarrow r=1$ or $r=\dfrac{3}{2}$ We reject $r = 1$, because if $r = 1$ then (1) (or (2) $\Rightarrow d = 0$ The required sum of the first 10 terms then follows the same working as in the first solution.	Note that the justification that r cannot be 1 has to be clearly stated as it is not given in the question.

2	**(i)**	By writing $\frac{2}{(2r+1)(2r+3)}$ in partial fractions, find an expression for $\sum_{r=2}^{n}\frac{2}{(2r+1)(2r+3)}$ in terms of n. [3]
	(ii)	Hence find the exact value of $\frac{2}{19}\times\frac{1}{21}+\frac{2}{21}\times\frac{1}{23}+\frac{2}{23}\times\frac{1}{25}+\cdots$. [2]

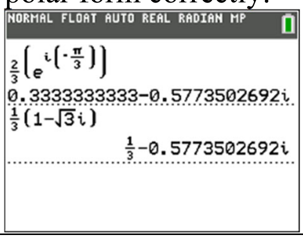
Solution		**Comments**
(i) **[3]**	Let $\frac{2}{(2r+1)(2r+3)}=\frac{A}{2r+1}+\frac{B}{2r+3}$ $2=A(2r+3)+B(2r+1)$ When $r=-\frac{1}{2}, A=\frac{2}{-1+3}=1$ When $r=-\frac{3}{2}, B=\frac{2}{-3+1}=-1$ $\frac{2}{(2r+1)(2r+3)}\equiv\frac{1}{2r+1}-\frac{1}{2r+3}$ $\sum_{r=2}^{n}\frac{2}{(2r+1)(2r+3)}=\sum_{r=2}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$ $=\left(\frac{1}{5}-\frac{1}{7}\right)$ $+\left(\frac{1}{7}-\frac{1}{9}\right)$ $+\left(\frac{1}{9}-\frac{1}{11}\right)$ $\vdots$ $+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$ $+\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$ $=\frac{1}{5}-\frac{1}{2n+3}$	Check your partial fractions by either 1) re-combining the 2 fractions into a single one OR 2) using GC to check both sides of the expression give the same graph (both graphs should overlap completely when sketched)  Do remember to write down the first 2 cancellations as well as the last one.

(ii) **[2]**	$\frac{2}{19}\times\frac{1}{21}+\frac{2}{21}\times\frac{1}{23}+\frac{2}{23}\times\frac{1}{25}+\cdots$ $=\sum_{r=9}^{\infty}\frac{2}{(2r+1)(2r+3)}$ $=\sum_{r=2}^{\infty}\frac{2}{(2r+1)(2r+3)}-\sum_{r=2}^{8}\frac{2}{(2r+1)(2r+3)}$ $=\lim_{n\to\infty}\left(\frac{1}{5}-\frac{1}{2n+3}\right)-\left(\frac{1}{5}-\frac{1}{2(8)+3}\right)$ $=\frac{1}{5}-\left(\frac{1}{5}-\frac{1}{19}\right)$ $=\frac{1}{19}$ Alternative: $\frac{2}{19}\times\frac{1}{21}+\frac{2}{21}\times\frac{1}{23}+\frac{2}{23}\times\frac{1}{25}+\cdots$ $=\sum_{r=9}^{\infty}\frac{2}{(2r+1)(2r+3)}$ $=\lim_{n\to\infty}\sum_{r=9}^{n}\frac{2}{(2r+1)(2r+3)}$ $=\lim_{n\to\infty}\sum_{r=9}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$ $=\lim_{n\to\infty}\left[\frac{1}{2(9)+1}-\frac{1}{2n+3}\right]$ $=\frac{1}{19}$	Do take note that the series is not finite. Hence the need to take limits or discuss what happens when n is large. Note: $\frac{1}{2n+3}\to 0$ as $n\to\infty$

3	For a curve with equation $x^2-3xy+y^3=8$, find	
	(i)	the x-coordinate of the point at which the tangent is parallel to the x-axis, [4]
	(ii)	the equation(s) of the normal(s) at the point(s) where $y=2$. [3]

Solution		**Comments**
(i) **[4]**	$x^2-3xy+y^3=8$ Differentiate w.r.t x $2x-3x\frac{\mathrm{d}y}{\mathrm{d}x}-3y+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3y-2x}{3y^2-3x}$ Since tangent // x-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $3y-2x=0$ $\Rightarrow y=\frac{2x}{3}$ Sub $y=\frac{2x}{3}$ into $x^2-3xy+y^3=8$ $x^2-3x\left(\frac{2x}{3}\right)+\left(\frac{2x}{3}\right)^3=8$ $\Rightarrow 27x^2-54x^2+8x^3=216$ $\Rightarrow 8x^3-27x^2-216=0$ $\Rightarrow x\approx 4.6329$ (using GC) x-coordinate of the point at which the tangent is parallel to the x-axis is 4.63 (3 s.f.)	Be careful of sign errors as you manipulate the equation after differentiating.
(ii) **[3]**	At $y=2$, $x^2-3x(2)+2^3=8$ $\Rightarrow x^2-6x=0$ $\Rightarrow x=0$ or 6 At $(0,2)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}$ Gradient of normal = -2 Equation of normal is $y=-2x+2$ At $(6,2)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=1$ Gradient of normal = -1 Equation of normal is $y=-x+8$	Do read the question carefully and remember that the equation of the **normal** is requested, not the tangent.

4	The polar form of a complex number z is given by $z=r\text{e}^{\text{i}\theta}$, where $r>0$ and $0<\theta\le\pi$, and the complex number $w=\left(\frac{1}{3}-\frac{\sqrt{3}}{3}\text{i}\right)z$.	
	(i)	Find w in exact polar form in terms of r and θ. [3]
	(ii)	Given that $\frac{z^5}{w^*}$ is real and positive, find the possible value(s) of θ in exact form, leaving your answer(s) in terms of π. [4]

	Solution	**Comments**
(i) **[3]**	$\frac{1}{3}-\frac{\sqrt{3}}{3}\text{i}=\frac{1}{3}\left(1-\sqrt{3}\text{i}\right)=\frac{1}{3}\sqrt{1+3}\,\text{e}^{-\tan^{-1}\frac{\sqrt{3}}{1}\text{i}}=\frac{2}{3}\text{e}^{-\frac{\pi}{3}\text{i}}$ $w=\left(\frac{2}{3}\text{e}^{-\frac{\pi}{3}\text{i}}\right)\left(r\text{e}^{\text{i}\theta}\right)=\frac{2}{3}r\,\text{e}^{\text{i}\left(\theta-\frac{\pi}{3}\right)}$ Alternative: Let $v=\frac{1}{3}-\frac{\sqrt{3}}{3}\text{i}$. Then $\lvert v\rvert=\frac{1}{3}\sqrt{1+3}=\frac{2}{3}$ and $\arg(v)=-\tan^{-1}\left(\frac{\frac{\sqrt{3}}{3}}{\frac{1}{3}}\right)=-\frac{\pi}{3}$ $w=vz\Rightarrow\lvert w\rvert=\lvert v\rvert\lvert z\rvert=\frac{1}{3}\sqrt{1+3}\lvert z\rvert=\frac{2}{3}r$ and $\arg(w)=\arg(v)+\arg z=\theta-\frac{\pi}{3}$ $w=\frac{2}{3}r\,\text{e}^{\text{i}\left(\theta-\frac{\pi}{3}\right)}$	The argument of a complex number is not found in general by simply taking tangent inverse of the imaginary part over the real part. You should **ALWAYS** sketch to see which quadrant the point corresponding to the complex number lies in, and work out the basic angle it makes with the real axis before finding the argument. You can also use your GC to check if you have converted to polar form correctly: 
(ii) **[4]**	$\frac{z^5}{w^*}=\frac{r^5\text{e}^{\text{i}5\theta}}{\frac{2}{3}r\text{e}^{-\text{i}\left(\theta-\frac{\pi}{3}\right)}}=\frac{3}{2}r^4\text{e}^{\text{i}\left(6\theta-\frac{\pi}{3}\right)}$ **OR:**	Most students realise that there is a need to work in polar form (hence (i)), but are not careful with the conjugate in the denominator.

	To find $\arg\left(\frac{z^5}{w^*}\right)$: $\arg\left(z^5\right)-\arg\left(w^*\right)$ $=5\arg(z)+\arg(w)$ $=6\theta-\frac{\pi}{3}$ Given that $\frac{z^5}{w^*}$ is real and positive, $6\theta-\frac{\pi}{3}=0,2\pi,4\pi$ since $0<\theta\leq\pi\Rightarrow-\frac{\pi}{3}<6\theta-\frac{\pi}{3}\leq5\frac{2}{3}\pi$ $6\theta=\frac{\pi}{3},\frac{7\pi}{3},\frac{13\pi}{3}$ $\theta=\frac{\pi}{18},\frac{7\pi}{18},\frac{13\pi}{18}$	Remember that for the complex number to be real and positive, the possible arguments are **even** multiples of π.

5 The function f is defined by

$$\mathrm{f}: x \mapsto 2x^2 + 5x - 3, \quad x \in \mathbb{R}, \quad x \le a.$$

(i) State the greatest value of a such that the function f^{-1} exists. [1]

For the rest of the question, use the value of a found in part **(i)**.

(ii) Find f^{-1} in a similar form. [3]

(iii) Find the exact solution of $\mathrm{f}^{-1}(x) = x$. [3]

(iv) The function g is defined for specific integer values of x as follows.

x	-5	-4	-1	0	2
$\mathrm{g}(x)$	6	4	$2\sqrt{3}$	0	-3

Find the value of b, where $\mathrm{f}^{-1}\mathrm{g}(b) = -3$. [1]

Solution	Comments

(i) [1]

$$2x^2 + 5x - 3 = 2\left(x + \frac{5}{4}\right)^2 - 2\left(\frac{5}{4}\right)^2 - 3$$

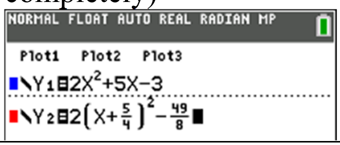
$$= 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8}$$

The greatest value of $a = -\dfrac{5}{4}$

Alternatively, the largest value of a is the x-coordinate of the minimum point of the quadratic, which occurs when

$$\mathrm{f}'(x) = 0 \Leftrightarrow 4x + 5 = 0 \Leftrightarrow x = -\frac{5}{4}$$

Comments: For students who complete the square, they should check their answers by either
1) re-expanding the completed square form
OR
2) using the GC to check both forms give the same graph (both graphs should overlap completely)



(ii) [3]

Let $y = 2\left(x + \dfrac{5}{4}\right)^2 - \dfrac{49}{8}$

$$2\left(x + \frac{5}{4}\right)^2 = y + \frac{49}{8}$$

$$\left(x + \frac{5}{4}\right) = \pm\sqrt{\frac{y}{2} + \frac{49}{16}} = \pm\sqrt{\frac{8y + 49}{16}}$$

$$x = -\frac{5}{4} \pm \frac{1}{4}\sqrt{8y + 49}$$

Since $x \le -\dfrac{5}{4}$, $x = -\dfrac{5}{4} - \dfrac{1}{4}\sqrt{8y + 49}$.

$$\mathrm{f}^{-1}: x \mapsto -\frac{5}{4} - \frac{1}{4}\sqrt{8x + 49}, \quad x \in \mathbb{R}, \quad x \ge -\frac{49}{8}$$

Comments: It is important to write $\pm$ before choosing the correct rule for f^{-1} by considering the domain of f. Justification is necessary.

Note that "similar form" (with rule and domain of f^{-1}) is required.

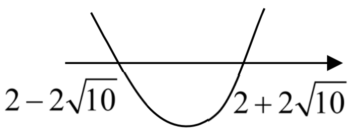
	Alternative (finding inverse by using the quadratic formula): Let $y = 2x^2 + 5x - 3$ $\Rightarrow 2x^2 + 5x - (3 + y) = 0$ $\Rightarrow x = \dfrac{-5 \pm \sqrt{25 + 8(3 + y)}}{2(2)}$ The rejection of one of the expressions follows the same argument as before.	This alternative is suitable for those who do not wish to complete the square.
(iii) **[3]**	$f^{-1}(x) = x \Rightarrow f(x) = x$ $2x^2 + 5x - 3 = x$ $2x^2 + 4x - 3 = 0$ $x = \dfrac{-4 \pm \sqrt{16 - 4(2)(-3)}}{4}$ $= \dfrac{-4 \pm \sqrt{40}}{4} = \dfrac{-2 \pm \sqrt{10}}{2}$ Since $x \le -\dfrac{5}{4}, x = -1 - \dfrac{\sqrt{10}}{2}$	It is much easier to solve $f(x) = x$ which is a simple quadratic equation, compared to the original equation which involves square root. Do take note of the restriction, $x \le -\dfrac{5}{4}$ (by considering domain of f) which is required to reject one of the solutions.
(iv) **[1]**	$f^{-1}g(b) = -3$ $g(b) = f(-3)$ $g(b) = 0$ From table, $b = 0$	

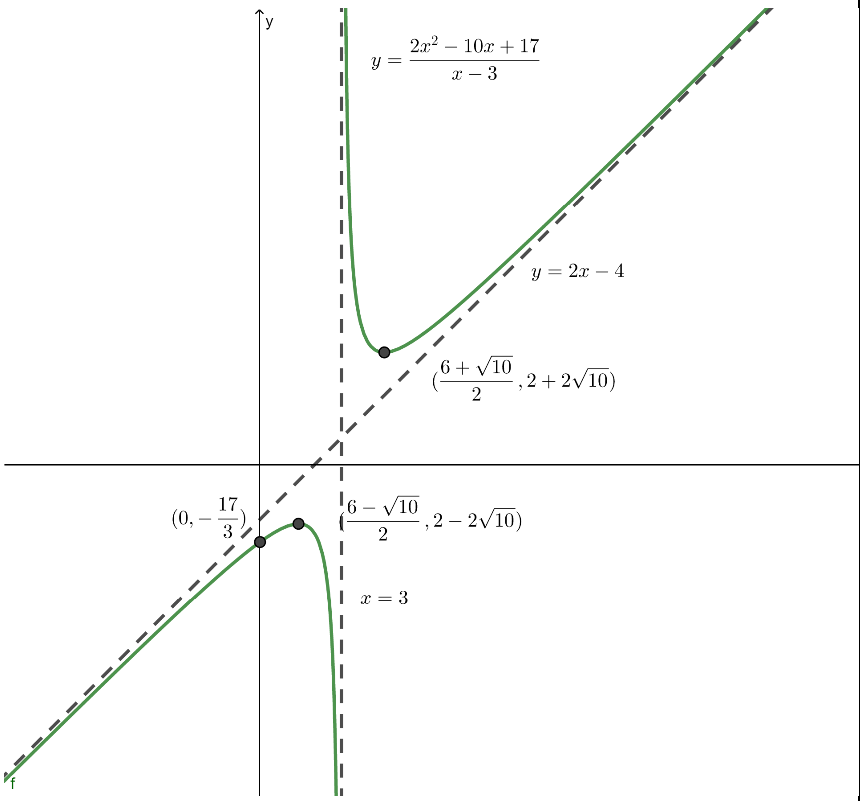
6

The curve C has equation $y=\dfrac{2x^2-10x+17}{x-3}$, $x\in\mathbb{R}$, $x\neq 3$.

(i) Without using a calculator, find the set of values of y that C can take. [4]

(ii) Sketch C, labelling the relevant features in exact form. [5]

Solution		Comments
(i) **[4]**	$y=\dfrac{2x^2-10x+17}{x-3}$ $\Rightarrow y(x-3)=2x^2-10x+17$ $\Rightarrow 2x^2-10x+17-xy+3y=0$ $\Rightarrow 2x^2+(-y-10)x+(17+3y)=0$ For quadratic equation to have real roots, discriminant ≥ 0 $\Rightarrow (-y-10)^2-4(2)(17+3y)\geq 0$ $\Rightarrow y^2+20y+100-136-24y\geq 0$ $\Rightarrow y^2-4y-36\geq 0$ Let $y^2-4y-36=0$  $y=\dfrac{4\pm\sqrt{4^2-4(1)(-36)}}{2}=2\pm 2\sqrt{10}$ $\Rightarrow\left[y-\left(2-2\sqrt{10}\right)\right]\left[y-\left(2+2\sqrt{10}\right)\right]\geq 0$ $\therefore y\leq 2-2\sqrt{10}$ or $y\geq 2+2\sqrt{10}$ Set of values of y is $\left(-\infty,2-2\sqrt{10}\right]\cup\left[2+2\sqrt{10},\infty\right)$	As the question states “without using a calculator”, you will need to justify the shape of the curve if you were to use calculus methods. The Cambridge markers’ report has stated clearly that full credit can only be awarded if the nature of the stationary points and the asymptotic behaviour are discussed for such questions. Hence it is strongly advised that you use the discriminant method.
(ii) **[5]**	$y=\dfrac{2x^2-10x+17}{x-3}=2x-4+\dfrac{5}{x-3}$ $\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=2-5(x-3)^{-2}$ When $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$, $\Rightarrow (x-3)^2=\dfrac{5}{2}$ $\Rightarrow x=3\pm\sqrt{\dfrac{5}{2}}$ $\Rightarrow x=\dfrac{6\pm\sqrt{10}}{2}$ When $x=0$, $y=-\dfrac{17}{3}$.	As the question asked for **exact form**, you need to show all working (including the differentiation) to obtain the x-coordinates of the stationary points. The corresponding y-coordinates have been found in part (i).

	When $y=0$, $2x^2-10x+17=0$ has no real roots since discriminant is $(-10)^2-4(2)(17)=-36<0$ 	All relevant features of the graph: (**SIA**) **S**hape (Stationary points) **I**ntercepts **A**symptotes

7 To build a Virtual Reality (VR) experience, a virtual world is created to contain geometric models. With reference to the origin O, every point in the virtual world is represented as coordinates (x, y, z). Perspective projection is a technique used to project points in the virtual world onto a virtual screen while respecting the scaling properties of objects at various distances. Getting this right in VR helps in the perception of depth and scale. In addition, only points that lie within a zone in front of the eye positioned at O, known as the viewing frustum (see **Fig. 1**), will be projected onto the virtual screen.

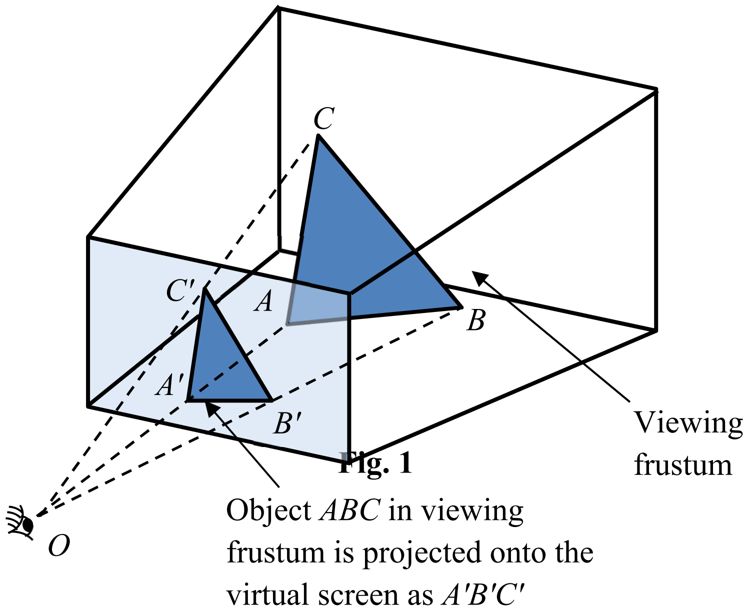


Fig. 1

In **Fig. 1**, a triangular object ABC in the viewing frustum with $A(36,36,12)$ and $B(24,48,72)$ is projected onto the virtual screen to give an image of $A'B'C'$. It is given that the coordinates of A' are $(3,3,1)$ and the line that passes through A' and B' is parallel to the vector $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

(i) Show that the coordinates of B' are $(1,2,3)$. [4]

(ii) Given that the coordinates of C' are $(1,4,2)$, find the equation of the virtual screen in scalar product form. [2]

(iii) Find the exact area of triangle $A'B'C'$. [2]

(iv) If the length of projection of $\overrightarrow{BC}$ onto the normal of the virtual screen is $\frac{57}{\sqrt{29}}$ units, find the possible position vector(s) of C, leaving your answer(s) in exact form. [4]

Solution		Comments
(i) **[4]**	Equation of line that passes through A' and B': $l_{A'B'}: \mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \mu \in \mathbb{R}$ Equation of line that passes through B and B': $l_{BB'}: \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$ Since $l_{A'B'}$ and $l_{BB'}$ intersects at B' $\lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ for some $\lambda, \mu \in \mathbb{R}$ $\lambda - 2\mu = 3$ $2\lambda - \mu = 3$ $3\lambda + 2\mu = 1$ Using GC, $\lambda = 1, \mu = -1$ $\Rightarrow \overrightarrow{OB'} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ The coordinates of B' are $(1,2,3)$.	For such contextual problems, to solve for the coordinates of a point, you need to ask yourself how the point is defined. Here it is defined by the intersection of 2 lines.
(ii) **[2]**	$\overrightarrow{A'C'} = \overrightarrow{OC'} - \overrightarrow{OA'} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ Normal to the plane = $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ Equation of plane is $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = 19$ $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = 19$.	Please check that the cross product is done correctly by using the dot product: $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = 0 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ Obtaining the wrong normal vector affects all the answers in subsequent parts of this question, which is costly.

(iii) **[2]**	Area of triangle $A'B'C'$ $=\frac{1}{2}\left\lvert\overrightarrow{A'C'}\times\overrightarrow{A'B'}\right\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}3\\2\\4\end{pmatrix}\right\rvert=\frac{1}{2}\sqrt{29}$	There is no need to re-perform the cross product as it is the same normal of the plane found in part (ii).
(iv) **[4]**	Let $\overrightarrow{OC}=\alpha\begin{pmatrix}1\\4\\2\end{pmatrix}$ for some $\alpha\in\mathbb{R}$. Length of projection of $\overrightarrow{BC}$ onto the normal of the virtual screen is $\left\lvert\frac{\overrightarrow{BC}\cdot\begin{pmatrix}3\\2\\4\end{pmatrix}}{\left\lvert\begin{pmatrix}3\\2\\4\end{pmatrix}\right\rvert}\right\rvert=\frac{57}{\sqrt{29}}$ $\Rightarrow\frac{\left\lvert\left[\alpha\begin{pmatrix}1\\4\\2\end{pmatrix}-\begin{pmatrix}24\\48\\72\end{pmatrix}\right]\cdot\begin{pmatrix}3\\2\\4\end{pmatrix}\right\rvert}{\sqrt{29}}=\frac{57}{\sqrt{29}}$ $\Rightarrow\left\lvert\alpha(3+8+8)-(72+96+288)\right\rvert=57$ $\Rightarrow 19\alpha-456=\pm57$ $\Rightarrow\alpha=21\text{ or }27$ Possible position vectors of C are $21\begin{pmatrix}1\\4\\2\end{pmatrix}$ or $27\begin{pmatrix}1\\4\\2\end{pmatrix}$.	Recall that length of projection formula involves a modulus sign. In fact the question suggested that there is more than 1 position vector that satisfy the condition with the (s).

Section B: PROBABILITY AND STATISTICS (48 marks)

8	The continuous random variable Y has the distribution $\mathrm{N}\left(\mu,\sigma^2\right)$. It is known that $\mathrm{P}\left(Y<-a\right)=\mathrm{P}\left(Y>5a\right)=0.1$. Express μ and σ in the forms ka and ma respectively, where k and m are constants to be determined. [4]

Solution		**Comments**
[4]	By symmetry, $\mu=\dfrac{-a+5a}{2}=2a$ $Y\sim \mathrm{N}(2a,\sigma^2)$. $\mathrm{P}(Y<-a)=0.1 \Rightarrow \mathrm{P}\left(Z<\dfrac{-a-2a}{\sigma}\right)=0.1$ $\Rightarrow \dfrac{-3a}{\sigma}=-1.28155$ $\Rightarrow \sigma=2.3409a$ Thus, $k=2$ and $m=2.34$ (3 s.f.) OR $\mathrm{P}(Y>5a)=0.1 \Rightarrow \mathrm{P}\left(Z>\dfrac{5a-2a}{\sigma}\right)=0.1$ $\Rightarrow \dfrac{3a}{\sigma}=1.28155$ $\Rightarrow \sigma=2.3409a$ Thus, $k=2$ and $m=2.34$ (3 s.f.)	The value of k is actually exact at 2 and should be found by exploiting the symmetry of the conditions given.

9	A card game called "Happy Family" is played with 28 cards, consisting of 7 sets of 4 cards. Each set consists of a father, a mother, a son and a daughter from the same family. The family names are Painter, Postman, Plumber, Butcher, Carpenter, Singer and Teacher. So for example, the complete set of Teacher family cards consists of father Teacher, mother Teacher, son Teacher and daughter Teacher. The objective of the game is to collect as many complete sets of family cards as possible. At the end of the game, all the cards are collected. Each player will either be left with no cards or complete sets of family cards. The winner is the one with the most number of complete sets of family cards. A and B play a game of "Happy Family".	
	(a)	At the end of the game, A has exactly 3 complete sets of family cards and one of the sets is the Teacher family cards. How many possible combinations of complete sets of family cards can B have? [1]
	(b)	If A is the winner at the end of the game, how many possible combinations of complete sets of family cards can A have? [2]
	(c)	Given that A has exactly 3 particular complete sets of family cards at the end of the game, and he arranges these 12 cards in a circle. How many different ways can these cards be arranged so that no two mothers are next to each other? [3]

	Solution	**Comments**
(a) [1]	Number of possible combinations $= {}^6C_4 = 15$	
(b) [2]	If A is the winner, then A can either have 4, 5, 6 or 7 complete sets. Number of possible combinations $= {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7 = 64$ **OR** **<u>Complement Method</u>** Number of possible combinations is $= 2^7 - \left({}^7C_0 + {}^7C_1 + {}^7C_2 + {}^7C_3\right) = 64$ **OR (Bijection)** In any situation, either A wins or B wins. For every game that A wins, B loses, and for every game that B wins, A loses. Hence the number of possible combinations is $= \frac{1}{2}(2^7) = 64$	

<table>
<tr><td>(c)
[3]</td><td>Excluding the mothers, we have 9 other cards to arrange in a circle which can be done in $(9-1)!$ ways.

There are 9 “slots” for the mothers, which can be performed in $^9C_3 \times 3!$ or $^9P_3$ ways.

Number of ways = $8! \times {}^9P_3 = 20321280$</td><td>When there are more than 2 objects that needs to be separated, it is much easier to do the slotting method.</td></tr>
<tr><td></td><td>Alternative: Complement method
(STRONGLY NOT ENCOURAGED for this situation)

Number of ways =
$11! - \underbrace{9!3!}_{\substack{\text{all 3 mothers together} \\ \text{9 cards + 1 group of 3 mothers} \\ \text{= 10 units}}} - \underbrace{3(9!2!)(8)}_{\substack{\text{2 mothers together, 1 mother separate (choose in 3 ways)} \\ \text{9 cards + 1 group of 2 mothers = 10 units} \\ \text{remaining mother slot in 8 possible ways to ensure separate}}}$</td><td>For students who attempt the complement method, the difficult case to handle is the one where 2 mother cards are next to each other, and the remaining one is separate from them. You are strongly discouraged from using this method.</td></tr>
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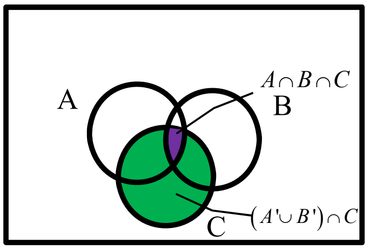
10	For events A and B, it is given that $\mathrm{P}(A)=0.38$, $\mathrm{P}(A \mid B)=0.5$ and $\mathrm{P}(A \cup B)=0.52$.	
	(i) Find $\mathrm{P}(B)$.	[3]
	For a third event C, it is given that $\mathrm{P}(C)=0.6$ and $\mathrm{P}\left(\left(A' \cup B'\right) \mid C\right)=0.85$.	
	(ii) Find $\mathrm{P}(A \cap B \cap C)$.	[3]
	(iii) State, with a reason, whether the events A and C are mutually exclusive.	[2]

	Solution	**Comments**
(i) **[3]**	$\mathrm{P}(A \mid B)=\dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}=0.5$ $\mathrm{P}(A \cap B)=0.5\mathrm{P}(B)$ $\mathrm{P}(A \cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A \cap B)$ $0.52=0.38+\mathrm{P}(B)-0.5\mathrm{P}(B)$ $\mathrm{P}(B)=0.28$	Make full use of the conditions of the problem given (and the appropriate definition of conditional probability) to form simultaneous equations to solve for $\mathrm{P}(B)$
(ii) **[3]**	$\mathrm{P}((A' \cup B') \mid C)$ $=\dfrac{\mathrm{P}\left(\left(A' \cup B'\right) \cap C\right)}{\mathrm{P}(C)}=0.85$ $\mathrm{P}((A' \cup B') \cap C)=0.85 \times 0.6=0.51$ $\mathrm{P}(A \cap B \cap C)=\mathrm{P}(C)-\mathrm{P}((A' \cup B') \cap C)$ $=0.6-0.51$ $=0.09$  Alternative: $\mathrm{P}((A' \cup B') \mid C)=1-P(A \cap B \mid C)=0.85$ $\Rightarrow P(A \cap B \mid C)=0.15$ $P(A \cap B \cap C)=0.15 \times 0.6=0.09$	Most students can use the definition of the conditional probability, but cannot make sense of the event $\left(A' \cup B'\right) \cap C$. These students are strongly encouraged to draw the Venn diagram to interpret $\left(A' \cup B'\right) \cap C$.
(iii) **[2]**	If A and C are mutually exclusive, then $A \cap C=\varnothing$ Since $A \cap B \cap C \subseteq A \cap C$, then $A \cap C=\varnothing \Rightarrow A \cap B \cap C=\varnothing \Rightarrow \mathrm{P}(A \cap B \cap C)=\mathrm{P}(\varnothing)=0$ But from part (iii), $\mathrm{P}(A \cap B \cap C)=0.09 \neq 0$, so $A \cap C \neq \varnothing$ Therefore A and C are not mutually exclusive. **OR** Since $A \cap B \cap C \subseteq A \cap C$, $\mathrm{P}(A \cap C) \geq \mathrm{P}(A \cap B \cap C)=0.09>0$, A and C are not mutually exclusive.	It is important to link the definition of A and C being mutually exclusive to what is found in part (ii).

11	In a game, a player tosses a fair die, whose faces are numbered from 1 to 6. If the player obtains a 6, he tosses the die a second time, and in this case, his score is the absolute difference of 6 and the second number. Otherwise, his score is the number obtained in the first toss. Let the player's score be denoted by X.	
	(i)	Show that $\mathrm{P}(X=1)=\frac{7}{36}$ and tabulate the probability distribution of X. [3]
	(ii)	Find the exact value of E(X). [1]
	Mr Lim plays this game 4 times.	
	(iii)	Find the probability that he obtained a score of 5 no more than 2 times. [2]
	(iv)	Find the probability that his total score is less than 2 in the 4 games that he played. [2]

	Solution	**Comments**
(i) **[3]**	$\mathrm{P}(X=1)=$ P(first throw = 1) + P(1st throw = 6, 2nd throw = 5) $=\frac{1}{6}+\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$ $=\frac{1}{6}+\frac{1}{36}$ $=\frac{7}{36}$ (Shown) Probability distribution of X x: 0, 1, 2, 3, 4, 5 $\mathrm{P}(X=x)$: $\frac{1}{36}$, $\frac{7}{36}$, $\frac{7}{36}$, $\frac{7}{36}$, $\frac{7}{36}$, $\frac{7}{36}$	Do check that the table is correct by using $\sum \mathrm{P}(X=x)=1$
(ii) **[1]**	E(X) = $\frac{7}{36}(1+2+3+4+5)=\frac{105}{36}=\frac{35}{12}$.	The question said exact value and thus 3 s.f. answers are not accepted.
(iii) **[2]**	Let the random variable Y denote the number of times that Mr Lim obtained a score of 5 when he played the game four times. $Y \sim \mathrm{B}\left(4,\frac{7}{36}\right)$ $\mathrm{P}(Y\le 2)=0.975$. (3 s.f.) **OR** (Without quoting Binomial Distribution) $\mathrm{P}(Y\le 2)=\left(\frac{29}{36}\right)^4+{}^4C_1\left(\frac{7}{36}\right)^1\left(\frac{29}{36}\right)^3+{}^4C_2\left(\frac{7}{36}\right)^2\left(\frac{29}{36}\right)^2$ $=0.975$ (3 s.f.)	

x	0	1	2	3	4	5
$\mathrm{P}(X=x)$	$\frac{1}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$

(iv) **[2]**	$\mathrm{P}(\text{total score less than 2})$ $= \mathrm{P}(\text{total score} = 0) + \mathrm{P}(\text{total score} = 1)$ $= \mathrm{P}(\text{scored 0 for all 4 games})$ $+ \mathrm{P}(\text{scored 1 for a game and 0 for remaining 3 games})$ $= \left(\frac{1}{36}\right)^4 + {}^4C_1 \left(\frac{1}{36}\right)^3 \left(\frac{7}{36}\right)$ $= \frac{29}{1679616}$ or 0.0000173 (3 s.f.)	Do remember to include the factor of 4 in the case where the total score is 1, since out of the 4 games, any of the 4 could be the game with score of 1.

12	A chocolate manufacturing company produces chocolate bars. The packaging of the chocolate bar states that the mass of each bar is 200 g. The production manager has been receiving complaints that the mass is overstated and he wants to carry out a hypothesis test.	
	(i)	Explain whether the manager should carry out a 1-tail test or a 2-tail test. State appropriate hypotheses for the test, defining any symbols you use. [2]
	The masses, x grams, of a random sample of 30 chocolate bars are summarised as follows. $n=30$ $\sum(x-200)=-54$ $\sum(x-200)^2=550$	
	(ii)	Calculate unbiased estimates of the population mean and variance of the mass of chocolate bars. [2]
	(iii)	Carry out this test, at the 1% level of significance. [2]
	The company replaces all the machines producing the chocolate bars. The supplier of the machines claims that the mass of each chocolate bar will have a normal distribution with mean 200 grams and standard deviation 2 grams.	
	(iv)	The manager takes a random sample of n chocolate bars, and the sample mean mass is found to be 199.2 grams. A test is carried out, at the 5% level of significance, to determine whether the mean mass of the chocolate bars is indeed 200 grams. Given that the null hypothesis is not rejected, find the maximum value of n. [4]

Solution		**Comments**
(i) **[2]**	Since the complaint is about the mass being overstated (that is, whether the actual mean mass is less than 200), he should carry out a one-tail test. Null hypothesis, $H_0 : \mu = 200$ Alternative hypothesis, $H_1 : \mu < 200$ where μ is the population mean mass of chocolate bars.	The definition of μ has to be accompanied with the word **population**
(ii) **[2]**	Unbiased estimate of population mean is $\bar{x} = \frac{-54}{30} + 200 = 198.2$ Unbiased estimate of population variance is $s^2 = \frac{1}{29}\left[550 - \frac{(-54)^2}{30}\right] = 15.614 = 15.6 \text{ (3 s.f.)}$	

(iii) **[2]**	$H_0 : \mu = 200$	Do use the at least 5 s.f. intermediate

	$H_1: \mu < 200$ Perform a one-tail test at 1% significance level. Under H0, since $n = 30$ is large, by Central Limit Theorem, $\bar{X} \sim N\left(200, \frac{15.614}{30}\right)$ approximately. Using a $z-$test, p-value $= P(\bar{X} < 198.2) \approx 0.0062973 < 0.01$ Since p-value $\approx 0.0062973 < 0.01$, we reject H_0 and conclude that there is sufficient evidence, at 1% level of significance, that the mass has been overstated.	answer for the unbiased estimate of population variance. Failure to do so will result in an inaccurate p-value. As the distribution of the masses is not stated in the question and sample size, 30, is sufficiently large, application of the Central Limit Theorem is necessary.
(iv) [4]	$H_0: \mu = 200$ $H_1: \mu \neq 200$ Perform a two-tail test at 5% significance level. Let Y be the new mass of a chocolate bar after the machines are replaced. Under H0, $Y \sim N(200, 4) \Rightarrow \bar{Y} \sim N\left(200, \frac{4}{n}\right)$. Given that H0 is not rejected, p-value $= 2P\left(\bar{Y} < 199.2\right) > 0.05$ $P\left(Z < \frac{199.2 - 200}{\sqrt{4/n}}\right) > 0.025$ $-0.4\sqrt{n} > -1.95996$ $0 < n < 24.009$ Maximum value of $n = 24$ Alternatively, by GC table of values,	The question now states that the manufacturer claims the distribution of mass of a chocolate bar is normal with given mean and standard deviation. Hence Central Limit theorem is not required in part (iv).

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to solve $2\mathrm{P}\left(\overline{Y} < 199.2\right) > 0.05$ which is equivalent to

$\mathrm{P}\left(\overline{Y} < 199.2\right) > 0.025$:

n	$\mathrm{P}\left(\overline{Y} < 199.2\right)$
24	0.025022 > 0.025
25	0.022750 < 0.025

Maximum value of $n = 24$

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<td>As the value of $\mathrm{P}\left(\overline{Y} < 199.2\right)$ at $n = 24$ is very close to 0.025, it is essential to provide this value to at least 4 or 5 s.f.</td>
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13	**In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

A cup of cheese tea is prepared with two ingredients, cream cheese foam and tea. The barista preparing the cheese tea is not allowed to remove cream cheese foam nor tea once they have been poured into the cup. The amounts, in ml, of cream cheese foam and tea in one cup of cheese tea made by an experienced barista have independent normal distributions with means and standard deviations as shown in the table.

	Mean amount (ml)	Standard deviation (ml)
Cream cheese foam	72	5
Tea	421	4

(i) Given that the amount of cheese tea in each cup is expected to be at least 490 ml, find the probability that a randomly chosen cup of cheese tea made by the experienced barista meets this expectation. [2]

A cup of cheese tea is considered well-made if the amount of cream cheese foam is between 65 ml and 75 ml and the amount of tea is between 416 ml and 424 ml.

(ii) Find the probability that, for ten randomly chosen cups of cheese tea made by the experienced barista, more than five cups are well-made. [3]

The cost price of cream cheese foam and tea is \$20 per litre and \$1.20 per litre respectively. A cup of cheese tea is sold at \$4.90.

(iii) Without factoring in other costs incurred in producing the cheese tea, find the probability that the profit for 100 randomly chosen cups of cheese tea made by the experienced barista is at least \$295. State any assumption(s) needed for your calculation. [5]

A trainee barista was engaged at the shop. The amount of cheese tea in a cup she made was measured to have mean 498 ml and standard deviation 30 ml. The amount of cheese tea in any cup may be assumed to be independent of the amount of cheese tea in any other cup.

(iv) Find the probability that the mean amount of cheese tea of 40 randomly chosen cups of cheese tea made by the trainee barista does not exceed 500 ml. [2]

Solution	**Comments**

(i) **[2]**	Let X and Y be the amount of cream cheese foam and tea in one cup of cheese tea, in ml, respectively. Then $X \sim \mathrm{N}\left(72, 5^2\right)$ and $Y \sim \mathrm{N}\left(421, 4^2\right)$ $X+Y \sim \mathrm{N}\left(72+421, 5^2+4^2\right)$ i.e $X+Y \sim \mathrm{N}(493, 41)$ $\mathrm{P}(X+Y \geq 490)=$ 0.680 (3 s.f)	It is important to define correctly and clearly the random variables.
(ii) **[3]**	P (a cup of cheese tea is well-made) $=\mathrm{P}(65 \leq X \leq 75) \mathrm{P}(416 \leq Y \leq 424)$ $=0.43067$ (5 s.f) Let W be the number of cups of cheese tea, out of 10, that are well-made. Then $W \sim \mathrm{B}(10, 0.43067)$ $\mathrm{P}(W>5)=1-\mathrm{P}(W \leq 5)$ $=0.222$ (3 s.f)	Do keep at least 5 s.f. for intermediate working. Failure to do so will result in an inaccurate final answer which will be penalized. Note that since the question asked for more than 5 cups, the complement should include "5".
(iii) **[5]**	Let C be the cost price of 100 cups of cheese tea. Then $C=\left(\frac{20}{1000}\right)\left[X_1+X_2+\cdots+X_{100}\right]+\left(\frac{1.2}{1000}\right)\left[Y_1+Y_2+\cdots+Y_{100}\right]$ $C \sim \mathrm{N}\left(100\left(\frac{20}{1000}\right) 72+100\left(\frac{1.2}{1000}\right) 421,\ 100\left(\frac{20}{1000}\right)^2 5^2+100\left(\frac{1.2}{1000}\right)^2 4^2\right)$ $C \sim \mathrm{N}(194.52, 1.002304)$ P(profit for 100 cups is at least \$295) $=\mathrm{P}(C<490-295)$ $=\mathrm{P}(C<195)$ $=0.684$ (3 s.f) **OR** Let D be the cost price of one cup of cheese tea.	Do be careful in working out the parameters of the distribution of the 100 cups. A common mistake is to have 100^2 in the variance, which <u>should not</u> happen.

	Then $D=\left(\frac{20}{1000}\right)X+\left(\frac{1.2}{1000}\right)Y$ $D\sim \mathrm{N}\left(\left(\frac{20}{1000}\right)72+\left(\frac{1.2}{1000}\right)421,\ \left(\frac{20}{1000}\right)^2 5^2+\left(\frac{1.2}{1000}\right)^2 4^2\right)$ $D\sim \mathrm{N}(1.9452, 0.01002304)$ Then average cost of 1 cup in a sample of 100 cups is $\bar{D}$, where $\bar{D}\sim \mathrm{N}\left(1.9452, \frac{0.01002304}{100}\right)$ P(profit for 100 cups is at least \$295) $=\mathrm{P}(D_1+D_2+\cdots+D_{100}<195)$ $=\mathrm{P}\left(\frac{D_1+D_2+\cdots+D_{100}}{100}<1.95\right)$ $=\mathrm{P}(\bar{D}<1.95)$ $=0.684$ (3 s.f) We assume that the amount of cream cheese foam in any cup is independent of the amount of cream cheese foam in any other cup. We also assume that the amount of tea in any cup is independent of the amount of tea in any other cup.	Note that total cost of 100 cups of cheese tea is $\sum_{i=1}^{100} D_i$, not $100D$. The assumption needs to be clear that it is the amounts or volumes of cream cheese foam AND tea in all cups are independent. For those who mentioned the price, it should be the cost prices that are independent, as the selling price of a cup of cheese tea is fixed. It is already mentioned in the question that X and Y are independent.
(iv) **[2]**	Let T be the amount of cheese tea in one cup, in ml. Given $\mathrm{E}(T)=498$, $\mathrm{Var}(T)=30^2$ Since $n=40$ is large, by Central Limit Theorem,	The question did not say that the distribution of the amount of cheese tea made by the

	$\overline{T} \sim N\left(498, \frac{30^2}{40}\right)$ approximately $P\left(\overline{T} < 500\right) = 0.663$ (3 s.f)	trainee barista is normally distributed. Hence, since sample size, 40, is sufficiently large as well, Central Limit Theorem is applied.