NANYANG JUNIOR COLLEGE

JC1 End of Year Examination

Higher 2

FURTHER MATHEMATICS **9649/01**

Paper 1 **29th September 2023**

3 Hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

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1 A $n\times n$ matrix $\mathbf{A}$ is said to be idempotent if $\mathbf{A}^2=\mathbf{A}$.

(a) Show that $\det(\mathbf{A})$ is 0 or 1. [2]

(b) Show that if $\det(\mathbf{A})=1$, then $\mathbf{A}=\mathbf{I}$. [1]

Let $\mathbf{A}=\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ be a 2×2 idempotent matrix where $w\neq 0$, $z\neq 0$.

(c) If $\det(\mathbf{A})=0$, find the value of $w+z$. [2]

2 Let $\mathrm{f}(x)=\sin 2x$. Prove by mathematical induction that $\dfrac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(\mathrm{f}(x)\right)=(-4)^n\sin 2x$ for $n\in\mathbb{Z}^+$. [5]

Hence, explain why the Maclaurin expansion of f(x) has only odd powers of x. [2]

3 **(a)** The curve C is defined parametrically by

$$x=a\left(\cos t+\ln\tan\frac{t}{2}\right),\ y=a\sin t \quad \text{where} \quad \frac{\pi}{4}\leq t<\frac{\pi}{3}.$$

and a is a positive constant.

Find the exact length of the arc of C, in terms of a. [4]

(b) Let $I_n=\displaystyle\int_0^1 \frac{x^n}{\sqrt{4-x^2}}\,\mathrm{d}x$, where n is a non-negative integer. Prove that for $n\geq 2$,

$$nI_n=4(n-1)I_{n-2}-\sqrt{3}.$$ [3]

Let R be the region bounded by the curve $y=\dfrac{x^3}{\sqrt{4-x^2}}$, the line $y=\dfrac{1}{\sqrt{3}}$ and the y-axis.

Find

(i) the exact area of R, [3]

(ii) the exact volume of the solid obtained when R is rotated 2π radians about the y-axis. [4]

4 **(a)** The sequence of positive numbers u_n for $n = 1, 2, 3, \ldots$ satisfy the recurrence relation

$$u_{n+1} = \frac{ku_n + a}{u_n + k}\text{, where } a \text{ and } k \text{ are positive real constants.}$$

(i) Given that the sequence converges, find its limit in terms of a. [2]

(ii) If $u_n > \sqrt{a}$, by considering $u_{n+1} - u_n$, show that $u_{n+1} < u_n$. [2]

(b) Two sequences $\{x_n\}$ and $\{y_n\}$ are related by

$$x_n = 0.8x_{n-1} + 0.1y_{n-1} \qquad \text{and} \qquad y_n = 0.2x_{n-1} + 0.9y_{n-1}$$

For $n \geq 1$.

(i) Given that $x_n + y_n = 1$, show that $x_n = 0.7x_{n-1} + 0.1$. [2]

(ii) Given that $x_1 = 0.9$, solve the recurrence relation in **(i)** to find x_n and y_n in terms of n. [4]

5 Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that

$$T\left(\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}4\\13\\9\\10\end{pmatrix},\quad T\left(\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right) = \begin{pmatrix}1\\5\\4\\6\end{pmatrix},\quad T\left(\begin{pmatrix}1\\1\\0\\3\end{pmatrix}\right) = \begin{pmatrix}6\\13\\7\\7\end{pmatrix} \text{ and } T\left(\begin{pmatrix}1\\0\\2\\4\end{pmatrix}\right) = \begin{pmatrix}11\\27\\16\\17\end{pmatrix}.$$

(a) Show that $\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\0\\2\\4\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^4$. [3]

Let **M** be the matrix representation of T.

(b) Write down the matrix **B** such that $\mathbf{M}\begin{pmatrix}1 & 1 & 1 & 1\\0 & -1 & 1 & 0\\1 & 1 & 0 & 2\\0 & -1 & 3 & 4\end{pmatrix} = \mathbf{B}$. Hence, with the aid of the graphing calculator, find **M**. [2]

(c) Find a basis for W, the range space of T. Hence, state the dimension of the kernel of T. [3]

(d) Given that $\begin{pmatrix}0\\b\\c\\d\end{pmatrix}$ belongs to the range space of T, find the conditions on b, c and d. [2]

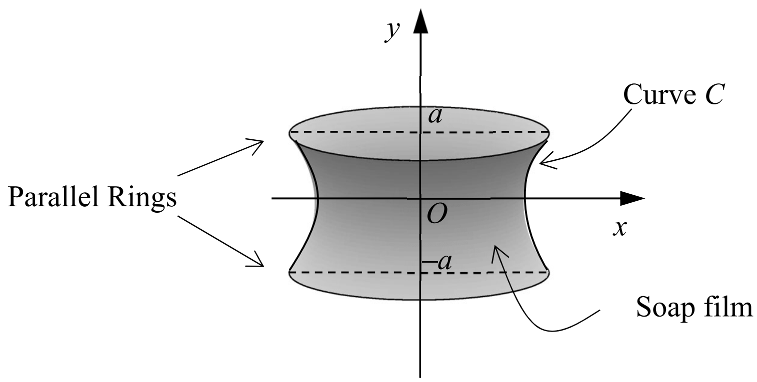
6 **(a)** Use the substitution $t=\tan\left(\frac{x}{2}\right)$ to find $\int \frac{1}{3-2\sin x+\cos x}\,dx$. [4]

(b) It is given that $f(x)=\cos 3mx+\sin 3nx$, where m and n are integers and $m>n>0$.

(i) Find $\int \cos 3mx \sin 3nx \, dx$ in terms of m and n. [2]

(ii) Find $\int_0^{\pi} \left(f(x)\right)^2 dx$ where m is even and n is odd, giving your answer in non-trigonometric form, in terms of m, n and π. [3]

7 When a frame consisting of two parallel rings with the same radius is dipped into soap solution, a soap film is formed *between* the rings. Coordinate axes have been superimposed so that one could model the soap film as the surface of revolution of a curve C around the y-axis (see figure).



The equation of curve C is given by

$$x=\frac{k}{2}\left(e^{\frac{y}{k}}+e^{-\frac{y}{k}}\right),\ -a\le y\le a \text{ , where } k \text{ is a positive constant.}$$

(a) Let S be the curved surface area of the soap film. Show that $S=\frac{k\pi}{2}\left(ke^{\frac{2a}{k}}+4a-ke^{-\frac{2a}{k}}\right)$. [5]

(b) Let A be the curved surface area of a cylinder with height $2a$ and the same radius as the rings. Determine, to 3 decimal places, the value of $\frac{a}{k}$ such that $S=2A$. [5]

(c) Let V be the volume of the region bounded by the soap film and the planes of the rings. Show that $V=\frac{1}{2}kS$. [3]

8 The matrices **A** and **B** are such that $\mathbf{A}=\begin{pmatrix} k & 0 & 1 \\ 2 & 0 & k \\ 1 & 2 & k \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} 0 & k & 1 \\ 2 & k & 0 \\ 1 & 2 & k \end{pmatrix}$.

(a) Find the exact values of k such that $\mathbf{ABx}=\mathbf{0}$ has non trivial solution only. [5]

(b) **(i)** Given that the linear transformation $T(\mathbf{x})=\mathbf{Ax}$ has an invariant point at (1, 0, –1), determine the value of k. [1]

(ii) Using the value of k found in **(b)(i)**, find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}=\mathbf{P}^{-1}\mathbf{DP}$. [6]

9 **(a)** Two vectors in $\mathbb{R}^3$ are said to be orthogonal if their dot product is zero. Let V be a subspace of $\mathbb{R}^3$. The orthogonal complement of V, is defined as

$$W=\left\{\mathbf{w}\in\mathbb{R}^3:\mathbf{w}\cdot\mathbf{v}=0 \text{ for every } \mathbf{v}\in V\right\}.$$

Show that W is a subspace of $\mathbb{R}^3$. [3]

(b) It is of interest during war to devise a plan for military assets to get as close as possible to the enemy command centre without being detected by the enemy's detection system. A new detection system under testing follows the model $\mathbf{Au}=\mathbf{b}$, where $\mathbf{A}=\begin{pmatrix} 1 & 2 & 4 \\ 5 & 10 & 20 \\ -2 & -4 & -8 \end{pmatrix}$ is the detection matrix constructed with reference to the terrain and weather, and $\mathbf{u}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the position vector of the location of the military asset relative to the enemy command centre.

(i) Find a basis for the null space of $\mathbf{A}$. [2]

(ii) The military asset is at a blind spot and is not detected by the detection system if $\mathbf{b}=\mathbf{0}$. Give a geometrical interpretation of the blind spots of the detection system. [1]

(iii) Give a geometrical interpretation of the orthogonal complement of the blind spots of the detection system. [1]

10 In the country of Everland, the production of Imperial carrots, in hundreds, is denoted by A_n and is defined by

$$A_1 = 5, \ A_2 = 8 \quad \text{and} \quad A_{n+1} = 3A_n + 2A_{n-1} - 8 \ \text{ for } n \geq 2.$$

(a) Determine the solution for A_n, giving A_n in the form of $p(k)^n + q(m)^n + 2$ where p, k, q and m are exact constants to be determined. [7]

However in the midst of farming, these carrots are devoured by wild rabbits. Peter hypothesized the number of wild rabbits is growing with a recurrence relation, B_n, in hundreds, in year n, taking year 2023 as year 1. The model proposed is given by

$$3B_n = 2A_n - 1, \ \text{ for } \ n \geq 1.$$

(b) **(i)** Calculate the values of B_n for $n = 1, \ 2, \ 3, 4$ and 5. [2]

(ii) Peter discovers that the sequence B_n follows a non-homogeneous second order recurrence relation. Find this recurrence relation. [2]

(c) Find the year when there are more than 2000 wild rabbits for the first time. [2]

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