ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

/100

CANDIDATE NAME

TUTORIAL/ FORM CLASS

INDEX NUMBER

MATHEMATICS 9758/01

Paper 1

21 August 2023

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on all the work you hand in.
Write in dark blue or black pen.

You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Question	Marks
1	/3
2	/4
3	/5
4	/6
5	/7
6	/8
7	/9
8	/10
9	/11
10	/12
11	/12
12	/13

This document consists of **27** printed pages and **1** blank page.

Anglo-Chinese Junior College

1 A curve has equation $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=x^3+ax^2+bx+c$. The curve $y=\mathrm{f}(x)$ has a maximum point at $(1,-32)$, while the curve $y=\dfrac{1}{\mathrm{f}(x)}$ has a vertical asymptote with equation $x=5$. Find the equation of the curve $y=\mathrm{f}(x)$. [3]

2 **Do not use a calculator in answering this question.**

The complex numbers z and w satisfy the following equations.

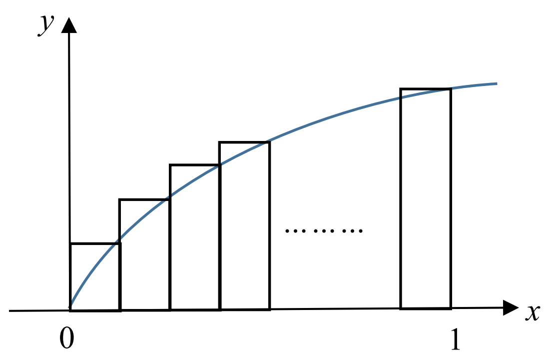
$$w+z^*=-2+4\mathrm{i}$$
$$z+2=3\mathrm{i}w$$

Find z and w, giving your answers in the form $a+\mathrm{i}b$, where a and b are real numbers. [4]

3 Let $\mathrm{f}(x)=\dfrac{x}{\sqrt{x+1}}$.

(i) Show that $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x=\dfrac{2}{3}\left(2-\sqrt{2}\right)$. [2]

(ii) The diagram below shows a sketch of the curve with equation $y=\mathrm{f}(x)$. Rectangles each of width $\dfrac{1}{n}$ are drawn as shown.



By considering the total area of these n rectangles, deduce that

$$\frac{1}{n\sqrt{n}}\left\{\frac{1}{\sqrt{n+1}}+\frac{2}{\sqrt{n+2}}+\frac{3}{\sqrt{n+3}}+\cdots+\frac{n}{\sqrt{2n}}\right\}>\frac{2}{3}\left(2-\sqrt{2}\right).$$ [3]

4 The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively, with respect to the origin O. The vector $\mathbf{p}$ is a unit vector and the vectors $\mathbf{q}$ and $\mathbf{r}$ are non-zero vectors.

(i) Write down the geometrical meaning of $|\mathbf{p}\cdot\mathbf{q}|$. [1]

It is given that $\mathbf{p}\cdot\mathbf{q}=2$ and that the point M is the foot of the perpendicular from Q to the line OP.

(ii) Show that $\overrightarrow{OM}=2\mathbf{p}$. [1]

It is further given that $\mathbf{r}=k(\mathbf{p}-\mathbf{q})$ for an arbitrary constant k.

(iii) Show that the area of triangle RQM is $a|\mathbf{p}\times\mathbf{q}|$ units^2, where a is a constant to be determined in terms of k. Hence describe the geometrical relationship of points R, Q and M when $k=-2$. [4]

5 **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x^2}$. Hence find $\int x^3\,\mathrm{e}^{-x^2}\,\mathrm{d}x$. [3]

(ii) By using the substitution $z=\mathrm{e}^{-x^2}y$, find the general solution of $\dfrac{\mathrm{d}y}{\mathrm{d}x}-2xy=x^3$, expressing y in terms of x. [4]

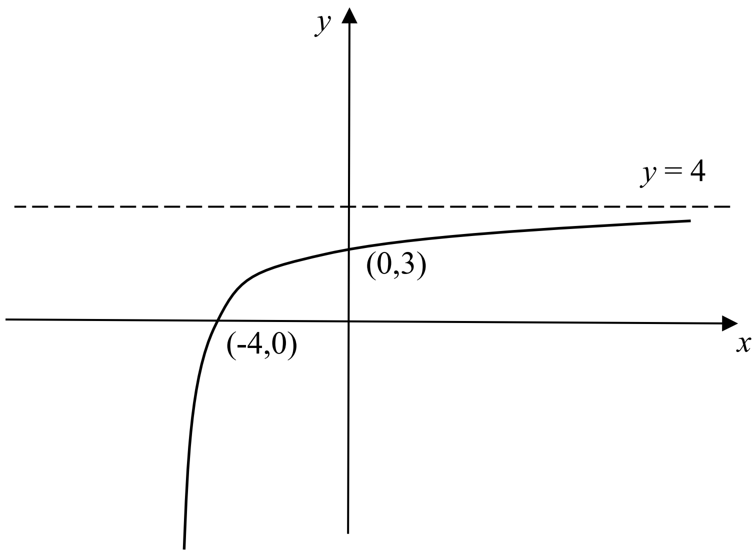
6 Two of the roots of the equation $z^4-kz^3+k^3z-k^4=0$, where k is a positive real number, are $z_1=k$ and $z_2=-k$. The other two roots are denoted z_3 and z_4, where $\mathrm{Im}(z_3)>0$ and $\mathrm{Im}(z_4)<0$.

(i) Show that $z_3=\dfrac{k}{2}+\dfrac{\sqrt{3}k}{2}\mathrm{i}$. [3]

(ii) Find $|z_3|$ and $\arg z_3$, giving your answer in terms of k if necessary. [2]

(iii) Find the two smallest positive integer values of n for which $\left(\dfrac{z_3}{-1+\mathrm{i}}\right)^n$ is purely imaginary. [3]

7 The graph below shows the graph of $y = \mathrm{f}(x)$.



(i) Sketch on the same diagram the graph of $y = \mathrm{f}^{-1}(x)$, clearly stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. State also the geometrical relationship between these two graphs. [3]

The function g is defined by $\mathrm{g} : x \mapsto x^2 + 6x + 5,\ x \in \mathbb{R}$.

(ii) Find the range of the composite function fg. [2]

(iii) It is given that $\mathrm{fg}(a) = 0$. Find the value of $\mathrm{g}(a)$, where a is a negative constant. [1]

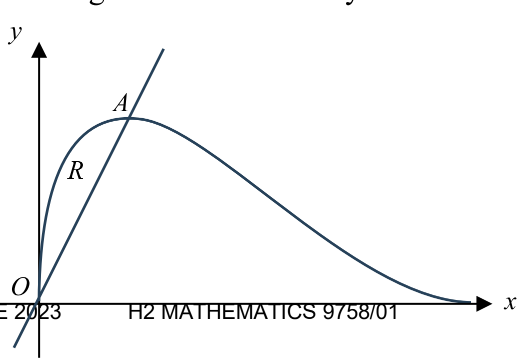
The function h is defined by $\mathrm{h} : x \mapsto x^2 + 6x + 5,\ x \in \mathbb{R},\ x < -4$.

(iv) Find h^{-1} and state the domain of h^{-1}. [3]

8 The curve C is defined by the parametric equations

$$x = t - \sin t,\ y = \sin^2 t, \quad 0 \le t \le \pi.$$

The point A on C has parameter $\frac{\pi}{2}$ and is a maximum point. The line l passes through the origin and the point A. The region R is bounded by C and l as shown in the diagram below.



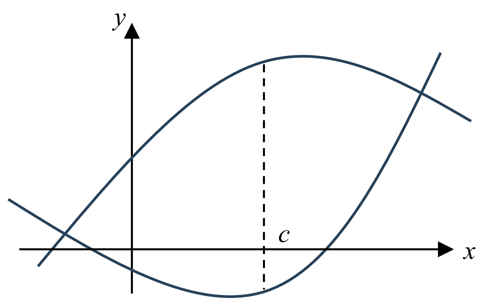
(i) Find the coordinates of A. [1]

(ii) Using an algebraic method, find the exact area of R. [5]

(iii) Show that the Cartesian equation of C is given by $x = \sin^{-1}\sqrt{y} - \sqrt{y}$. [1]

(iv) Find the volume of the solid formed when R is rotated by 2π radians about the y-axis. [3]

9 **(a)** The following diagram shows the graphs of two differentiable functions $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$. At $x = c$, the distance between the two functions is a maximum. Show that $\mathrm{f}'(c) = \mathrm{g}'(c)$ where $\mathrm{f}'(x)$ and $\mathrm{g}'(x)$ are derivatives of $\mathrm{f}(x)$ and $\mathrm{g}(x)$ respectively. Using the second derivative, explain clearly why the distance is a maximum. [3]



(b) Two runners Andy and Ben, with different strategies and abilities, participate in a 10000 m race. The distance in metres ran in x minutes by Andy and Ben are given by the functions A(x) and B(x) respectively as defined below.

$$\mathrm{A}(x) = \begin{cases} -0.16x^3 + 12x^2, & 0 \le x \le 50 \\ 10000, & x > 50 \end{cases}.$$

$$\mathrm{B}(x) = 5000\log_3(x+9) - 10000, \ 0 \le x \le 72.$$

(i) Sketch the graphs of $y = \mathrm{A}(x)$ and $y = \mathrm{B}(x)$ on the same diagram for $0 \le x \le 72$. [2]

(ii) Andy and Ben are furthest apart at $x = c$. By using **(a)** or otherwise, show that $x = c$ satisfies the equation $Px^2 + Qx = \dfrac{R}{x+9}$, where P, Q and R are constants to be determined, and find the value(s) of c. [4]

(iii) Hence or otherwise, find the furthest distance between Andy and Ben and state who was leading at that moment. [2]

10 It is given that the curve C has equation $y = \dfrac{-4x^2 - 6x - 11}{2x+2}, x \in \mathbb{R}, x \neq -1$.

(i) Without using a calculator, find the set of values that y cannot take. [3]

(ii) Sketch C, clearly stating the equations of any asymptotes, the coordinates of the stationary points and the point(s) where the curve crosses the axes. [3]

(iii) Deduce the range of values of a such that the equation

$$(x+1)^2(2x+2)^2 - a^2(-4x^2-8x-13)^2 = a^2(2x+2)^2$$

has no root, where a is a positive constant. [3]

(iv) Describe fully a sequence of transformations which transform the curve $y = 2x + \dfrac{9}{2x}$ onto the curve $y = \dfrac{-4x^2 - 6x - 11}{2x+2}$. [3]

11 A famous artist wants to create an artwork that uses squares only. On the first day, he draws the perimeter of a square with length x m as shown in Figure 1. On the second day, he divides the square into 9 equal squares and draws the perimeter of the middle square as shown in Figure 2. On the third day, he divides the remaining 8 empty squares from Day 2 into 9 equal squares, and draws the perimeters of the middle squares as shown in Figure 3. This procedure of "drawing the perimeters of the middle squares" is then repeated for subsequent days (see Figure 4 for his finished effort on Day 4).

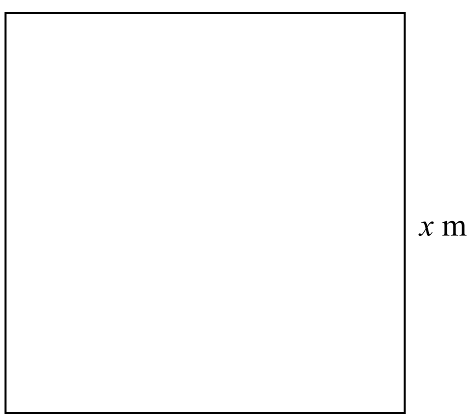


Figure 1 (Day 1)

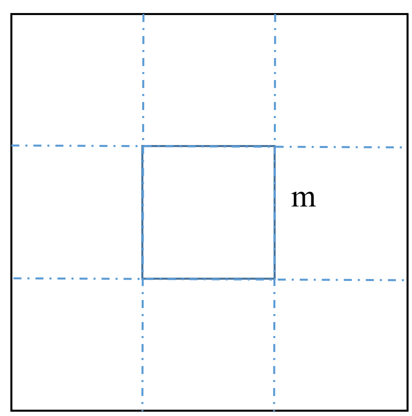


Figure 2 (Day 2)

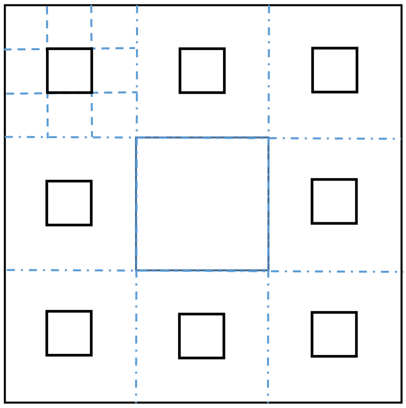


Figure 3 (Day 3)

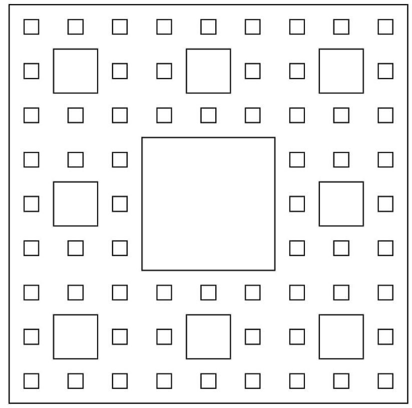


Figure 4 (Day 4)

(i) Write down the number of new squares the artist draws on Day 3 and Day 4.

Hence write down the number of new squares the artist draws on Day N $(N \geq 2)$. [2]

(ii) Write down the length of one side of a new square the artist draws on Day 3 and Day 4.

Hence write down the length of one side of a new square the artist draws on Day N $(N \geq 2)$. [2]

(iii) Show that the total perimeter of new squares the artist draws on Day N $(N \geq 2)$ is $\frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$. [1]

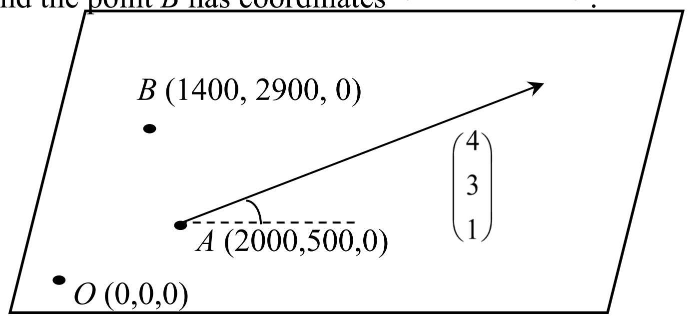
(iv) Hence find the total perimeter of all the squares the artist would have drawn by the end of the N-th day $(N \geq 2)$. [3]

(v) A company wants to support the artist. It decides to pledge \$10 for the drawing of the first square. For each subsequent square the artist draws, the company will pledge \$1 more than what was pledged for the previous square. (For the second square drawn, the artist will receive \$11. For the third square drawn, he will receive \$12, and so on.) The artist decides to stop drawing after he completes all the squares on Day 6.

A director of the company hears about the pledge, and he decides to personally pay for the 1st, 4th, 7th,10th, … square that the artist draws. Find the amount of money the director will personally pay. [4]

12 An airfield on a horizontal flat piece of ground must manage the take-off flight paths of aircraft from its runways. These flight paths may be assumed to be straight lines, and the aircraft may be assumed to be point objects.

In the diagram below, the labelled points are defined relative to an origin O with coordinates $(0,0,0)$ on the airfield. The x-, y- and z- axes are in the directions east, north and vertically upwards respectively, with units in metres. The direction vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ correspond to the east, north and vertically upwards directions respectively. The point A has coordinates $(2000,500,0)$ and the point B has coordinates $(1400,2900,0)$.



Aircraft Alpha takes off from Point A in the direction $4\mathbf{i}+3\mathbf{j}+\mathbf{k}$ as shown.

(i) Find the initial take-off angle θ, the angle that Aircraft Alpha's flight path makes with the horizontal ground. [2]

The peak of a majestic mountain has coordinates $(18000,4500,2000)$.

(ii) Find the coordinates of the closest point on the flight path of Aircraft Alpha to the peak. Hence find the closest distance between the flight path and the peak, giving your answer to the nearest metre. [5]

(iii) Aircraft Bravo takes off from Point B. For aviation safety reasons, the flight paths of Aircraft Alpha and Aircraft Bravo should never cross each other. Given that Aircraft Bravo will take off in the direction $4\mathbf{i}+2\mathbf{j}+h\mathbf{k}$, find the value of h which will result in the flight paths crossing each other, giving your answer to 3 decimal places. [2]

The value of h is now given to be 1.

(iv) Find the cartesian equation of the plane containing point B and the flight path of Aircraft Alpha. Using this answer, provide another mathematical explanation as to why the flight path of Aircraft Bravo will not cross the flight path of Aircraft Alpha. [4]