JURONG JUNIOR COLLEGE

Preparation for 2018 GCE A Level Examination

H2 Mathematics 9758

Revision Package (Topical)

Specially Prepared For:

Contents

SCHEME OF EXAMINATION PAPERS

For the examination in H2 Mathematics, there will be two 3-hour papers, each carrying 50% of the total mark, and each marked out of 100, as follows:

PAPER 1 (3 hours)	**A Levels:**	**Friday**	**9 November 2018**	**08:00-11:00**

A paper consisting of about 10 to 12 questions of different lengths and marks based on the Pure Mathematics section of the syllabus.

There will be at least two questions on application of Mathematics in real-world contexts, including those from sciences and engineering. Each question will carry at least 12 marks and may require concepts and skills from more than one topic.

Candidates will be expected to answer all questions.

PAPER 2 (3 hours)	**A Levels:**	**Wednesday**	**14 November 2018**	**08:00-11:00**

A paper consisting of 2 sections, Sections A and B.

Section A (Pure Mathematics – 40 marks) will consist of about 4 to 5 questions of different lengths and marks based on the Pure Mathematics section of the syllabus.

Section B (Probability and Statistics – 60 marks) will consist of about 6 to 8 questions of different lengths and marks based on the Probability and Statistics section of the syllabus.

There will be at least two questions on application of Mathematics in real-world contexts, including those from sciences and engineering. Each question will carry at least 12 marks and may require concepts and skills from more than one topic.

Candidates will be expected to answer all questions.

USE OF GRAPHIC CALCULATOR (GC)

The use of GC without computer algebra system will be expected. The examination papers will be set with the assumption that candidates will have access to GC. As a general rule, unsupported answers obtained from GC are allowed unless the question states otherwise. Where unsupported answers from GC are not allowed, candidates are required to present the mathematical steps using mathematical notations and not calculator commands. For questions where graphs are used to find a solution, candidates should sketch these graphs as part of their answers. Incorrect answers without working will receive no marks. However, if there is written evidence of using GC correctly, method marks may be awarded.

Students should be aware that there are limitations inherent in GC. For example, answers obtained by tracing along a graph to find roots of an equation may not produce the required accuracy.

PURE MATHEMATICS

Algebraic series

Binomial expansion:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n$$, where n is a positive integer and

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \qquad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \qquad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \qquad (\text{all } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1} x^r}{r} + \ldots \qquad (-1 < x \leq 1)$$

Partial fractions decomposition

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$

Trigonometry

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A \equiv 2\sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

$$\sin P + \sin Q \equiv 2\sin\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$

$$\sin P - \sin Q \equiv 2\cos\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$

$$\cos P + \cos Q \equiv 2\cos\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$

$$\cos P - \cos Q \equiv -2\sin\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$

Principal values:

$$-\tfrac{1}{2}\pi \leqslant \sin^{-1}x \leqslant \tfrac{1}{2}\pi \qquad (|x| \leqslant 1)$$

$$0 \leqslant \cos^{-1}x \leqslant \pi \qquad (|x| \leqslant 1)$$

$$-\tfrac{1}{2}\pi < \tan^{-1}x < \tfrac{1}{2}\pi$$

Derivatives

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$

Integrals

(Arbitrary constants are omitted; a denotes a positive constant.)

$f(x)$	$\int f(x)\,dx$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{x-a}{x+a}\right)$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right)$	$(\lvert x\rvert < a)$
$\tan x$	$\ln(\sec x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$
$\cot x$	$\ln(\sin x)$	$(0 < x < \pi)$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x + \cot x)$	$(0 < x < \pi)$
$\sec x$	$\ln(\sec x + \tan x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$

Vectors

The point dividing AB in the ratio $\lambda : \mu$ has position vector $\dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$

Vector product:

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

PROBABILITY AND STATISTICS

Standard discrete distributions

Distribution of X	$\mathrm{P}(X = x)$	Mean	Variance
Binomial $\mathrm{B}(n,p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Poisson $\mathrm{Po}(\lambda)$	$\mathrm{e}^{-\lambda} \dfrac{\lambda^x}{x!}$	λ	λ
Geometric $\mathrm{Geo}(p)$	$(1 \quad p)^{x \ 1} p$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$

Standard continuous distribution

Distribution of X	p.d.f.	Mean	Variance
Exponential	$\lambda \mathrm{e}^{\ \lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$

Sampling and testing

Unbiased estimate of population variance:

$$s^2 = \frac{n}{n-1}\left(\frac{\Sigma(x-\bar{x})^2}{n}\right) = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$$

Unbiased estimate of common population variance from two samples:

$$s^2 = \frac{\Sigma(x_1-\bar{x}_1)^2 + \Sigma(x_2-\bar{x}_2)^2}{n_1+n_2-2}$$

Regression and correlation

Estimated product moment correlation coefficient:

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\{\Sigma(x-\bar{x})^2\}\{\Sigma(y-\bar{y})^2\}}} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{n}}{\sqrt{\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)\left(\Sigma y^2 - \frac{(\Sigma y)^2}{n}\right)}}$$

Estimated regression line of y on x:

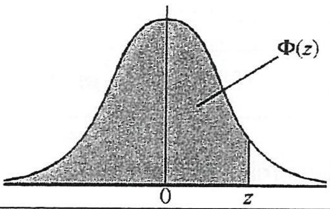
$$y - \bar{y} = b(x - \bar{x}), \quad \text{where } b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

THE NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1 then, for each value of z, the table gives the value of $\Phi(z)$, where

$$\Phi(z) = \mathrm{P}(Z \leqslant z).$$

For negative values of z use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
														ADD					
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0

Critical values for the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that

$$\mathrm{P}(Z \leqslant z) = p.$$

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Topic 1: Equations and Inequalities

1. **RI/I/1**

(a) Using a graphical approach, solve the inequality $\sqrt{x} < 2x$. [2]

(b) Solve the inequality $\frac{(x-1)(x-3)}{(x-2)^2} < 0$. [3]

2. **2013/DH/I/2**

Without the use of a calculator, solve the inequality $\frac{1}{x} + 1 \le x(1+x)$. [3]

Hence, solve $1 - e^{2x} + e^{-x} - e^{x} \le 0$. [2]

3. **2012/AJC/I/3**

Without using a calculator, solve the inequality $\frac{6x-4}{x-3} \le 1-2x$, $x \ne 3$. [4]

Hence find the exact range of values of θ for which $\frac{6-4\operatorname{cosec}\theta}{1-3\operatorname{cosec}\theta} \le 1-2\sin\theta$ where $0 < \theta < 2\pi$. [3]

4. **TJC/I/2**

Given that a is a positive constant, solve the inequality $|x-2a| < 2x+a$. [4]

Hence solve the inequality $|x+4| < 2-2x$. [2]

5. **JJC/I/2**

Solve the inequality $x+5 < |2x-1| + |x-3|$. [5]

6. **PJC/II/1**

The first four terms of a sequence are given by $T_1 = 1$, $T_2 = 2$, $T_3 = 4$, $T_4 = 8$.

Given that T_r is a cubic polynomial in r, find T_r in terms of r. [4]

7. **ACJC/II/1**

The graph of $y = f(x)$ passes through the points $(1,2)$, $(-1,3)$ and $(2,2)$. Given that $f'(x) = 3ax^2 + 2bx + 2$, where a and b are constants, find $f(x)$. [4]

8. **2014/TPJC/P1/Q1**

Two curves with equations $y = \sqrt{a}\,x + bx^2$ and $y = \ln(cx)$ intersect at the points where $x = 1$, $x = 2$ and $x = 3$. Find the values of a, b and c, correct to 3 decimal places. [4]

9. **HCI/I/1**

John has a total of 88 blue, red and green matchsticks. All the matchsticks are indistinguishable apart from their colour. He uses all the blue matchsticks to form triangles, all the red matchsticks to form squares and all the green matchsticks to form hexagons. The total number of triangles and hexagons formed is four times the number of squares formed. If John were to exchange $\frac{1}{6}$ of his green matchsticks for red matchsticks, he will have the same number of blue and red matchsticks.

Find the number of blue matchsticks John has. [3]

10. Angeline, Benjamin and Clement are 3 tenants of a house, and they have agreed to share the monthly utility bill comprising of 3 components, namely electricity, water and gas, according to their usage patterns. The percentages of share that each tenant has agreed to pay across the 3 utility components are given in the following table:

	Percentage of share across 3 components		
	electricity	water	gas
Angeline	40	60	0
Benjamin	50	30	20
Clement	70	20	10

In a particular month, they received a utility bill with a total amount of $400 and the breakdown by percentages of cost for electricity, water and gas is 50%, 40% and 10% respectively. Write down and solve equations to find the share of the bill for each of tenants for that month. [4]

Topic 2: Graphing Techniques

1. AJC/I/8

The curve C has the equation $y=\dfrac{ax^2}{x+a}$ where a is a negative constant.

(i) Show that the curve C has two stationary points for all negative values of a. [2]

(ii) Sketch the curve C, showing clearly all the asymptotes, axial intercepts and turning points. [3]

(iii) Using the sketch in (ii), find the range of values of k for which $x^4=\left(k-x^2\right)(x-1)^2$ has exactly two roots. [3]

2. 2013 MI/I/6(i) modified

The curve C has equation $y=\dfrac{x^2+2x+3}{x+1}$, $x\neq -1$

Find the range of values of y for which C does not lie within, in exact form. [3]

3. RVHS/I/10(i)-(iv)

The curve C has equation $y=\dfrac{ax^2+bx-5}{x+c}$ where a, b, c are constants and $x\neq -c$.

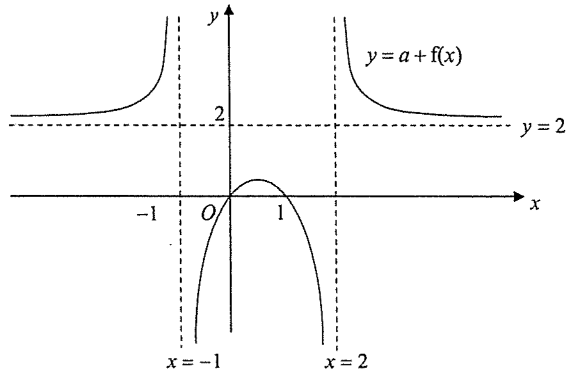
(i) Given that $x=1$ is an asymptote of C and C has a turning point on the y-axis, determine the values of b and c. [3]

(ii) Given also that C has no x-intercept, show that $a<-\dfrac{5}{4}$. [2]

(iii) Sketch the curve C for $-\dfrac{5}{2}<a<-\dfrac{5}{4}$, stating clearly the coordinates of any stationary point, point of intersection with the axes, and the equations of any asymptotes. [3]

(iv) By adding an additional line on the same diagram, determine in terms of a, the set of values of x which satisfies the inequality $\dfrac{ax^2+bx-5}{x+c}>ax+1$ for $-\dfrac{5}{2}<a<-\dfrac{5}{4}$. [3]

4. 2013/JJC/1/6 modified



The diagram shows the graph of $y=a+\mathrm{f}(x)$, where a is a constant and $\mathrm{f}(x)$ is a proper rational function. The curve has a maximum point at $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$ and it crosses the x-axis at the points $(0,0)$ and $(1,0)$. The lines $x=-1$, $x=2$ and $y=2$ are the asymptotes of the curve.

(i) State the value of a. [1]

(ii) Describe a transformation which would transform the above curve to the curve $y=\mathrm{f}(x)$. [2]

(iii) State the equations of the asymptotes of the curve $y=\dfrac{1}{a+\mathrm{f}(x)}$. [2]

(iv) Determine the number of real roots of the equation $\mathrm{f}(|x|)+1=0$. [2]

5. NJC/II/4(ii)

Sketch the curve C, with equation $\dfrac{(y+2)^2}{4}-(x-3)^2=1$, indicating clearly the equations of any asymptotes. [2]

6. TPJC/II/1

By expressing the equation $y=\dfrac{2x+1}{x-4}$ in the form $y=A+\dfrac{B}{x-4}$, where A and B are constants, state a sequence of transformations which transform the graph of $y=\dfrac{1}{x}$ to the graph of $y=\dfrac{2x+1}{x-4}$. [3]

7. **ACJC/2015/Prelim/I/9**

The curve C is given by the equation $y=\frac{1}{x}+\frac{2}{x^2}, x\neq 0$.

(i) Without using a calculator, find the set of values that y can take. [2]

(ii) Sketch the curve C, stating the equations of any asymptotes and the coordinates of any turning points and points of intersection with the axes. [3]

Given that the solution of the inequality $ax^2+bx+c>\frac{1}{x}+\frac{2}{x^2}$ is the set

$$\{x\in\mathbb{R}: -1.5<x<-1 \text{ or } 1<x<k\},$$

find the values of a, b and c. [3]

Hence find the value of k. [1]

8. **DHS/II/3(iii)**

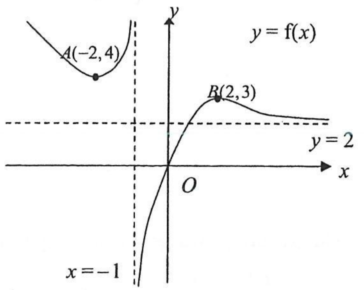
A curve C has parametric equations $x=at-\frac{1}{t}$, $\quad y=bt+\frac{1}{t}$, $\quad t>0$.

For $a=1$ and $b=1$, the curve C has two oblique asymptotes $y=x$ and $y=-x$.

By considering the curve of C, sketch the graph of $y=f'(x)$. [3]

9. **DHS/I/7(a) modified**

The diagram shows the graph of $y=f(x)$, which has turning points at $A(-2,4)$ and $B(2,3)$. The horizontal and vertical asymptotes are $y=2$ and $x=-1$ respectively.



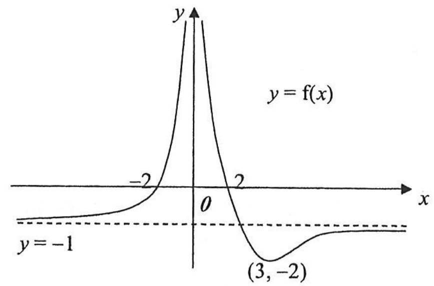
Sketch the graph of $y=-f(|x|)$, showing clearly all relevant asymptotes, intercepts and turning point(s), where possible. [3]

10. **JJC/I/12(b)**

The curve G given below has equation $y=f(x)$. Sketch, on separate diagrams, the graphs of

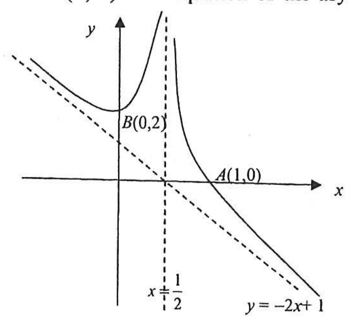
(i) $y=f(3-x)$, [3]

(ii) $y=\frac{1}{f(x)}$. [3]



11. **AJC/I/10 (modified)**

The sketch below shows the graph of $y=f(x)$. The curve passes through the point $A(1, 0)$ and has a minimum point at $B(0, 2)$. The equation of the asymptotes are $y=-2x+1$ and $x=\frac{1}{2}$.



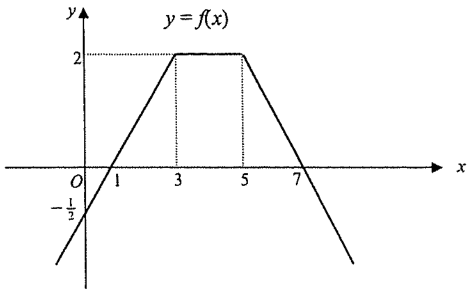
On separate diagrams, sketch the following graphs indicating the points corresponding to A, B and asymptotes where necessary.

(i) $y=\frac{1}{f(x)}$ [3]

(ii) $y=f'(2x)$ [3]

12. TJC/I/11(a)

The graph of $y = \text{f}(x)$ is shown below. The curve cuts the y-axis at $\left(0, -\frac{1}{2}\right)$ and the x-axis at $(1, 0)$ and $(7, 0)$. Sketch the graph of $y = \frac{1}{\text{f}(x)}$, showing clearly the main relevant features of the curve. [3]



13. 2013 NYJC/I/4(i)

Sketch the graphs of $y = \frac{x-b}{x-a}$ and $y = \frac{x-b}{b}$ on a single diagram, where a and b are positive constants and $1 < a < b$, showing all asymptotes and axial intercepts clearly. [3]

Using the diagram, solve $\frac{x-b}{x-a} > \frac{x-b}{b}$. [2]

14. PJC/I/10(b)

A graph with the equation $y = \text{f}(x)$ undergoes, in succession, the following transformations:

A: A translation of 1 unit in the direction of the x-axis.

B: A stretch parallel to the x-axis by a scale factor $\frac{1}{2}$.

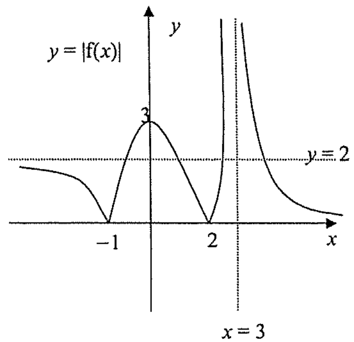
C: A reflection in the y-axis.

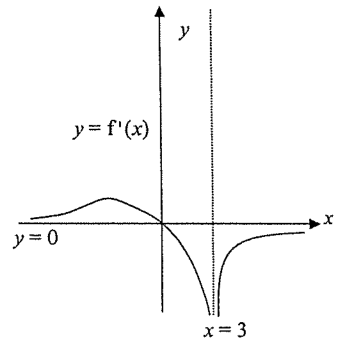
The equation of the resulting curve is $y = \frac{4}{4x^2 + 4x + 1}$.

Determine the equation of the graph $y = \text{f}(x)$, giving your answer in the simplest form. [4]

15. NYJC/II/4(a)(b)

(a) The diagrams below show the graphs of $y = |\text{f}(x)|$ and $y = \text{f}'(x)$.

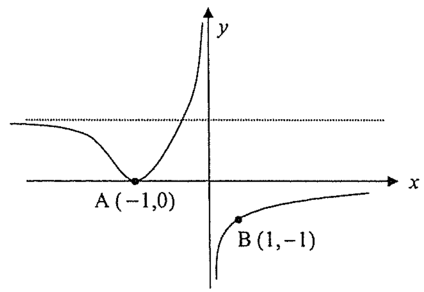




Sketch the graph of $y = \text{f}(x)$, stating the equations of any asymptotes and the coordinates of any axial intercepts and turning points. [3]

Hence, find the range of values of k if there is exactly 1 real root to the equation $\text{f}(x) - k = 0$. [2]

(b) The diagram below shows the graph of $y = \text{f}(2x-1)$. The curve passes through the point $A(-1, 0)$ and $B(1, -1)$. The asymptotes are $x = 0$ and $y = 0$ and $y = 3$.



Sketch the graph of $y = \text{f}(x)$ describing the sequence of transformations involved. [5] Your sketch should show clearly the equations of any asymptotes and the coordinates of the points corresponding to A and B.

16. RVHS/II/5 and 2013 RVHS/I/6(b)

(a) Describe a sequence of transformations which transforms the graph of $x^2+y^2=1$ to that of $x^2+(2y+2)^2=4$. [3]

(b) The curve C_1 has equation $x^2+y^2+6x-8y+16=0$. Express the equation in the form $(x+a)^2+(y+b)^2=c^2$, where a, b and c are constants to be found. [1]

Another curve C_2 has equation $x^2+\frac{y^2}{9}=1$.

State a sequence of transformations that transforms C_2 to C_1. [3]

Topic 3: Functions

1. 2011/JJC/II/2

Functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{1}{x^2-2}, \quad x>0,\ x \neq \sqrt{2},$$

$$\mathrm{g}: x \mapsto 2x+3, \quad x>0.$$

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

(ii) Show that fg exists. Find fg in a similar form and the range of fg. [6]

2. 2010/SRJC/I/5

The function f is defined by $\mathrm{f}: x \mapsto \left|3+x-2x^2\right|$ for $x \in \mathbb{R}$, $x \geq k$.

(i) Find the least value of k such that f has an inverse. [2]

(ii) Using the value of k in part **(i)**, find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

(iii) Hence, find the exact solution of the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [2]

3. 2011/HCI/II/2(a)

The function f is defined by

$$\mathrm{f}: x \to x^2+\lambda x+2, \quad x \in \mathbb{R},\ x \leq 2,$$

where λ is a constant.

(i) Find the range of values of λ such that f^{-1} exists. [2]

(ii) Given that $\lambda=-4$, obtain f^{-1} in a similar form. [3]

4. 2011/TJC/I/3

Functions f and g are defined by

$$\mathrm{f}: x \mapsto 2\sqrt{1-x^2}, \quad x \in \mathbb{R}, -1<x<1,$$

$$\mathrm{g}: x \mapsto \ln x, \quad x \in \mathbb{R}^+.$$

(i) Show that composite function gf exists and find the range of gf in exact form. [4]

(ii) Describe the transformations such that the graph of $y=\mathrm{g}(x)$ is mapped onto the graph of $y=\ln(2x-1)$. [2]

5. **2011/DH/I/4(a)** and **2011/NJC/I/7**

(a) Given that f is a one-one function, determine if ff^{-1} exists. Justify your answer. [1]

(b) The function f is defined by $f: x \mapsto 1-(x+3)^2,\ x \in \mathbb{R},\ x \geq -2$.

(i) Show that the inverse function of f exists. Find $f^{-1}(x)$ and state its domain. [4]

(ii) On the same diagram, sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = f^{-1}f(x)$, showing clearly the relationship between the graphs. [3]

(iii) Hence, find the exact solution of $f^{-1}f(x) \leq f(x)$. [3]

6. **2011/YJC/I/12**

The function f is defined as $f: x \mapsto 4x^2 - 12x + 23$ for $x \in \mathbb{R}$.

(i) Prove that f does not have an inverse function. [2]

(ii) If the domain of f is further restricted to $x \leq a$, state the largest value of $a \in \mathbb{Z}$ for which the function f^{-1} exist. [1]

In the rest of the question, the domain of f is $x \in \mathbb{R},\ x \leq a$.

(iii) Find $f^{-1}(x)$. State the domain and range of f^{-1}. [5]

The function h is defined as $h: x \mapsto -e^x,\ x \in \mathbb{R}$.

(iv) Find hf in a similar form and determine the exact range of hf. [4]

7. **2011/ACJC/I/9**

Sketch the curve given by the equation $y^2 + ax^2 = 5$ for $x \geq 0$ and $y \geq 0$, where a is a positive constant. [1]

The functions f and g are defined by

$$f: x \mapsto \sqrt{5-ax^2},\ x \in \mathbb{R},\ 0 \leq x \leq \sqrt{\frac{5}{a}}$$

$$g: x \mapsto 1 + e^{-x},\ x \in \mathbb{R}, x \geq 0.$$

Show that f^{-1} exists and define f^{-1} in a similar form. [4]

Given that $f^2(x) = x$, for all $x \in \mathbb{R},\ 0 \leq x \leq \sqrt{\frac{5}{a}}$,

(i) show that $a = 1$ without evaluating $f^2(x)$, [2]

(ii) using the result in **(i)**, show that fg exists and find its corresponding range. [3]

8. **2011/IJC/I/9**

The function f is defined as $f: x \mapsto \dfrac{2}{1+(5x-1)^2},\quad x \in \mathbb{R}$.

(i) Find the range of f. [2]

(ii) Give a reason why f does not have an inverse. [1]

(iii) If the domain of f is restricted to $x \geq k$, state the least value of k for which the function f^{-1} exists, and find $f^{-1}(x)$ for this domain. [3]

(iv) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram if the domain of f is restricted to $x \geq k$, where k is the value found in **(iii)**. Your diagram should show clearly the relationship between the two graphs. [2]

(v) Show algebraically that the solution of the equation $f(x) = f^{-1}(x)$ satisfies the equation $25x^3 - 10x^2 + 2x - 2 = 0$. [2]

9. **2011/TPJC/II/4**

The functions f and g are defined by

$$f: x \to \frac{2x}{x-1}, \qquad \text{for } x \in \mathbb{R}, x > 1$$

$$g: x \to e^{x^2-4}, \qquad \text{for } x \in \mathbb{R}.$$

(i) Sketch the graph of $y = g(x)$. [1]

(ii) If the domain of g is further restricted to $x \geq k$, state with a reason the least value of k for which the function g^{-1} exists. [2]

(iii) Find $g^{-1}(x)$, stating the domain of g^{-1}. [2]

For the rest of the question, the domain of g is $x \in \mathbb{R}$, as originally defined.

(iv) Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist. [3]

(v) Solve the inequality $f^2(x) > x$. [3]

10. 2013/CJC/II/3

Functions f and g are defined by

$$f : x \mapsto (x-2)^2 - 1, \quad x \in \mathbb{R},\ x < 2$$
$$g : x \mapsto \ln(x^2+1), \quad x \in \mathbb{R}.$$

(i) Find $f^{-1}(x)$ and state the domain of f^{-1}. [2]

(ii) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram.

Your sketch should indicate the position of the graphs in relation to the origin. [3]

(iii) Write down the equation of the line in which the graph of $y = f(x)$ must be reflected in order to obtain the graph of $y = f^{-1}(x)$, and hence find the exact solution of the equation $f(x) = f^{-1}(x)$. [3]

(iv) Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite function that exists, and explain why the other does not exist. [4]

11. 2013/IJC/I/13

It is given that

$$f(x) = \begin{cases} \dfrac{3}{(2x+3)^2 - 2} & \text{for } -2 \le x < -1, \\ -3 & \text{for } -1 \le x < 0, \end{cases}$$

and that $f(x) = f(x+2)$ for all real values of x.

(i) Sketch the graph of $y = f(x)$ for $-\frac{5}{2} \le x \le \frac{5}{2}$. [3]

(ii) Find the exact value of $\int_{-\frac{3}{2}}^{1} |f(x)|\, dx$. [4]

The function h is defined by

$$h : x \mapsto \frac{3}{(2x+3)^2 - 2} \quad \text{for } -2 \le x < a.$$

(iii) Write down the greatest value of a such that h^{-1} exists. [1]

Assume that a takes the value found in part **(iii)**.

(iv) Sketch, on a single diagram, the graphs of $y = h(x)$ and $y = h^{-1}(x)$. [2]

(v) Explain why the x-coordinate of the point of intersection of the curves in part **(iv)** satisfies the equation $4x^3 + 12x^2 + 7x - 3 = 0$ and find the value of this x-coordinate, correct to 4 significant figures. [3]

12. 2010/IJC/I/10

The functions f and g are defined as

$$f : x \to \sqrt{1-x} \quad \text{for } x \le 1$$
$$g : x \to e^{-x} - 1 \quad \text{for } x > 0.$$

(i) Define f^{-1} in a similar form, including its domain. [3]

(ii) State the relationship between f and f^{-1}, and sketch the graphs of f and f^{-1} on the same diagram. [3]

(iii) Find the exact solutions of the equation $f(x) = f^{-1}(x)$. [2]

(iv) Show that the composite function fg exists. [2]

(v) Given that $h'(x) = fg(x)$ for $x > 0$, show that h is an increasing function for $x > 0$. [2]

13. 2013/PJC/I/3

A function f is said to be self-inverse if $f(x) = f^{-1}(x)$ for all x in the domain of f.

Functions f and g are defined by

$$f : x \mapsto \frac{1-ax}{a-x}, \quad x \in \mathbb{R},\ x \ne a,$$
$$g : x \mapsto \ln(a-x), \quad x \in \mathbb{R},\ x < a,$$

where a is a positive constant and $a > 1$.

(i) Sketch the graph of f and write down its range. [2]

(ii) Show that f is self-inverse. Hence, or otherwise, find $f^{2013}(5)$ in terms of a. [4]

(iii) Show that the composite function gf does not exist. [1]

(iv) If the domain of f is restricted to the set $\{x \in \mathbb{R} : x < k\}$, find the greatest value of k for which gf exist. [1]

14. 2013/RI/II/2(a)

The functions f and g are defined as follows:

$$f : x \mapsto \frac{2x+1}{x^2+2x+3} + k, \quad x \in \mathbb{R},$$
$$g : x \mapsto (x-5)^2, \quad x < 2k,$$

where k is a constant.

(i) Find the range of values of k for which the function gf exists. [3]

(ii) For $k = 5$, find the range of gf. [2]

15. 2013/AJC/I/3

The function f is a strictly increasing function such that $y = \mathrm{f}(x)$ for $x \geq 0$. The coordinates of certain points on the curve of $y = \mathrm{f}(x)$ are as follows:

x	0	$\frac{1}{\sqrt{3}}$	1	2	$\frac{7}{2}$	4	6	8	11	14
y	-2	-1	$-\frac{1}{2}$	1	$\frac{7}{2}$	8	14	19	24	26

(i) State the value of $\mathrm{ff}(6)$ and the value of $\mathrm{f}^{-1}(8)$. [2]

Another function g is defined by $\mathrm{g}: x \mapsto \tan 2x$ for $0 \leq x \leq \frac{\pi}{8}$.

(ii) Explain why the composite function fg exists. [1]

(iii) Find the range of fg. Hence find the set of values of x such that the composite function fg satisfies the inequality $|\mathrm{fg}(x)| < 1$. Leave your answer in exact form. [3]

16. HCI/2013/I/6(i)-(iii)

The function f is defined by $\mathrm{f}: x \mapsto \ln(x-a)$, $x \in \mathbb{R}$, $a < x \leq a+2$,

where a is a positive constant.

(i) Show that f^{-1} exists. [2]

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [2]

(iii) Find an expression for $\mathrm{g}(x)$ for each of the following cases.

(a) $\mathrm{fg}(x) = x$, [1]

(b) $\mathrm{gf}(x) = x^2$. [1]

17. 2009/HCI/I/8

The functions f and g are defined as follows:

$$\mathrm{f}: x \to \frac{5-x}{1-x}, \quad x \in \mathbb{R}, x \neq 1$$

$$\mathrm{g}: x \to 2x^2 + 4x + \lambda, \quad x \in \mathbb{R}, \ x > -2.$$

(i) Explain why f has an inverse, f^{-1}, and show that $\mathrm{f}^{-1} = \mathrm{f}$. [4]

(ii) Evaluate $\mathrm{f}^{51}(4)$. [2]

(iii) Find the range of values of λ such that fg exists.

For these values of λ, find the range of fg in terms of λ. [3]

18. 2016/CJC/I/11

The function f is defined by

$$\mathrm{f}: x \mapsto \frac{2x+k}{x-2}, x \in \mathbb{R}, \ x \neq 2,$$

where k is a positive constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the x and y axes. [3]

(ii) Describe fully a sequence of transformations which would transform the curve $y = \frac{1}{x}$ onto $y = \mathrm{f}(x)$. [4]

(iii) Find f^{-1} in a similar form and write down the range of f^{-1}. [3]

(iv) Hence or otherwise, find f^2.

Find the value of $\mathrm{f}^{2017}\left(\frac{1}{2}\right)$, leaving your answer in terms of k. [4]

The function g is defined by

$$\mathrm{g}: x \mapsto a + \sqrt{x-3}, \quad x \in \mathbb{R}, \ x > 3,$$

where a is a real constant.

(v) Given that fg exists, write down an inequality for a and explain why gf does not exist. [3]

Topic 4: Sequences and Series

1. **2011/IJC/I/7**

An arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of a geometric series are equal to the ninth, fifth and second terms respectively of the arithmetic series.

(i) Prove that the geometric series is convergent and find, in terms of a, the sum to infinity. [5]

(ii) Given that $a > 0$, find the largest value of n for which the sum of the first n terms of the geometric series is less than four fifths of the sum to infinity. [3]

2. **2011/TPJC/I/3**

The n^{th} term, T_n, of a sequence is given by $\log 2x^{n-1}$, where x is a positive constant. Show that the sequence is an arithmetic progression for all positive integers n. [2]

When log 2 is subtracted from each of the tenth, fourth and second terms of the above arithmetic progression, these terms become the first three consecutive terms of a geometric progression respectively.

Find the range of values of x for which the sum of the first 20 terms of the arithmetic progression exceeds the sum to infinity of the geometric progression. [5]

3. **2016/RI(JC)/I/9**

(a) The nth term of a series is given by $T_n = \mathrm{e}^{2+nx(x+1)}$, where x is a constant.

(i) Show that this series is geometric. [2]

(ii) Find the set of values of x for the sum to infinity to exist. [2]

(b) An arithmetic progression A has $2N$ terms with first term a and fifth term b, $N \geq 3$. The sum of all its terms is three times the sum of its first N terms, S.

(i) Show that $b = \left(\dfrac{N+9}{N+1}\right)a$. [3]

(ii) When $N = 39$, it is known that $S = \dfrac{1521}{4}$. Find the third term of A. [3]

4. **2011/TJC/I/9**

(a) Find the sum of all the integers between 1102 and 2011 (inclusive) which are not divisible by 3. [5]

(b) A geometric progression G has first term a and common ratio r. A sequence H is formed by squaring the terms of G. Prove that H is also a geometric progression. The sum to infinity of G is S. Find the value of r such that the sum to infinity of H is $2S^2$. [4]

5. **2011/PJC/I/3**

Each time a ball falls vertically onto a horizontal floor, it rebounds to three-quarters of the height from which it fell. It is initially dropped from a point 4 m above the floor.

(i) Show that the total distance the ball travels until it is about to touch the floor for the $(n+1)$th time is given by $28 - 24\left(\dfrac{3}{4}\right)^n$. [3]

(ii) Find the least number of times the ball must bounce for it to travel more than 24m. [2]

(iii) Explain why the ball will not travel more than 28 m. [1]

6. **2016/NYJC/I/11**

A bank provides a special loan for customers. At the beginning of each month, it charges the customer interest at a fixed rate of 0.8% of the outstanding amount owed. John borrowed \$30,000 on 1st January 2016 to fund his son's studies in the university. He made a partial payment of \$800 to the bank on the 15th of each month, starting in January 2016. Let January 2016 be the first month.

(i) Show, with sufficient working, that on 1st March 2016, when interest for that month has been charged, he will owe the bank \$29,106.52. [2]

(ii) Show that in the beginning of the n^{th} month, after interest has been charged, he will owe the bank a total of $\$[100800 - 70000(1.008)^n]$. [3]

(iii) On what date will John finish paying his loan? [3]

John decides to change his monthly payment so that he is able to pay off the loan on 15th Jan 2018.

(iv) Find the new minimum monthly payment. [4]

7. 2011/NJC/I/6

Altitude Insurance (AI) offers two investment plans for its clients as follows:

Plan A: A fixed interest of \$100 is added to the account at the end of each year, if at least \$1500 was invested at the beginning of each year.

Plan B: Interest is added to the account at the end of each year, at a fixed rate of 3% of the total invested amount at the beginning of each year.

Should a client, at any point in time, decide to draw part or all of his invested amount, he will have to close his account.

(i) Mr. Wei is a client who invests \$2000 under Plan A in the first year. Subsequently, his investment at the beginning of each year is \$400 more than the previous year's. Show that, at the end of n years, he accumulates a total amount of $\$\left[2100n+200n(n-1)\right]$. [2]

(ii) Mr. Wei intends to stay with AI until his total amount in (i) reaches \$100,000. What is the least number of years he needs to keep his account open for? [2]

(iii) Ms. Goon decides to take up Plan B to accumulate \$60,000 by the end of the 11th year. Given that she puts the same amount into her AI account at the beginning of each year, what is the minimum sum of money, correct to the nearest dollar, she will need to add to her account each year? [3]

8. 2011/RVHS/I/5

A circle with radius 10 cm is cut into 10 sectors whose areas follow a geometric progression. The first sector, which is the largest, has an area of a cm^2. The second sector has an area of ar cm^2, the third sector has an area of ar^2 cm^2, and so on. Given also that the total area of the odd-numbered sectors is 10π cm^2 more than that of the remaining sectors, show that $11r^{11}-9r^{10}-11r+9=0$. Hence find the area of the smallest sector. [7]

Suppose the cutting of the circle is done such that starting with the biggest sector, the subsequent sectors have areas following a geometric progression with common ratio 0.7. Find the maximum area of the biggest sector such that the cutting may be continued infinitely. [2]

9. 2016/AJC/I/11

A publisher tracks the sales of a new book. It is found that 3^n copies of the book are sold in the first week, where n is a positive integer ($n \geq 3$). In week 2 he sells 3^{n-1} copies more than in week 1, and in week 3 he sells $(3^{n-1}+3^{n-2})$ copies more than in week 2.

(i) By finding and simplifying the number of copies sold in each week from week 1 to week 3 in the form $a(3^n)$, show that the numbers of copies sold in each of the first three weeks form a geometric progression. [3]

In fact, starting from week 3, the number of copies sold in each week forms an arithmetic progression with common difference -48.

(ii) Given that the number of copies sold falls to zero in week K, show that

$$K=3+3^{n-3}.$$ [3]

(iii) Hence find the total number of copies sold from week 1 up to and including week K, expressing your answer in terms of n. [3]

(iv) The publisher printed 20,000 copies of the book. What is the greatest value of n such that the books will not be out of stock? [3]

10. 2016/HCI/I/9

The government of a city started to invest in the tourism of the city in 2004. According to the government budget, the investment in 2004 is x million dollars. In each subsequent year, the investment will be 20% less than that of the previous year.

(i) What is the theoretical maximum amount of investment that the government can invest? [2]

It is given that $x=8$ and the annual tourism income in 2004 is 1.1 million dollars.

(ii) Due to the government investment, the annual tourism income in each subsequent year increased by 5%. With this rate of increment, by which year would the total tourism income (starting from 2004) first exceed the total investment? [4]

Economists predicted that the trend of 5% yearly increase in the annual tourism income will end in 2020 and the annual tourism income would instead decrease by k million dollars in each subsequent year from 2021 onwards despite investment from the government.

(iii) Find the maximum value of k, correcting to 4 decimal places, such that the annual tourism income for 2030 will still be more than the annual investment according to the government budget. [3]

11. 2011/DHS/I/10

The terms in the arithmetic sequence $\{3n+1, n=1,2,3,...\}$ are grouped into sets such that the r^{th} bracket contains 2^r terms as shown below:

$$\{4,7\}, \{10,13,16,19\}, \{22,25,28,31,34,37,40,43\},\dots .$$

(i) Show that the total number of terms in the first N brackets is $2(2^N-1)$. [2]

(ii) Find the sum of the terms in the first N brackets. [2]

(iii) Show that the first term in the N^{th} bracket is $3(2^N)-2$ and find also the last term in the N^{th} bracket. [4]

(iv) Find the least N such that the sum of terms in the N^{th} bracket is more than 10^{12}. [2]

12. 2011/DHS/I/1

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, show that $\sum_{r=1}^{n}(2r+1)(2r+3) = \frac{n}{3}\left(4n^2+18n+23\right)$.

Hence find the exact sum of the 25th to the 75th term of the series. [5]

13. 2011/YJC/I/4

(i) Express $\frac{2}{4r^2-1}$ in partial fractions. [1]

(ii) Hence find $\sum_{r=1}^{N}\frac{1}{4r^2-1}$, expressing your answer as a single algebraic fraction. [3]

(iii) Using (ii), find $\sum_{r=1}^{N}\frac{1}{(2r+1)(2r+3)}$, expressing your answer as a single algebraic fraction. [5]

14. 2011/NJC/I/2

By considering $u_k - u_{k+1}$, where $u_k = \frac{1}{k^2(k+1)^2}$, find $\sum_{k=1}^{n}\frac{1}{k^2(k+1)(k+2)^2}$ in terms of n.

Hence, find $\sum_{k=3}^{\infty}\frac{1}{k^2(k+1)(k+2)^2}$, leaving your answer as a fraction in its lowest terms. [7]

15. 2011/PJC/I/4

(i) By writing $\frac{1}{(r-3)(r-2)}$ in partial fractions, find $\sum_{r=4}^{N}\frac{1}{(r-3)(r-2)}$. [4]

(ii) Hence find $\sum_{r=0}^{N-6}\frac{1}{(r+2)(r+1)}$. [2]

(iii) Give a reason why the series $\sum_{r=4}^{\infty}\frac{1}{(r-3)(r-2)}$ converges, and write down its value. [2]

16. 2011/RVHS/I/4

(i) Express $(2r+3)$ in the form $2(r+1)+Ar+B(r-1)$, where A and B are constants to be found. [2]

(ii) Hence, or otherwise, find $\sum_{r=1}^{n}(2r+3)2^r$ in terms of n. [4]

(iii) Using your answer to (ii), find $\sum_{r=1}^{n}(2r+5)2^r$. [3]

17. 2016/CJC/I/6

(i) Show that $\ln\left[\frac{r(r+2)}{(r+1)^2}\right] = A\ln(r)+B\ln(r+1)+C\ln(r+2)$, where A, B and C are constants to be determined. [1]

(ii) Prove by the method of differences that $\sum_{r=1}^{N}\ln\left[\frac{r(r+2)}{(r+1)^2}\right] = \ln\left(\frac{N+2}{2N+2}\right)$. [3]

(iii) Hence, state the value of $\sum_{r=1}^{\infty}\ln\left[\frac{r(r+2)}{(r+1)^2}\right]$. [1]

(i) Using the result in **(ii)**, find $\sum_{r=6}^{N}\ln\left[\frac{r(r-2)}{(r-1)^2}\right]$ in terms of N. [3]

18. 2011/TJC/I/1

It is given that $u_1 = 2$ and $u_{r+1} - u_r = 2(r+1)$ where $r \in \mathbb{Z}^+$.

By using the method of difference, find u_n in terms of n. [5]

19. 2016/RI(JC)/I/6(i)(ii)

(i) Using the formulae for $\cos(A \pm B)$, show that

$$\cos(A-B)-\cos(A+B)=2\sin A\sin B.$$

Deduce that $\cos 2(k-1)\theta - \cos 2k\theta = 2\sin(2k-1)\theta\sin\theta$. [2]

(ii) Find a formula for $\sum_{k=1}^{n}\sin(2k-1)\theta\sin\theta$ in terms of $\sin n\theta$. [3]

20. 2016/DHS/I/12

The sum of the first n terms of a sequence is denoted by S_n. It is given that S_1, S_2, S_3 and S_4 are 2, 6, 18 and 44 respectively.

(i) Find the first 3 terms of the sequence. Hence or otherwise, explain why S_n cannot be a quadratic polynomial in n. [4]

It is known that S_n is a cubic polynomial in n.

(ii) Find S_n in terms of n. [3]

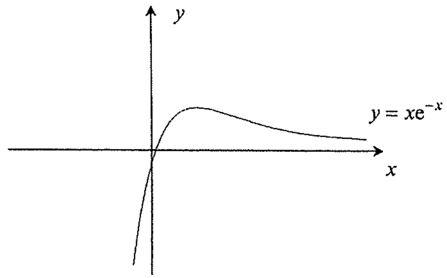
(iii) Show that $S_n - S_{n-1} = 3n^2 - 7n + 6$. Hence use the method of differences to show that

$$\sum_{n=1}^{N} n^2 = \frac{N}{6}(N+1)(2N+1).$$ [4]

Topic 5: Techniques and Applications of Differentiation

1. 2011 NYJC/I/9(i)(ii)

The diagram shows a sketch of the curve $y = xe^{-x}$.



(i) By differentiation, find the range of values of x for which the graph of $y = xe^{-x}$ is decreasing. [3]

(ii) Determine the range of values of x for which the graph of $y = xe^{-x}$ is decreasing and concave downwards. [3]

2. 2016/TPJC/I/1

A curve C has equation $e^{x+y} + e = (3y+1)^2$.

(i) By considering $\frac{dy}{dx}$, show that C has no stationary points. [5]

(ii) Write down an equation relating x and y at which the tangent is parallel to the y-axis. [1]

3. 2012/IJC/I/4

The equation of a curve is given by $xy - 2y^2 + 4x^2 = 66$.

(i) Find the exact coordinates of the points on the curve where the tangent is parallel to the y-axis. [4]

(ii) Show that every line parallel to the x-axis cuts the curve at two distinct points. [3]

4. **2013/JJC/I/10**

The equation of a curve is $x^2 - 4xy + 2y^2 = k$, where k is a constant.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [2]

(ii) For the case where $k = -2$, find the coordinates of each point on the curve at which the tangent is parallel to the x-axis. [3]

(iii) For the case where $k = 2$, a point $P(x, y)$ moves along the curve in such a way that its x-coordinate is increasing at a constant rate of 2 units per second. Find the exact rate of change of its y-coordinate at the instant when $x = 4$ and $y = 7$. [2]

(iv) Show that for $k > 0$, every line parallel to the y-axis cuts the curve at two distinct points. [3]

5. **2014/MJC/I/10**

The parametric equations of a curve C are $x = \cos^3\theta,\ y = 2\sin^3\theta$ where $-\frac{\pi}{2} \le \theta \le 0$.

(i) Sketch the graph of C. [2]

(ii) Find $\frac{dy}{dx}$ in terms of θ. [2]

(iii) The tangent to the curve C at the point $P\left(\cos^3 t,\ 2\sin^3 t\right)$ intersects the x-axis and the y-axis at the points U and V respectively. Show that the coordinates of U are $\left(a\cos t,\ 0\right)$, where a is a constant to be found. Find the coordinates of V. [5]

(iv) Find a cartesian equation of the locus of the mid-point of UV as t varies. [3]

6. **2012/RVHS/I/8**

The parametric equations of a curve are $x = \sec t,\ y = \tan t$, where $0 < t < \frac{\pi}{2}$.

(i) Find $\frac{dy}{dx}$ in terms of t. [2]

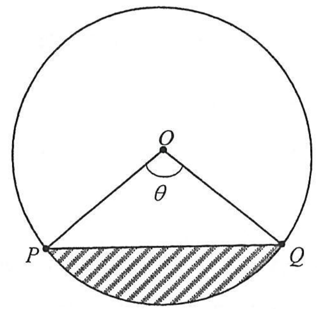
(ii) Show that the equation of the tangent to the curve at the point $P\left(\sec\theta,\ \tan\theta\right)$, is of the form $y = mx - \cot\theta$ where m consists of a single trigonometric term. [3]

(iii) The tangent at the point P intersects the x-axis and the y-axis at the points A and B respectively. Given that $\theta = \frac{\pi}{6}$, find the exact area of triangle AOB. [3]

(iv) Find a cartesian equation of the locus of the mid-point of AB as θ varies. [3]

7. **2013/TJC/II/1**

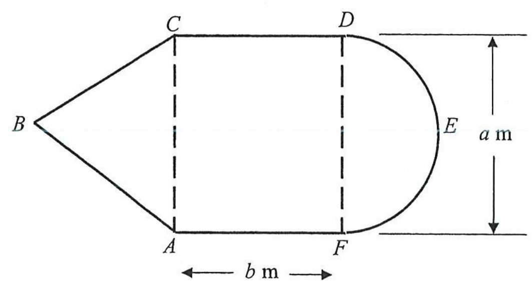
The diagram below shows the points P and Q on the circumference of a circle with centre O, and radius $2a$ cm, where $\angle POQ = \theta$. Points P and Q are moving on the circumference so that θ is increasing at a constant rate.



Find the acute angle θ at the instant when the rate of change of the area of the shaded segment is $\frac{a}{2}$ times the rate of change of the length of the minor arc PQ. [5]

8. **2014/HCI/I/8**

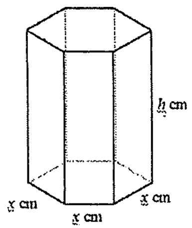
The diagram shows a field consisting of an equilateral triangle ABC, a rectangle $ACDF$ and a semi-circle DEF.



Suppose $DF = a$ m, $AF = b$ m, and the area of the field is fixed at 400 m^2.
Find, using differentiation, the values of a and b which give a field of minimum perimeter, giving your answers correct to 2 decimal places. [8]

9. 2012/CJC/I/8

The diagram below shows the points P and Q on the circumference of a circle with centre O, Candy is being stored in a closed container in the form of a regular hexagonal prism, with sides x cm and height h cm (as shown in the figure below).



Given that the container has a volume of 972 cm^3,

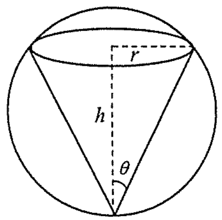
(i) show that its base area given by $\frac{3\sqrt{3}x^2}{2}$ cm^2. [1]

(ii) Using differentiation, find the minimum area of the material, A cm^2, that is used to make the container, leaving your answer to 2 decimal places. [6]

(iii) Given that the cost of the material for the packaging is \$ 0.05 per 100 cm^2, find the minimum cost required for the container. [1]

10. 2014/MJC/II/4

The diagram shows a right inverted cone of radius r, height h and semi-vertical angle θ, which is inscribed in a sphere of radius 1 unit.



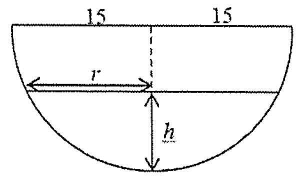
Prove that $r^2 = 2h - h^2$. [1]

(i) As r and h varies, determine the exact maximum volume of the cone. [5]

(ii) Show that $h = 2\cos^2\theta$. The volume of the cone is increasing at a rate of 6 cm^3/s when $h = \frac{3}{2}$. Determine the rate of change of θ at this instant, leaving your answer in the exact form. [7]

[The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.]

11. 2012/DHS/II/1



A hemispherical goldfish tank with radius 15 cm (as shown in the figure above) was initially filled with water. The tank has a defect and water is leaking at a constant rate of 20 cm^3 per min. The volume of water in the tank is given by $V = \frac{\pi}{3}\left(45h^2 - h^3\right)$, where h is the depth of water at the centre of the tank in cm.

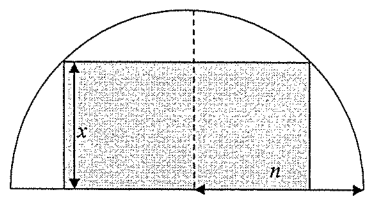
Show that r, the radius of the water surface in cm, is given by $r = \sqrt{30h - h^2}$. [1]

Given that the minimum depth of water needed for the goldfish to survive is 5 cm, find, at this instant

(i) the rate of change of the depth of water, and [2]

(ii) the rate of decrease of the radius of the water surface. [3]

12. 2012/PJC/I/5



The diagram shows the cross sectional view of a cylinder open on both ends inscribed in a hemisphere with fixed radius n cm.

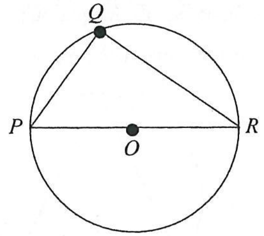
If the diameter of the cylinder is x cm, show that the surface area A of the cylinder is

$$2\pi x\sqrt{n^2 - x^2} \text{ cm}^2.$$ [2]

Given that as x varies, the maximum value of A occurs when the ratio of the diameter of the cylinder to the height of the cylinder is $\frac{1}{k}$. Find the value of k. [5]

13. 2012/MJC/I/8

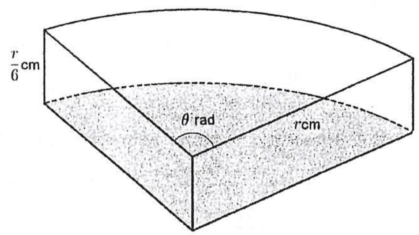
In the diagram below, triangle PQR is inscribed inside a circle of centre O and constant radius r. PR is a line that passes through the centre of the circle O. As point Q moves along the circle, the area of triangle PQR changes



(i) Using differentiation, find the length of the sides PQ and QR such that the maximum area of triangle PQR is obtained. Leave your answers in terms of r. [7]

(ii) Given that QR increases at a rate of 0.2 units per second, find the rate of change of $\angle QPR$ when $\angle QPR = \frac{\pi}{3}$ and $r = 2$. [3]

14. 2015/NJC/I/9



The diagram shows a right container with the top face removed. Its base is a circular sector of radius r cm, with angle θ radians at the centre, where $0 < \theta < \pi$. The height of the container is $\frac{r}{6}$ cm. The thickness of its wall and base can be assumed to be negligible.

(i) It is given that the total exterior surface area of this container is a fixed value A cm^2, and the volume is a maximum. Use differentiation to show that $\theta = 1$. [6]

(ii) It is given instead that the volume of the container is 72 cm^3 and its total exterior surface area is 108 cm^2. Find the value of r and justify why this is the only possible value. [2]

15. 2015/MI/II/2

(a) [Two circles are concentric when they have the same centre but different radii.]
Two concentric circles have radii R and r, where $R > r$. If R increases at a constant rate of 2 cm/s and the area between the circles is constant at 20 cm^2, find the rate at which r is increasing when $R = 5$ cm. [4]

(b) A curve C has parametric equations $x = 2t - 1$, $y = \frac{1}{2t+1}$, $t \in \mathbb{R}$.

(i) Determine if there is a tangent to the curve that is parallel to the line $9y = 4x + 3$. [3]

(ii) Find the exact coordinates of the point at which the normal to the curve at $t = \frac{1}{2}$ meets the curve again. [4]

16. 2015/ACJC/I/3

$ABCD$ is a rectangular field whose sides, AB and BC, measure $2a$ m and a m respectively. A road runs along the side AB. A man, starting from A, wishes to reach the opposite corner C in the shortest possible time. He can walk along the road at 100 m per minute and across the field at 60 m per minute. Find an expression for the time, in minutes, he will take if he walks along the road to P, a point x m from B, and then across the field from P to C. [2]
Use differentiation to find, in terms of a, the value of x for the time taken to be the shortest possible. Find, also, the shortest possible time taken, and prove that it is the minimum. [4]

17. 2016/IJC/I/4

[It is given that the volume of a pyramid is $\frac{1}{3} \times (\text{base area}) \times (\text{height})$.]
A right pyramid of vertical height h m has a square base with side of length $2x$ m and volume $\frac{8}{3}$ m^3.

(i) Express h in terms of x. [1]

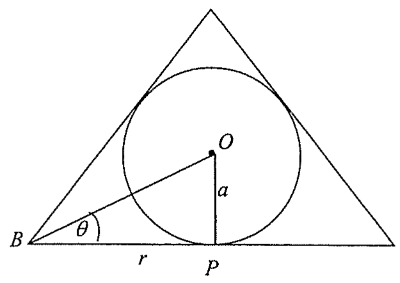
(ii) Show that the surface area S m^2 of the pyramid is given by

$$S = 4x^2\left[1 + \sqrt{\left(1 + \frac{4}{x^6}\right)}\right].$$ [3]

(iii) Use differentiation to find the value of x, correct to 2 decimal places, that gives a stationary value of S. [3]

18. **2016/RI/I/5**

[A right circular cone with base radius r, height h and slant height l has curved surface area πrl.]



A right circular cone of base radius r is designed to contain a sphere of fixed radius a.
The sphere touches both the curved surface and the base of the cone.
(See diagram for a cross-sectional view.)
The point O is the centre of the sphere, the point B is on the circumference of the base of the cone, the point P is the centre of the circular base of the cone and θ is the angle OB makes with the base.

(i) Show that $\cos 2\theta = \dfrac{r^2 - a^2}{r^2 + a^2}$. [2]

(ii) Use differentiation to find, in terms of a, the minimum total surface area of the cone (consisting of the curved surface area and the base area), proving that it is a minimum. [6]

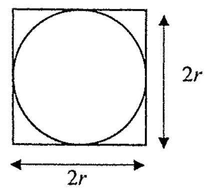
19. **2017/JJC/I/8**

(a) When a liquid is poured onto a flat surface, a circular patch is formed. The area of the circular patch is expanding at a constant rate of 6π cm^2/s.

(i) Find the rate of change of the radius 24 seconds after the liquid is being poured. [3]

(ii) Explain whether the rate of change of the radius will increase or decrease as time passes. [1]

(b)

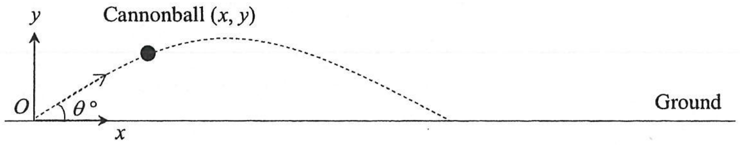


A cylindrical can of volume 355 cm^3 with height h cm and base radius r cm is made from 3 pieces of metal. The curved surface of the can is formed by bending a rectangular sheet of metal, assuming that no metal is wasted in creating this surface. The top and bottom surfaces of the can are cut from square sheets of metal with length $2r$ cm, as shown below. The cost of the metal sheets is \$ K per cm^2.

(i) Show that the total cost of metal used, denoted by \$ C, is given by
$$C = K\left(\frac{710}{r} + 8r^2\right).$$ [3]

(ii) Use differentiation to show that, when the cost of metal used is a minimum, then $\dfrac{h}{r} = \dfrac{8}{\pi}$. [5]

20. **2017/DHS/I/10**



The diagram shows the trajectory of a cannonball fired off from an origin O with an initial speed of v ms^{-1} and at an angle of $\theta°$ above the ground.

At time t seconds, the position of the cannonball can be modelled by the parametric equations $x=(v\cos\theta)t$, $y=(v\sin\theta)t-5t^2$,

where x m is the horizontal distance of the cannonball with respect to O and y m is the vertical distance of the cannonball with respect to ground level.

(i) Find the horizontal distance, d m, that a cannonball would have travelled by the time it hits the ground. Leave your answer in terms of v and θ. [4]

Use $v = 200$ to answer the remaining parts of the question.

An approaching target is travelling at a constant speed of 10 ms^{-1} along the ground. A cannonball is fired towards the target when it is 3000 m away. You may assume the height of the moving target is negligible.

(ii) Show that in order to hit the target, the possible angles at which the cannonball should be fired are $22.7°$ and $69.5°$. [2]

(iii) Explain at which angle the cannonball should be fired in order to hit the target earlier. [2]

Topic 6: Techniques and Applications of Integration

1. **2016/MI PU2 P1/Promo/5**

(a) Find $\int \frac{1}{3t^2+1}\,dt$. [2]

(b) Use the substitution $u = e^x$ to find the exact value of $\int_0^1 e^{e^x+x}\,dx$. [4]

2. **2016/RVHS/Promo/1**

(i) Show that $x = a(4x+4)+b$ where a and b are constants to be determined. [1]

(ii) Hence, find the integral $\int \frac{x}{\sqrt{6-4x-2x^2}}\,dx$. [4]

3. **2016/NYJC/Promo/3**

(a) Find $\int \frac{1}{1+\cos 4\theta}\,d\theta$. [2]

(b) Find $\int \cos 2x \sin 3x\,dx$. [3]

4. **2016/HCI/Promo/4**

(a) Find the exact value of $\int_{-1}^{3} x e^{|4-x^2|}\,dx$. [3]

(b) Find $\int (\ln x)^2\,dx$. [4]

5. **2016/ACJC/Promo/9**

(a) By using the substitution $x=\frac{1}{t}$, find $\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx$, leaving your answer in exact form. [4]

(b) Show that $\int \frac{x+1}{x^2+4x+7}\,dx = \frac{1}{2}\ln\left(x^2+4x+7\right)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}}+c$. [4]

Hence find $\int_{-2}^{1} \frac{|x+1|}{x^2+4x+7}\,dx$ exactly. [3]

6. 2016/PJC/Promo/7

(a) By using the substitution $u=\sqrt{x}$, find $\int \frac{1}{(\sqrt{x})(x+2\sqrt{x}+2)}\,dx$. [4]

(b) Find $\frac{d}{dx}(x\sin x)$. [1]

Hence find $\int (\sin x + x\cos x)\ln x\,dx$. [3]

7. 2016/MJC/Promo/7

(a) Use the substitution $u=\sqrt{(x+1)}$ to find $\int \frac{x^2}{\sqrt{(x+1)}}\,dx$. [4]

(b) Find $\int \frac{e^{5x}}{\left(e^{5x}-e\right)^4}\,dx$. [3]

(c) Show that $\frac{x^2}{1+x^2}=1-\frac{1}{1+x^2}$. [1]

Hence find the exact value of $\int_0^{\sqrt{3}} x\tan^{-1}x\,dx$. [4]

8. 2017/RI/I/3

(a) Find $\int \frac{x+2}{\sqrt{\left(1-8x-4x^2\right)}}\,dx$.. [4]

(b) Use the substitution $x=2\sec\theta$ to find the exact value of $\int_2^4 \frac{1}{x}\sqrt{\left(x^2-4\right)}\,dx$. [4]

9. 2016/ACJC/Promo/2

Find the area of the region bounded by the curve with equation $x^2-y^2=1$ and the line with equation $y=7-2x$. [4]

10. 2017/ACJC/II/3

(i) Find $\int \frac{x}{\left(1+x^2\right)^2}\,dx$. [2]

(ii) By using the substitution $x=\tan\theta$, show that

$$\int \frac{1}{\left(1+x^2\right)^2}\,dx = k\left(\frac{x}{1+x^2}+\tan^{-1}x\right)+c,$$

where c is an arbitrary constant, and k is a constant to be determined. [5]

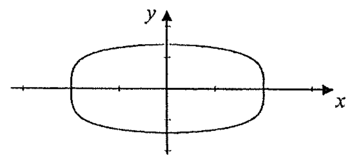
(iii) Hence find $\int \frac{x^2}{\left(1+x^2\right)^2}\,dx$. [3]

(iv) Using all of the above, find $\int \frac{x^2+2x+5}{\left(1+x^2\right)^2}\,dx$, simplifying your answer. [2]

11. 2016/AJC/Promo/3

By considering integration by parts, or otherwise, show that

$$\int \sqrt{4-x^2}\,dx = \frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\left(\frac{x}{2}\right)+c.$$ [4]



The diagram shows the curve C with equation $y^4=4-x^2$. The region enclosed by C where $y\geq 0$ is rotated through 360° about the x-axis. Find the exact value of the volume of solid thus formed. [3]

12. 2016/ACJC Promo/12

The curve C is part of a circle with constant radius r, centred at $(r, 0)$, and is defined by the parametric equations $x = r(1+\sin 2\theta)$, $y = r\cos 2\theta$, $-\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{4}$.

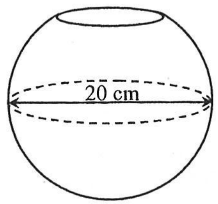
(i) Find the equation of the normal at the point $\left(r(1+\sin 2p),\ r\cos 2p\right)$, leaving your answer in the form $y = m(x-a)$. Hence, or otherwise, write down the coordinates of the point that all normals to C pass through. [4]

(ii) Show that the Cartesian equation of C can be written as $y = \sqrt{2rx - x^2}$. [2]

The region bounded by the part of C from $x = 0$ to $x = h$, the x-axis, and the line $x = h$ where $0 < h \le 2r$, is rotated 2π radians about the x-axis.

(iii) Show that the volume of the solid obtained is $\frac{1}{3}\pi\left(3rh^2 - h^3\right)$, and deduce that the volume of a sphere with radius r is $\dfrac{4}{3}\pi r^3$. [3]

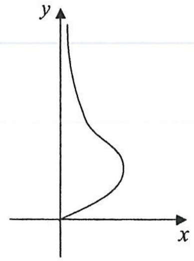
A fish bowl is constructed out of a spherical bowl with diameter 20 cm, with the top portion removed, as shown in the diagram below.



The bowl was initially empty and water flows into the bowl at a rate of $10\text{ cm}^3\text{ s}^{-1}$. Let V and h be the volume and depth of water in the bowl at time t s respectively.

Find $\dfrac{dh}{dt}$ when h is increasing at the slowest rate. [4]

13. 2016/DHS/Promo/7



The diagram shows the graph of curve C with equation $ye^{-y} - \frac{1}{2}x = 0$ for $y \ge 0$. The y-axis is an asymptote to C.

(i) Find $\dfrac{dy}{dx}$ in terms of y. Hence find the equation of the tangent to C which is parallel to the y-axis. [4]

(ii) Obtain a formula for $\int_0^n xe^{-x}\,dx$ in terms of n, where $n > 0$. Hence find the area of the region between C and the positive y-axis. [5]

[You may assume that $ne^{-n} \to 0$ as $n \to \infty$.]

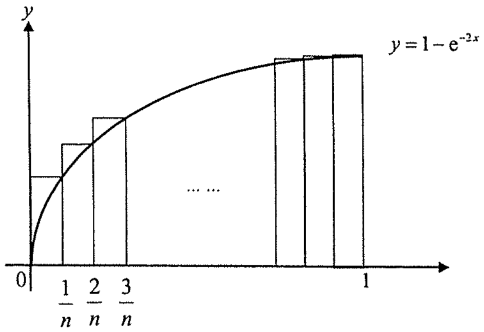
14. 2016/DHS/Promo/9

The volume of revolution, V units^3, is formed when the region bounded by the curve $y = \dfrac{2}{x^2+1}$, the x-axis, the y-axis and the line $x = \tan\alpha$, where $0 < \alpha < \frac{1}{2}\pi$, is rotated through 2π radians about the x-axis.

(i) Use the substitution $x = \tan\theta$ to show that $V = 4\pi\int_0^{\alpha}\cos^2\theta\,d\theta$. Hence evaluate this integral in terms of α and show that $2\alpha\pi < V \le (1+2\alpha)\pi$. [7]

(ii) Given that α is increasing at a constant rate of 0.5 radians per second, find the exact rate of increase of V when $\alpha = \frac{1}{6}\pi$. [2]

15. 2016/HCI/Promo/3



The diagram above shows the graph of $y = 1 - e^{-2x}$ with n rectangles constructed to approximate the area bounded by the curve, the x-axis and the line $x = 1$. Each rectangle has width $\frac{1}{n}$ with its top right-hand corner always in contact with the curve.

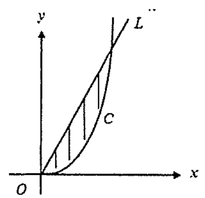
(i) Show that the total area of the n rectangles, A_n, can be expressed as

$$\frac{1}{n}\left[n - e^{\frac{-2}{n}}\left(\frac{e^{-2}-1}{e^{\frac{-2}{n}}-1}\right)\right].$$ [3]

(ii) Find the exact value of $\lim_{n\to\infty} A_n$. [3]

16. 2016/NJC/Promo/7

The diagram below shows a shaded region which is bounded by the curve C with equation $y = \sec x \tan^3 x$ and the line L with equation $y = \frac{4\sqrt{2}}{\pi}x$.



(i) Verify that the point $\left(\frac{\pi}{4}, \sqrt{2}\right)$ lies on both C and L [1]

(ii) Find the exact volume of the solid generated by revolving the shaded region about the x-axis through 2π. Express your answer in the form $\frac{1}{42}\left(a\pi^2 - b\pi\right)$, where a and b are integers. [6]

17. 2016/MJC/Promo/8

A curve C is defined by the equation $y = e^{\sin^{-1}(2x)}$.

Show that $\sqrt{(1-4x^2)}\frac{dy}{dx} = 2y$ and by further differentiation of this result, find the Maclaurin series for y up to and including the term in x^2. [5]

(i) Hence obtain an approximation for the area bounded by the curve C, the lines $x = 0$, $x = 0.3$ and the x-axis. Without performing further calculations, explain how you would obtain a better approximation. [3]

(ii) Write down the equation of the tangent to the curve C at $x = 0$. Hence find the volume of revolution when the region bounded by the curve C, the tangent to the curve C at $x = 0$ and the line $y = 3$ is rotated completely about the y-axis. [4]

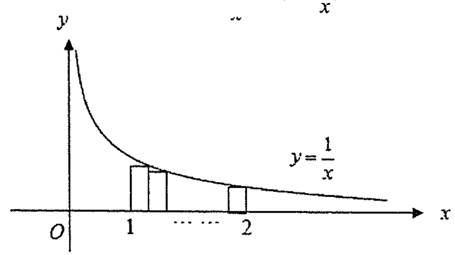
18. 2016/MI PU2 P1/Promo/12

(a) A curve C has parametric equations $x = t^3$, $y = e^t$.

(i) Find the exact area of the region bounded by C, the x-axis and the lines $x = 1$ and $x = 8$. [4]

(ii) By obtaining the cartesian equation of C, find the volume of revolution when the region bounded by C, the y-axis and line $y = 3$ is rotated completely about the y-axis. Give your answer correct to 2 decimal places. [3]

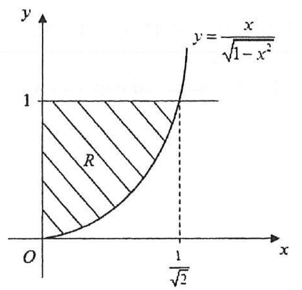
(b) The diagram below shows the curve $y = \frac{1}{x}, x > 0$.



(i) Show that the total area of n rectangles, each of equal width, under the curve between $x = 1$ and $x = 2$ is equal to $\sum_{r=1}^{n}\frac{1}{n+r}$. [2]

(ii) Give an interpretation of $\sum_{r=1}^{\infty}\frac{1}{n+r}$ and find its exact value. [2]

19. 2016/PJC/Promo/8



The diagram shows the region R bounded by the curve $y=\frac{x}{\sqrt{1-x^2}}$, the line $y = 1$ and the y-axis.

(i) Find the exact area of R. [2]

(ii) Find the numerical value of the volume of revolution obtained when R is rotated completely about the x-axis. Give your answer correct to 4 significant figures. [3]

(iii) Find the exact value of the volume of revolution obtained when R is rotated completely about the y-axis. [5]

20. RVHS/Promo/11

(a) By using the substitution $x = 3\tan\theta$, find $\int \frac{x^2}{(x^2+9)^2}\,dx$. [5]

(b) An hourglass is a mechanical device used to measure the passage of time. It comprises two glass bulbs connected vertically by a narrow neck that allows a regulated trickle of sand from the upper bulb to the lower one. The size of the hourglass can be generated when the curve $y=\frac{x}{x^2+9}$ is rotated through 2π about the x-axis, from $x=-a$ to $x=a$. Two hourglasses, A and B, are generated.

(i) Find the exact volume of the hourglass A generated when $a = 3$. [3]

(ii) It is found that the actual volume of the hourglass A produced is larger than the theoretical volume found in part **(i)**. State a possible reason. [1]

(iii) A new hourglass B is to be generated such that its volume is double that of A. Find the value of a. [3]

21. 2016/NJC/Promo/11

The curve C has parametric equations $x = t + e^t$, $y = 2t$, for $t \in \mathbb{R}$.

(i) Sketch C, labelling any intersections between C and the axes. [3]

(ii) Show that the tangent to the curve at $t = 1$, denoted by l, has equation $y=\frac{2}{1+e}x$. [3]

(iii) Find the exact value of $\int_{-1}^{1} w\left(1+e^{w}\right) dw$.

Hence, find the exact area bounded by the curve C, the lines l and $x=-1+e^{-1}$. [8]

22. 2017/HCI/I/8

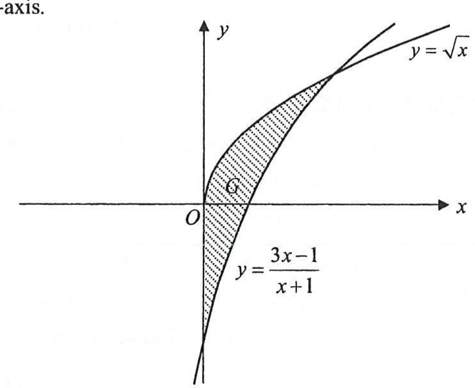
(a) A curve is defined parametrically by the equations

$$x=\sin t \quad \text{and} \quad y=\cos^3 t, \quad -\pi \le t \le \pi.$$

(i) Show that the area enclosed by the curve is given by $k\int_0^{\frac{\pi}{2}} \cos^4 t\, dt$, where k is a constant to be determined. [3]

(ii) Hence find the exact area enclosed by the curve. [3]

(b) In the diagram, the region G is bounded by the curves $y=\frac{3x-1}{x+1}$, $y=\sqrt{x}$ and the y-axis.



Find the exact volume of the solid generated when G is rotated about the y-axis through 2π radians. [6]

Topic 7: Maclaurin's Series and Binomial Expansion

1. 2012/CJC/I/9

Let x and y be variables such that $\ln(y+1) = 1 + \tan^{-1} x$.

(i) Show that $(1+x^2)\dfrac{d^2 y}{dx^2} = (1-2x)\dfrac{dy}{dx}$. [2]

(ii) Find the Maclaurin series for y, up to and including the term in x^3, giving the coefficients in terms of e. [4]

(iii) Given that the first two non-zero terms in the Maclaurin series for y are equal to the first two non-zero terms in the series expansion of $\dfrac{1}{a+bx}$, where a and b are non-zero constants, find a and b in terms of e. [3]

2. 2012/DHS/I/6

It is given that $y = e^{\cos^{-1} x}$.

(i) Show that $\left(1-x^2\right)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = y$. [2]

(ii) Find the Maclaurin's series for y with exact coefficients, up to and including the term in x^3. [3]

(iii) Hence find the expansion of $-\dfrac{e^{\cos^{-1} x}}{\sqrt{1-x^2}}$ up to and including the term in x^2 and estimate the gradient of the tangent of $y = e^{\cos^{-1} x}$ at $x = 0.5$. [3]

3. 2013/MJC/I/3

Let $f(x) = \dfrac{1}{\sqrt[3]{x-1}}$.

(i) Show that the series expansion of f(x) in descending powers of x, up to and including the third term, is $x^{-\frac{1}{3}} + \frac{1}{3}x^{-\frac{4}{3}} + \frac{2}{9}x^{-\frac{7}{3}}$. [3]

Denote the answer in part **(i)** by h(x).

(ii) By evaluating h(8), find an approximation for $\sqrt[3]{7}$ as a fraction in its lowest form. [2]

(iii) Given that $x \in \mathbb{Z}^+$, find the minimum value of x such that the value of $\left|h(x) - f(x)\right|$ is less than 0.001. [2]

4. 2012/HCI/I/5

(i) Use the standard series for $\ln(1+x)$ to find the first three terms of the Maclaurin's series for $y = \dfrac{1}{\sqrt{1+\ln(1+2x)}}$. [3]

(ii) Find the range of values of x for the above expansion to be valid. [3]

(iii) Use your answer in part **(i)** to calculate an approximate value of $\int_0^2 y \, dx$.

Explain why the answer obtained is not a good approximation. [2]

5. 2012/RI/I/2

Expand $\dfrac{1}{(1+2x^2)^2}$ as a series in ascending powers of x, up to and including the term in x^6, giving the coefficients in their simplest form. [2]

Find the coefficient of x^{2r} and the range of values of x for the expansion to be valid. [3]

6. 2013/ACJC/I/5

In the triangle PQR, $PQ = 3$, $QR = \sqrt{2}$ and angle $PQR = \theta + \dfrac{\pi}{4}$ radians.

Given that θ is a sufficiently small angle, show that $PR \approx \left(5 + 6\theta + 3\theta^2\right)^{\frac{1}{2}} \approx a + b\theta + c\theta^2$, for constants a, b and c to be determined. [5]

7. 2012/NYJC/I/6

(a) In a triangle with vertices A, B and C, angle BAC is a right-angle and angle $ABC = \dfrac{\pi}{3} - x$.

(i) Show that $\dfrac{AB}{AC} = \dfrac{1+\sqrt{3}\tan x}{\sqrt{3} - \tan x}$. [1]

(ii) Hence, show that when x is small enough for x^2 and higher powers of x to be neglected, then $\dfrac{AB}{AC} \approx a + bx$, where a and b are exact constants to be determined. [3]

(b) A curve is defined by the equation $\left(1+x^2\right)\dfrac{dy}{dx} + xy = \sqrt{1+x^2}$ and (0, 1) is a point on the curve.

(i) Find the Maclaurin's expansion of y up to and including the term in x^2. [3]

(ii) Hence, find the series expansion of e^y, up to and including the term in x^2. [3]

8. 2012/PJC/II/1 (modified)

(i) Given that $\mathrm{h}(\theta)=\dfrac{1-\sqrt{2(1-\cos 3\theta)}}{2+\sin\theta}$, and that θ is a sufficiently small angle, show that $\mathrm{h}(\theta)\approx\dfrac{1-3\theta}{2+\theta}$. [2]

(ii) Given that n is a positive integer, show that $(\mathrm{h}(\theta))^n \approx a+b\theta$, where a and b are constants to be found in terms of n. [2]

(iii) State the range of values of θ for which the expansion is valid. [1]

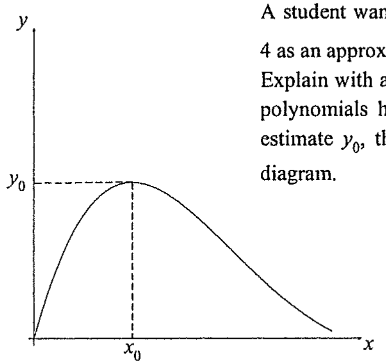
(iv) Verify your answers in **(ii)** by finding a suitable Maclaurin's series. [2]

9. 2013/SRJC/I/1

Let $y=\mathrm{e}^{-2x}\sin 3x$.

(i) Using the standard series expansions of e^x and $\sin x$, show that the Maclaurin's series for y is given by $y=ax+bx^2+cx^3+5x^4+\ldots$, where a, b and c are constants to be determined. [3]

(ii) The graph of $y=\mathrm{e}^{-2x}\sin 3x$ for $0\le x\le 1$ is given below:



A student wants to use a polynomial of degree up to 4 as an approximation for $y=\mathrm{e}^{-2x}\sin 3x$. Explain with a clear diagram, which of the following polynomials he should use if he wants to use it to estimate y_0, the maximum value of y in the above diagram.

$y=ax+bx^2$

$y=ax+bx^2+cx^3$

$y=ax+bx^2+cx^3+5x^4$ [2]

10. 2016/TPJC/I/6(i)(ii)

It is given that $\mathrm{f}(x)=\dfrac{x+3}{(1-x)^n}$, where $-1<x<1$ and n is a positive integer.

(i) Find the binomial expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^2. [4]

(ii) Given that the coefficient of x^2 in the above expansion is 21, find the value of n. [3]

11. 2014/CJC/II/4

(i) Find the expansion of $(4+8x)^{\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^3. [3]

(ii) Given that $y=\ln(\cos x)$, $-1<x<1$.

(a) Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1=0$. [3]

(b) By further differentiation of the result in **(i)**, find the Maclaurin's expansion for y up to and including the term in x^4. [4]

(iii) By using your answers in **(i)** and **(ii)**, find the series expansion of $\sqrt{4+16x}-\tan x$, up to and including the term in x^3. [3]

12. 2014/NYJC/II/3

Given that $y=\dfrac{\ln\sqrt{1+x}}{1+x}$, where $-1<x<1$, show that $2(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y=\dfrac{1}{1+x}$. [1]

(i) By further differentiation, find the Maclaurin series for y up to and including the term in x^3. [5]

(ii) Verify that the same result is obtained if the standard series expansions are used. [3]

(iii) Deduce the approximate value of $\int_0^{\frac{1}{4}\pi} y\,\mathrm{d}x$. Explain why the approximation is not good. [2]

(iv) State the equation of the tangent to the curve y at $x=0$. [1]

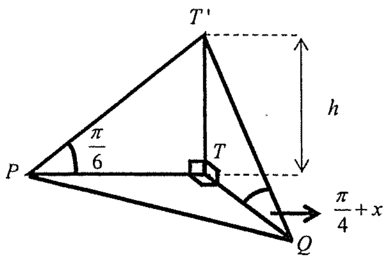
13. 2015/IJC/II/3

(i) Given that $\mathrm{f}(x)=\dfrac{\sin x+2}{\cos 2x+3}$, where x is sufficiently small, find the series expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^2. [4]

(ii) Use your answer to part **(i)** to give an approximation for $\int_0^n \mathrm{f}(x)\,\mathrm{d}x$ in terms of n. Evaluate this approximation in the case where $n=0.5$, leaving your answer in 6 decimal places. [3]

(iii) Use your calculator to find an accurate value for $\int_0^{0.5}\mathrm{f}(x)\,\mathrm{d}x$, correct to 6 decimal places. Explain why this value is more than the approximation obtained in part **(ii)**. [2]

14. 2015/JJC/I/4



Two ground spotlights P and Q are shining at the top of a tower TT' of height h m. P is due west and Q is due south of the tower. The angle TPT' is $\frac{\pi}{6}$ radians and the angle TQT' is $\left(\frac{\pi}{4}+x\right)$ radians, where x is small.

Show that $QT = h(1-x)(1+x)^{-1}$. [3]

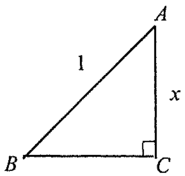
Hence, by using the standard results in MF26, show that $PQ^2 \approx 4h^2\left(1-x+2x^2\right)$. [3]

15. 2016/MJC/I/4

Let $\mathrm{f}(x)=\cos^{-1}x$, where $-1<x<1$ and $0<\mathrm{f}(x)<\pi$.

Show that $\left(1-x^2\right)\mathrm{f}''(x)=x\,\mathrm{f}'(x)$. [2]

By further differentiation of this result, or otherwise, find the first three non-zero terms in the expansion of $\mathrm{f}(x)$ in ascending powers of x. [3]



The diagram shows a triangle ABC.
Given that the lengths of AB and AC are 1 and x units respectively, show that $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$. [1]

Hence find the series expansion of $\sin^{-1}x$ in ascending powers of x, up to and including the term in x^3. [1]

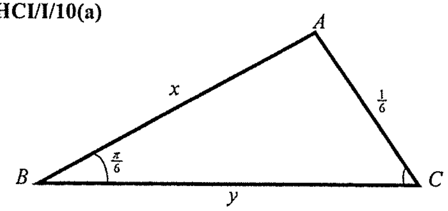
16. 2016/NJC/I/7

(i) Given that $y=\ln(\sec x)$, show that $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=2\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$. [2]

(ii) Hence, by further differentiation, find the first two non-zero terms in the Maclaurin's series for y. [3]

(iii) The equation $\frac{1}{12}x^2+\ln(\sec x)=\cos 2x$ has a positive root α close to zero. Use the result in part **(ii)** and the first three terms of the Maclaurin series for $\cos 2x$ to obtain an approximation to α, leaving your answer in surd form. [3]

17. 2016/HCI/I/10(a)



In the triangle ABC, $AB=x$, $BC=y$, $AC=\frac{1}{6}$, angle $ABC=\frac{\pi}{6}$ radians and angle $ACB=\theta$ radians (see diagram).

(i) Show that $\frac{x}{y}=\frac{2\sin\theta}{\cos\theta+\sqrt{3}\sin\theta}$. [3]

(ii) Given that θ is sufficiently small, express $\frac{x}{y}$ as a cubic polynomial in θ. [3]

18. 2017/VJC/I/4(a)

It is given that $y=\mathrm{f}(x)$ is such that $my^2\frac{\mathrm{d}y}{\mathrm{d}x}-y^3=-\mathrm{e}^x\sin x$ and that the Maclaurin series for $\mathrm{f}(x)$ is given by $1+\frac{1}{3}x+nx^2+\ldots$, where m and n are some real constants.

(i) State the values of $\mathrm{f}(0)$ and $\mathrm{f}'(0)$. [2]

(ii) Find the values of m and n. [3]

Topic 8: Differential Equations

1. **2011/TJC/Promo**

Given the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{y}{2}$, find the general solution for y in terms of x. [4]

Hence find the equation of the solution curve that passes through the point $(0,-1)$. [2]

2. **2011/DHS/Promo**

Find the solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=(1-y)^2$ for which $y=0$ when $x=-1$.

Give your answer in the form $y=\mathrm{f}(x)$. [4]

3. **2011/NJC/Promo**

A differential equation is given by $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-y}{x}+xy^2$. By using the substitution $z=xy$, show that $\frac{\mathrm{d}z}{\mathrm{d}x}=1+z^2$. [2]

Hence find y in terms of x. [3]

4. **2009/MI/Prelim Modified**

(i) Find the general solution of the differential equation $\left(\frac{x^2+1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2$. [2]

(ii) Describe the gradient of every solution curve as $x\to\infty$. [1]

(iii) Find the particular solution of the differential equation in (i) for which $y=0$ and $x=0$. [1]

5. **2013 Jun AQA MPC2/Q7**

The height of the tide in a certain harbour is h metres at time t hours. Successive high tides occur every 12 hours. The rate of change of the height of the tide can be modelled by a funciton of the form a cos (kt), where a and k are constants. The largest value of this rate of change is 1.3 metres per hour. Write down a differential eqaution in the variables h and t. State the values of the constants a and k. [3]

6. **2012/NJC/II/Q3(b)**

A curve $y=\mathrm{f}(x)$, defined for $x>-\frac{1}{3}$, has gradient value of 2 at the point $(0,1)$. Given further that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{2}{1+3x}$, find the equation of the curve. [6]

7. **2015/TJC/II/3(a)**

By considering a standard series expansion, find the general solution of the differential equation $x=1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\frac{1}{2!}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\frac{1}{3!}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3+\dots+\frac{1}{r!}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^r+\cdots$. [4]

8. **2011/PJC/Promo**

A worker pours hot water and cold water into an empty rectangular tank. At time t hours the depth of water in the tank is x metres. The tank has a horizontal base area of 2 m^2.

(i) The worker's friend suggests pouring hot water at a constant rate of 2 $\mathrm{m}^3\mathrm{h}^{-1}$, and pouring cold water at a rate which is proportional to the square of the depth of water in the tank. Show that $\frac{\mathrm{d}x}{\mathrm{d}t}=1+kx^2$, where k is a positive constant. [1]

Given that $x=5$ when $t=1.5$, find x in terms of t. [4]

(ii) The worker's cousin thinks that the previous model is not realistic. Instead, the cousin suggests that x and t should be related by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{1+t^2}$. Find the general solution of this differential equation. Explain in simple terms what will eventually happen to the depth of water using this model. [3]

9. **2011/TPJC/I/6**

Some years ago an island was populated by grey squirrels. The population y, in thousands, of grey squirrels is modelled by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t}=2y-y^2$.

When $t=0$, there are 1000 squirrels on the island.

(i) Express y in terms of t. [8]

(ii) What is the long-term population of grey squirrels predicted by the model? Explain your reasons clearly. [1]

10. 2011/RVHS/Promo Modified

(a) Find the general solution of the differential equation $\frac{d^2y}{dx^2} = \sec x \tan x$. [2]

Hence, find the particular solution of the differential equation for which $y = 2$, $\frac{dy}{dx} = 1$ when $x = 0$. [2]

(b) The rate of change in the population of a colony of insects of size x thousands at time t days after it is first discovered, satisfies the differential equation $\frac{dx}{dt} = 40 - 0.1x$.

Find x in terms of t, given that $x = 300$ when $t = 0$. [5]

State what will happen to the population of the colony of insects eventually. [2]

11. 2012/RI/P1/Q9

By using the substitution $z = ye^{2x}$, find the general solution of the differential equation $\frac{dy}{dx} + 2y = (x+1)e^{-2x}$, expressing your answer in the form $y = f(x)$. [4]

It is given that $y = 1$ when $x = 0$.

(i) Find the particular solution. [1]

(ii) By repeated differentiation of the given differential equation, find the Maclaurin expansion for y up to and including the term in x^3. [4]

(iii) Without carrying out the calculation, describe briefly how you would use the answer in **(i)**, to check the correctness of your answer in **(ii)**. [2]

12. 2012/MJC/P1/Q10(b)

There was an island where initially there was no one living on it. The total capacity of the island is 9 000. The population increases at a rate which is inversely proportional to the remaining capacity of the island. At the same time, the rate at which the population decreases is $\frac{1}{20}$ of the population size. When the population reaches 4 000, it remains at this value. The population size (in thousands) is x at time t months, show that

$$\frac{dx}{dt} = \frac{(x-4)(x-5)}{20(9-x)}.$$

Find t in terms of x. Hence find the time when the population reaches 2 000. [7]

13. 2012/RVHS/P2/Q1 Modified

Use the substitution $u = xy$, where u is a function of x, to reduce the differential equation $x\frac{dy}{dx} + (1 - x^3)y - x^2 = 0$ to $\frac{du}{dx} = (u+1)x^2$. Hence show that the general solution to the differential equation is of the form $y = \frac{Ae^{f(x)} - 1}{x}$, where f is a function in x. [5]

14. 2011/HCI/I/9

(a) By using the substitution $y = ux^2$, find the general solution of the differential equation $x^2\frac{dy}{dx} - 2xy = y^2$. [4]

(b) Water is poured into a filtration device at a constant rate of k litres per minute. The device discharges water at a rate proportional to the volume of water in the device. At time t minutes after the device is activated, the volume of water in the device is v litres. When the volume is 1 litre, it remains at this value. Show that $\frac{dv}{dt} = k(1 - v)$, and hence find the general solution for v in terms of t. [5]

The rate of discharge is always less than the rate of water poured into the device at any time. Sketch a member of the family of solution curves of this differential equation. [2]

15. 2011/NJC/I/5

The number of rats in Nat City is increasing due to changes in the environment that favour its rapid growth. The birth rate of these rats is proportional to the number of rats, x, at time t months after the change in the environment, and the death rate is a constant at 50 rats per month. The population growth rate is 250 rats per month when the population size reaches 500.

(i) Form a differential equation relating x and t, and show that the general solution may be expressed in the form $x = \frac{250}{3} + Ae^{\frac{3t}{5}}$, where A is an arbitrary constant. [5]

(ii) If the number of rats in Nat City reaches 5000, an epidemic is likely to occur. Two models are proposed to represent the situation in (i) for $A = 5$ and 15. Determine which is a better model if an epidemic occurs in the City within 10 months since the change in the environment. [2]

16. 2012/TJC/P1/Q11(b)

In an experiment to study the spread of a soil disease, an area of 15 m^2 of soil was exposed to infection. In a simple model, it is assumed that the infected area grows at a rate which is proportional to the product of the infected area and the uninfected area. Initially, 5 m^2 was infected and the rate of growth of the infected area was 0.1 m^2 per hour. At time t hours after the start of the experiment, an area x m^2 is infected.

(i) Show that $\frac{dx}{dt} = \frac{x(15-x)}{500}$. [2]

(ii) Solve the differential equation and express t in terms of x. [4]

(iii) Find the minimum time in hours needed for 95% of the soil area to become infected. [1]

17. 2011/AJC/II/1

At a water treatment plant, sewage water is being pumped into a processing tank, with an unknown capacity, at a constant rate of 100 litres per hour. The tank processes the water and discharges it at a rate proportional to the volume of sewage water currently in it. Due to a crack in the tank, sewage water is leaking out of the tank at a constant rate of 3 litres per hour. At time t hours, the volume of sewage water in the tank is u litres.

(i) By setting up and solving a differential equation, show that the general solution is $u = \frac{1}{k}\left(97 - Ae^{-kt}\right)$ where A and k are constants, and $k > 0$. [3]

(ii) The processing container is initially empty. After an hour, the volume of sewage water in the container is 70 litres. Given that the sewage water that leaked out of the tank is not processed, find the volume of water processed by the container when $t = 2$. [4]

(iii) Sketch, on a single diagram, the graph of the solution found in part (ii) and find the minimum tank capacity for the model to be valid. Give your answer to the nearest litre. [2]

18. 2012/SAJC/P2/Q4

A virus is found to be present in Country A, containing a population of 10,000 people. The rate at which the number of infected people, x, in thousands, is increasing at any time t is proportional to the product of the infected people and the number that have yet to be infected. Initially, there are 2,000 people infected. Show that the number of infected people at time t is

$x = \frac{10e^{10kt}}{A + e^{10kt}}$, where $A > 0$ is a constant to be determined. [4]

State the range of values of k such that the virus will be eliminated. Give a possible reason for that to happen. [2]

19. 2011/ACJC/I/11

Detectives arrive at a crime scene at 12 p.m. and found an unfinished cup of coffee at $45^\circ C$. In order to estimate what time the coffee was brewed, a fresh cup of coffee was made from the coffee machine in the same room, and its temperature was found to be $110^\circ C$. After leaving the new cup of coffee in the room for 5 minutes, the temperature dropped to $80^\circ C$. It is known that the rate at which the temperature of the coffee falls is proportional to the amount by which its temperature exceeds that of the room. Given that the crime scene is an air-conditioned room with temperature controlled at $25^\circ C$, at what time was the unfinished cup of coffee brewed? Leave your answer to the nearest minute. [6]

20. 2011/PJC/I/8

In a chemical plant, the amount of substance X in a chemical reaction is being observed. The amount of substance X, in kg at any time t minutes after the start of the chemical reaction is denoted by x and it satisfies the differential equation $\frac{dx}{dt} = k(1-2x)$, where k is a positive constant. It is known that at the start of the chemical reaction, the amount of substance X is 1 kg and is decreasing at a rate of 0.05 kg per minute.

(i) Obtain an expression for x in terms of t. [4]

(ii) Using a non-graphical method, show that the amount of substance X is always decreasing. [2]

(iii) State, with a reason whether substance X will be used up in the long run. [1]

(iv) Sketch a graph to show how the amount of substance X varies with time. [2]

21. 2011/TJC/II/5

At time $t = 0$, there are 8000 fish in a lake. At time t days the birth-rate of fish per day is equal to one-fortieth $\left(\frac{1}{40}\right)$ of the number N of fish present. Fish are taken from the lake at the rate of 150 per day.

Modelling N as a continuous variable, show that $40\frac{\mathrm{d}N}{\mathrm{d}t} = N - 6000$. [2]

Solve the differential equation to find N in terms of t. Find the time taken for the population of fish in the lake to increase to 12000. [5]

When the population of fish has reached 12000, the number of fish taken from the lake is increased from 150 per day to r per day.

Write down, in terms of r, the new differential equation satisfied by N. [1]

Find the maximum value of r so that the population of fish in the lake will not decrease. [2]

22. 2015/CJC/II/4

In a research project, the population is modelled by the following *logistic* differential equation, $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.64P\left(1-\frac{P}{10}\right)$, where P is the population function of time t.

(i) Solve the differential equation by expressing P in terms of t, given that $P = 1$ when $t = 0$. [5]

Sketch the solution curve for $t \geq 0$. Comment on the population in the long run. [2]

An alternative model for the population is the *Gompertz* function, which is the solution to the following differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.4P(\ln 10 - \ln P)$.

(ii) By solving the differential equation, show that the general solution is $P = 10\mathrm{e}^{-A\mathrm{e}^{-0.4t}}$, where A is a constant. [3]

Given the same initial condition that $P = 1$ when $t = 0$, sketch the solution curve of the particular solution for $t \geq 0$ on the same diagram in part **(i)**. Comment on the similarity and difference between the two models. [4]

23. NYJC/2014/I/6 Modified

In a bid to analyse the path of an insect, an entomologist decides to fit a mathematical model for the path of the insect. The path travelled by the insect measured with respect to the origin in the horizontal and vertical directions, at time t seconds, is denoted by the variables x and y respectively. It is given that when $t = 0$, $x = 1$, $y = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$. The variables are related by the differential equations $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \mathrm{e}^{-t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} + y = \frac{\mathrm{e}^{-t}}{t-30}$ for $0 \leq t \leq 20$.

(i) Find x in terms of t. [5]

(ii) Using the substitution $w = y\mathrm{e}^{t}$, find y in terms of t. [5]

(iii) Sketch the path travelled by the insect. [2]

24. 2017/AJC/II/1

At the intensive care unit of a hospital, patients of a particular condition receive a certain treatment drug through an intravenous drip at a constant rate of 30mg per hour. Due to the limited capacity for absorption by the body, the drug is lost from a patient's body at a rate proportional to x, where x is the amount of drug (in mg) present in the body at time t (in hours). It is assumed that there is no presence of the drug in any patient prior to admission to the hospital.

(i) Form a differential equation involving x and t and show that $x = \frac{30}{k}\left(1-\mathrm{e}^{-kt}\right)$ where k is a positive constant. [4]

(ii) If there is more than 1000mg of drug present in a patient's body, it is considered an overdose. Suppose the drug continues to be administered, determine the range of values of k such that a patient will have an overdose. [2]

For a particular patient, $k = \frac{1}{50}$.

(iii) Find the time required for the amount of the drug present in the patient's body to be 200mg. [3]

25. 2017/JJC/II/4

To determine whether the amount of preservatives in a particular brand of bread meets the safety limit of preservatives present, the Food Regulatory Authority (FRA) conducted a test to examine the growth of fungus on a piece of bread over time after its expiry date.

The piece of bread has a surface area of 100 cm^2. The staff from FRA estimate the amount of fungus grown and the rate at which it is growing by finding the area of the piece of bread the fungus covers over time. They believe that the area, A cm^2, of fungus present t days after the expiry date is such that the rate at which the area is increasing is proportional to the product of the area of the piece of bread covered by the fungus and the area of the bread not covered by the fungus. It is known that the initial area of fungus is 20 cm^2 and that the area of fungus is 40 cm^2 five days after the expiry date.

(i) Write down a differential equation expressing the relation between A and t. [1]

(ii) Find the value of t at which 50% of the piece of bread is covered by fungus, giving your answer correct to 2 decimal places. [6]

(iii) Given that this particular brand of bread just meets the safety limit of the amount of preservatives present when the test is concluded 2 weeks after the expiry date, find the range of values of A for any piece of bread of this brand to be deemed safe for human consumption in terms of the amount of preservatives present, giving your answer correct to 2 decimal places. [2]

(iv) Write the solution of the differential equation in the form $A = \text{f}(t)$ and sketch this curve. [3]

Topic 9: Vectors

1. Mathematics-The Core Course for A-Level (1985) p491 Q6 modified

The line passes through the point $(2, 7, -1)$ and has direction cosines $\frac{4}{5}, 0, \frac{3}{5}$.

(i) State the angle between the line and the positive y-axis.

(ii) Find the vector equation and the cartesian equation of the line.

2. Haese Mathematics for the international student HL (Core) p428 Ex14K.2 Q5(a),(b)

A, B and C are three distinct points with non-zero position vectors **a**, **b** and **c** respectively.

(i) If $\mathbf{a}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}$, what can be said about $\overrightarrow{OC}$ and $\overrightarrow{AB}$?

(ii) If $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$, what relationship exists between $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{c}$?

3. 2011/MI/I/11

(a) Two vectors **a** and **b** of magnitude 2 and 3 respectively are perpendicular to each other. Vectors $\mathbf{a} + k\mathbf{b}$ and $\mathbf{a} - 2\mathbf{b}$ are also perpendicular, for a constant value of k. Evaluate k. [4]

(b) $OCDE$ is a parallelogram with $\overrightarrow{OC} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\overrightarrow{OE} = \begin{pmatrix}3\\6\\0\end{pmatrix}$. The point X divides CD in the ratio 1:2.

(i) Find $\overrightarrow{OX}$. [3]

(ii) Find the position vector of the point of intersection of the lines OD and EX. [4]

4. 2011/NJC/I/9

The position vectors of the points A, B and C with respect to the origin O are **a**, **b** and **c** respectively, such that $(2+\mu)\mathbf{b} = \mathbf{a} + \lambda\mathbf{c}$, where $\mu - \lambda + 1 = 0$ for non-zero constants λ, μ.

(i) Show that A, B, C are collinear. [2]

(ii) Let $\mu = 1$ and $\lambda = 2$. Given that $\mathbf{a} = \mathbf{i}+5\mathbf{j}+2\mathbf{k}$ and $\mathbf{b} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$, find the position vector of a point P that lies on the line segment BC such that $BP : BC = 2 : 3$. [3]

Another point X has position vector $2\mathbf{i}+7\mathbf{j}+3\mathbf{k}$.

(iii) Find angle PBX. [2]

(iv) Using a vector product, find the exact area of the triangle BAX. [2]

5. 2011/IJC/II/4

Relative to the origin O, the position vectors of two points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. The vector $\mathbf{a}$ is a unit vector which is perpendicular to $\mathbf{a}+3\mathbf{b}$. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{2\pi}{3}$.

(i) Show that $|\mathbf{b}|=\frac{2}{3}$. [4]

(ii) By expanding $(\mathbf{b}-2\mathbf{a})\cdot(\mathbf{b}-2\mathbf{a})$, find the exact value of $|\mathbf{b}-2\mathbf{a}|$. [4]

(iii) The point P divides the line AB in the ratio $\lambda:1-\lambda$. Find the area of triangle OAP in terms of λ. [5]

6. PJC 2014 J1 Mid-Yr Exam Q2

The position vectors **a**, **b** and **c** are given by

$$\mathbf{a}=3\mathbf{i}-2\mathbf{j}+6\mathbf{k},\quad \mathbf{b}=4\mathbf{i}+7\mathbf{j}-3\mathbf{k}\quad\text{and}\quad \mathbf{c}=4p\mathbf{i}+7p\mathbf{j}-4p\mathbf{k}$$

where $p<0$. Given that **c** is a unit vector,

(i) find the exact value of p, [2]

(ii) give a geometrical interpretation of $|\mathbf{c}\cdot\mathbf{a}|$ and find its exact value, [3]

(iii) find the area of triangle OAB, where O is the origin and the points A and B are such that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. Hence, or otherwise, find the perpendicular distance from O to AB. [4]

[4]

7. ACJC 2014 J2 Mid-Yr Exam Q7

$OABC$ is a parallelogram having OB as a diagonal, where O is the origin. The points A and B are such that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. The points D and E lie on AB and BC respectively such that $5AD=2DB$ and $BE=EC$. The line segments OD and AE intersect at F.

Giving your answer in terms of **a** and **b**,

(a) (i) find $\overrightarrow{OE}$ and $\overrightarrow{OD}$, [2]

(ii) find the equation of line through points A and E, [1]

(iii) hence find the position vector of the point F. [3]

(b) It is now given that $|\mathbf{a}|=\sqrt{50}$ and unit vector $\mathbf{b}$ is given by $\mathbf{b}=2p\mathbf{i}-6p\mathbf{j}+3p\mathbf{k}$, where p is a positive constant.

(i) Find the exact value of p. [1]

(ii) Given that the area of triangle OBF is 2, find the angle between OA and OB. [4]

8. 2016/TP/I/2

Referred to the origin O, the points A and B have position vectors given by

$\mathbf{a}=\begin{pmatrix}\cos t\\ -\sin t\\ 0.5\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}\cos 2t\\ \sin 2t\\ -1\end{pmatrix}$ respectively, where t is a real parameter such that $0\le t<\pi$.

(i) Show that $\mathbf{a}\cdot\mathbf{b}=p+\cos(qt)$, where p and q are constants to be determined. [2]

(ii) Hence find the exact value of t for which $\angle AOB$ is a maximum. [3]

9. 2011/JJC/II/4

The points A and B are $(1, -2, 4)$ and $(6, 0, 8)$ respectively. Line l is defined by the equation $\mathbf{r}=\overrightarrow{OB}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$.

(i) Find the length of projection of $\overrightarrow{AB}$ on line l. [2]

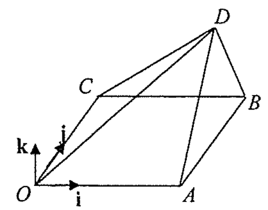
(ii) Find the acute angle between $\overrightarrow{AB}$ and the x-axis. [3]

(iii) Find the shortest distance from A to line l. [2]

(iv) Given that N is the foot of perpendicular from A to line l, and B is the foot of perpendicular from a point P $(5, -4, 6)$ to line l, identify the shape of the quadrilateral $PANB$, justifying your answer. [3]

(v) Find $|\overrightarrow{OA}\times\overrightarrow{OB}|$, and state the geometrical meaning of this expression. [3]

10. 2011/PJC/II/4



The diagram shows a pyramid with a square base $OABC$ of side 4 cm. The vertex D is 6 cm vertically above M, the mid-point of AB. Taking O as the origin and unit vectors **i**, **j** and **k** as indicated, find

(i) the position vector of L, the mid-point of CD, [2]

(ii) the area of triangle OAL, giving your answer in exact form, [2]

(iii) the acute angle between the lines ML and OD. [3]

11. ACJC 2014 J2 Mid-Yr Exam Q5(i)-(iii)

A plane p_1 has equation $\mathbf{r}\cdot(-2\mathbf{i}+3\mathbf{j})=2$. Referred to the origin O, the points A and B have position vectors $4\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $2\mathbf{i}+5\mathbf{j}+q\mathbf{k}$ respectively where $q\in\mathbb{R}^+$.

(i) Find the position vector of the foot of perpendicular F, from A to p_1. [3]

(ii) A plane p_2, which is parallel to vector $\overrightarrow{AB}$, has equation $x+y-z+1=0$. Find the exact value of q. [2]

(iii) A point C has position vector given by $\mathbf{c}=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$. Find the length of projection of vector $\overrightarrow{AC}$ onto p_1. [3]

12. HCI 2014 J2 Block Test 2 Paper 1 Q9

The point A has coordinates $(3,0,-2)$ and the plane Π has equation $2x-y+3z=7$. The line through A parallel to the line $\frac{3-x}{2}=y=\frac{z+2}{2}$ meets Π at the point B. The perpendicular from A to Π meets Π at point C.

(i) Show that the coordinates of B are $(-11,7,12)$ and find the coordinates of C. [6]

(ii) Find the equation of the image of the line AB after a reflection in Π. [4]

13. RI 2014 Yr 6 Term 3 Common Test Q6(i)-(iii)

The plane p_1 has equation $\mathbf{r}\cdot(\mathbf{i}+\mathbf{k})=7$, and the points A and B have position vectors $\mathbf{i}+6\mathbf{k}$ and $2\mathbf{i}+\alpha\mathbf{j}+2\mathbf{k}$ respectively, where $\alpha\in\mathbb{R}$.

(i) Find, in terms of α, the position vector of N, the foot of the perpendicular from B to p_1. [3]

The plane p_2 contains the points A, B and N.

(ii) Show that the equation of p_2 is $\mathbf{r}\cdot(-\alpha\mathbf{i}+5\mathbf{j}+\alpha\mathbf{k})=5\alpha$. [2]

(iii) Find, in terms of α, the equation of the line of intersection between p_1 and p_2. [2]

14. 2013/JJC/II/4(i)-(iii)

The line l has equation $\frac{x-5}{2}=\frac{y+3}{4}=\frac{z-7}{-3}$, and the plane p has equation $2x-y+z=8$

(i) Find the acute angle between l and p. [3]

(ii) Find the point of intersection between l and p. [3]

(iii) The plane q is perpendicular to p, and contains l. Find a cartesian equation of q. [4]

15. 2011/IJC/I/10 (modified)

The line l passes through the points A and B with coordinates (1, 2, 4) and $(-5,4,-2)$ respectively. The plane p has equation $\mathbf{r}=\begin{pmatrix}0\\-17\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\17\\7\end{pmatrix}+\mu\begin{pmatrix}1\\3\\0\end{pmatrix}$, where $\lambda,\mu\in\mathbb{R}$.

(i) Show that the plane p has Cartesian equation $3x-y+2z=17$. [3]

(ii) Find the length of projection of $\overrightarrow{AB}$ on p. [2]

(iii) Find the coordinates of the point of intersection of l and p. [4]

(iv) Find the acute angle between l and p. [2]

(v) Find the perpendicular distance from B to p. [2]

16. 2011/CJC/I/10

The line l has equation $\frac{x+2}{-2}=\frac{y-1}{4}=\frac{z-3}{6}$ and the plane π has equation $\mathbf{r}\cdot\begin{pmatrix}1\\-2\\-3\end{pmatrix}=1$.

(i) Show that l is perpendicular to π. [2]

(ii) Find the coordinates of P, the point of intersection of l and π. [3]

(iii) The point A has position vector $2\mathbf{i}-7\mathbf{j}-9\mathbf{k}$. Find the position vector of the point B on AP produced, such that $AP:AB=1:3$. [3]

(iv) Find the equation of the plane OAB (where O is the origin) in scalar product form. [3]

(v) Find the exact area of triangle OAB. [2]

17. 2010/MJC/I/9(i)-(iv)

A line l passes through the points A and B with coordinates $(0,-1,2)$ and $(1,0,1)$ respectively.

(i) Find the angle between OA and the line l. [2]

(ii) Hence, find the shortest distance from the origin to the line l. [1]

A plane π_1 has equation $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=4$.

(iii) Show that the line l lies on the plane π_1. [2]

A second plane π_2 contains the line l and is perpendicular to the plane π_1.

(iv) Find a vector equation of π_2. [2]

18. 2011/TJC/I/8

The vector equations of lines l_1 and l_2 are given by

$\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{k})$, and

$\mathbf{r} = 3\mathbf{j} + 3\mathbf{k} + \mu(q\mathbf{i} + 2\mathbf{k})$, respectively, where λ, μ and $q \in \mathbb{R}$.

It is given that l_2 passes through the points A and B with position vectors $3\mathbf{j}+3\mathbf{k}$ and $q\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ respectively.

(i) If the acute angle between l_1 and l_2 is $60°$, find the exact values of q. [3]

(ii) If $q=1$, find the position vector of the point C on l_1 such that A is the foot of perpendicular from C to l_2. [4]

(iii) Let T be any point on l_1. If the area of triangle ABT is constant for any λ, find the value of q. [2]

19. 2011/MI/II/4 (modified)

The plane π_1 contains the points $A(2,2,-2)$, $B(1,3,-3)$ and is parallel to the line $\mathbf{r}=\begin{pmatrix}5\\6\\7\end{pmatrix}+\lambda\begin{pmatrix}-2\\0\\1\end{pmatrix}$. Find

(i) the equation of π_1 in scalar product form, [2]

(ii) the position vector of the foot of the perpendicular from the point $P(-1,7,6)$ to π_1, [3]

(iii) the position vector of the point P', which is the reflection of P about π_1. [2]

The plane π_2 has equation $x+y-2z=2$. Find the equation of the line of intersection of π_1 and π_2. [2]

20. 2011/TPJC/I/4(i)(ii)

Two planes p_1 and p_2 have equations $-3x+4y-5z=0$ and $y+z=0$ respectively, and they intersect in a line l.

(i) Find the cosine of the acute angle between planes p_1 and p_2. [2]

(ii) Find a vector equation of l. [3]

21. 2011/NYJC/I/1

Relative to the origin O, two points A and B have position vectors $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $3\mathbf{i}+5\mathbf{j}+6\mathbf{k}$ respectively. The plane π has equation $\mathbf{r}\cdot(2\mathbf{i}+\lambda\mathbf{j})=\mu$.

(a) If the plane π cuts the line segment AB in the ratio 3:1, find a relation between λ and μ. [3]

(b) If instead the plane π contains the line AB, find the values of λ and μ. [4]

22. 2011/JJC/I/10

The equations of two planes π_1 and π_2 and the equation of the line l_1 are as follows:

π_1: $2x-5y+2z=2$,

π_2: $6x-15y+\alpha z=\beta$,

l_1: $\mathbf{r}=2\mathbf{i}+4\mathbf{j}+\lambda(3\mathbf{i}-2\mathbf{k})$, $\lambda\in\mathbb{R}$.

(i) Find the position vector of A, the point of intersection of π_1 and l_1. [2]

(ii) Another line l_2, which lies in π_1, passes through A and is perpendicular to line l_1. Find a vector equation of l_2. [3]

(iii) The plane π_3 contains lines l_1 and l_2. Find the acute angle between π_1 and π_3. [2]

(iv) Find the values of α and β such that π_1 and π_2 do not intersect. [2]

23. PJC 2014 J2 CT1 P1 Q12(i)-(iii)

The planes p_1 and p_2 have equations $\mathbf{r}\cdot\begin{pmatrix}-2\\-1\\5\end{pmatrix}=3$ and $\mathbf{r}\cdot\begin{pmatrix}a\\b\\1\end{pmatrix}=c$ respectively, where a. b and c are constants. p_1 is perpendicular to p_2 and the two planes meet in a line with an equation given by $\mathbf{r}=\begin{pmatrix}0\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}, \lambda\in\mathbb{R}$.

(i) Find the values of a, b and c. [4]

(ii) A plane intersects p_1 and p_2 at a unique point. State the z-coordinate of this point. [1]

The plane p_3 has equation $\mathbf{r}\cdot\begin{pmatrix}1\\-2\\d\end{pmatrix}=4$, where d is a positive constant.

(iii) If p_3 makes an angle of $60°$ with p_1, show that $d=\sqrt{\frac{15}{7}}$. [3]

24. 2016/VJ/II/4(b)

Plane π has equation $3x+2y+5z=45$.

Obtain a vector equation of π in the form $\mathbf{r}=\mathbf{t}+\lambda\mathbf{u}+\mu\mathbf{v}$, $\lambda,\mu\in\mathbb{R}$, given that $\mathbf{t}$ and $\mathbf{u}$ are of the form $p\mathbf{i}+p\mathbf{j}$ and $2\mathbf{i}+q\mathbf{j}$ respectively, where p and q are constants to be determined, and $\mathbf{u}$ is perpendicular to $\mathbf{v}$. [5]

25. **2016/TP/I/8**

The plane p_1 passes through the points $A(4, 1, 1)$ and $B(2, 1, 0)$ and is parallel to the vector $4\mathbf{i}-\mathbf{j}+2\mathbf{k}$. A line l has equation $\frac{x-2}{-2}=y+1=z+4$.

(i) Show that a vector perpendicular to the plane p_1 is parallel to $\mathbf{i}-2\mathbf{k}$. Find the equation of p_1 in scalar product form. [3]

(ii) Find the coordinates of the point C at which l intersects p_1. [3]

(iii) The point D with coordinates $(2, -1, -4)$ lies on l. Find the position vector of the foot of the perpendicular from D to p_1. Find the coordinates of the point E which is the mirror image of D in p_1. [4]

Jurong Junior College
H2 Mathematics (9758)

Topic 10: Complex Numbers

1. **2009/MJC/I/7(a)**

Do not use a calculator in answering this question.

Express $(1-4\mathrm{i})^2$ in the form $a+b\mathrm{i}$. Hence, find the roots of the equation $\left(\frac{z}{2}+3\right)^2=15+8\mathrm{i}$. [4]

2. **2011/MI/I/12**

(a) It is given that $1-i$ is a root to the equation $2z^4-5z^3+8z^2-6z+4=0$. Find the exact values of the other roots of the equation. [5]

(b) It is given that $z=-\sqrt{3}+i$ and $w=1+i$.

(i) Find the exact values of $\left|\frac{z}{w}\right|$ and $\arg\left(\frac{z}{w}\right)$. [3]

(ii) Find the least positive integer value of n such that z^n is purely imaginary. [3]

3. **2011/RI/II/1**

Given that one of the roots of the equation $z^4-az^3+10z-25=0$ is $1+2\mathrm{i}$ where a is real, show that $a=2$. Without using the graphic calculator, find the other roots of the equation in exact form. [4]

Hence find the roots of the equation $(w-1)^4+2(w-1)^3-10(w-1)-25=0$, giving your answers in exact form. [2]

4. **2016/RVHS/I/6**

(a) The complex numbers z and w are such that $z=\frac{3a-5\mathrm{i}}{1+2\mathrm{i}}$ and $w=1+13b\mathrm{i}$, where a and b are real. Given that $z^*=w$, find the exact values of a and b. [4]

(b) Without using a graphic calculator, find the modulus and argument of the complex number $z=\frac{(1-\mathrm{i})^2}{(-1+\sqrt{3}\mathrm{i})^4}$, giving your answers in exact form. Hence evaluate z^6 exactly. [5]

5. **2011/NYJC/II/1**

(a) The complex numbers z and w are such that $z=-1+2\text{i}$ and $w=1+b\text{i}$, where b is real.

(i) Given that the imaginary part of $\frac{w}{z}$ is $-\frac{3}{5}$, find the value of b. [2]

(ii) Find, in radians, the argument of zw. [1]

(b) Let the complex number z be given by $a+\text{i}b$, where $a,b\in\mathbb{R}$ and $-\pi<b\le\pi$.

Find the exact values of a and b, given that $\frac{1}{\text{e}^{\text{i}z}}=2+\text{i}$. [5]

6. **2009/PJC/I/3**

(i) Express $1-\text{i}$ in the form $re^{\text{i}\theta}$, giving the values of r and θ in exact form. [1]

(ii) Hence find the exact value of $1+z+z^2+\ .\ .\ .\ +z^{10}$ where $z=1-\text{i}$. [4]

7. **2012/VJC/I/12(a)**

Let z be the complex number $-1+\text{i}\sqrt{3}$. Find

(i) arg (z), [1]

(ii) the real number a such that $\arg(z(z+a))=\frac{5\pi}{6}$. [4]

8. **2005/JJC/CT/2(c) modified**

Do not use a calculator in answering this question.

Express $z=\frac{1-\sqrt{3}i}{1+i}$ in modulus argument form and also in the form $a+bi$.

Hence show that $\cos\frac{7\pi}{12}=\frac{1-\sqrt{3}}{2\sqrt{2}}$ and find $\sin\frac{7\pi}{12}$ in a similar form. [7]

9. **2012/HCI/I/4**

(a) Find the complex number z in the form $x+\text{i}y$, where $x,\ y\in\mathbb{R}$ such that $\frac{3\text{i}z}{z^*-1+2\text{i}}=-\text{i}$. [3]

(b) The two roots z_1, z_2 of the equation $z^2+z+p=0$, where $p\in\mathbb{R}$, are such that $z_1=a+\text{i}b$, where $a,\ b\in\mathbb{R}$, $b\neq0$ and $|z_1-z_2|=\sqrt{3}$. Find the value of p. [3]

10. **2016/TJC/I/12(b)**

It is given that $w=\frac{1}{2}-\frac{1}{2}\text{i}$. Find the modulus and argument of w, leaving your answers in exact form. [2]

It is also given that the modulus and argument of another complex number v is 2 and $\frac{\pi}{6}$ respectively.

(i) Find the exact values of the modulus and argument of $\frac{v}{w^*}$. [3]

(ii) By first expressing v in the form $\sqrt{c}+d\,\text{i}$ where c and d are integers, find the real and imaginary parts of $\frac{v}{w^*}$ in surd form. [3]

(iii) Deduce that $\tan\left(\frac{\pi}{12}\right)=2-\sqrt{3}$. [2]

11. **2011/PJC/I/9**

The polynomial $\text{P}(z)$ has real coefficients. The equation $\text{P}(z)=0$ has a root $re^{\text{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$.

(i) Write down a second root in terms of r and θ, and hence show that a quadratic factor of $\text{P}(z)$ is $z^2-2rz\cos\theta+r^2$. [3]

(ii) z_1 and z_2 are roots of the equation $\text{P}(z)=0$, where $z_1=2\text{e}^{\text{i}\frac{\pi}{3}}$ and $z_2=\text{i}z_1$.

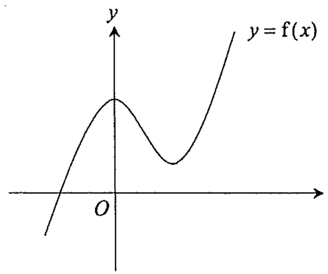
Write down the exact modulus and argument of z_2. State the geometrical relationship between z_1 and z_2 and illustrate this relationship clearly on an Argand diagram. [3]

(iii) Given further that $\text{P}(z)$ is of degree four, express $\text{P}(z)$ as a product of two quadratic factors with real coefficients, giving each factor in exact non-trigonometrical form. [3]

12. 2017/DHS/I/8

Do not use a graphic calculator in answering this question.

(a)



It is given that $f(x)$ is a cubic polynomial with real coefficients. The diagram shows the curve with equation $y = f(x)$. What can be said about all the roots of the equation $f(x) = 0$? [2]

(b) The equation $2z^2 - (7+6i)z + 11 + ic = 0$, where c is a non-zero real number, has a root $z = 3+4i$. Show that $c = -2$. Determine the other root of the equation in cartesian form. Hence find the roots of the equation $2w^2 + (-6+7i)w - 11 + 2i = 0$. [6]

(c) The complex number z is given by $z = 1 + e^{i\alpha}$.

(i) Show that z can be expressed as $2\cos\left(\frac{1}{2}\alpha\right)e^{i\left(\frac{1}{2}\alpha\right)}$. [2]

(ii) Given $\alpha = \frac{1}{3}\pi$ and $w = -1 - \sqrt{3}i$, find the exact modulus and argument of $\left(\frac{z}{w^3}\right)^*$. [5]

13. 2017/HCI/II/2

The complex numbers z and w satisfy the following equations

$$2z + 3w = 20,$$

$$w - zw^* = 6 + 22i.$$

(i) Find z and w in the form $a+bi$, where a and b are real, $a \neq 0$. [5]

(ii) Show z and w on a single Argand diagram, indicating clearly their modulus. State the relationship between z and w with reference to the origin O. [2]

14. 2017/RVHS/I/6

Do not use a graphic calculator in answering this question.

(a) Solve the simultaneous equations

$$z - 4w = 11 + 6i \text{ and } 3z + 6iw = 27$$

giving z and w in the form $x + iy$ where x and y are real. [4]

(b) (i) The complex numbers z and w are given as $z = 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$ and $w = 1 + i\sqrt{3}$. w^* denotes the conjugate of w. Find the modulus r and the argument θ of $\frac{w^*}{z^2}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

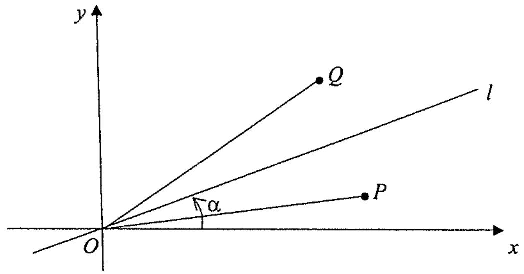
(ii) Find the set of possible values of n such that $\left(\frac{w^*}{z^2}\right)^n$ is purely imaginary. [2]

15. 2017/AJC/I/6

The diagram below shows the line l that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$.

Point P represents the complex number z_1 where $0 < \arg z_1 < \alpha$ and length of OP is r units.

Point P is reflected in line l to produce point Q, which represents the complex number z_2.



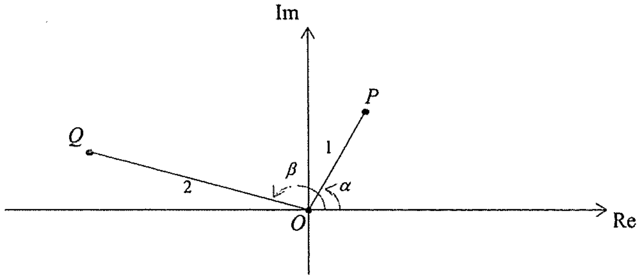
Prove that $\arg z_1 + \arg z_2 = 2\alpha$. [2]

Deduce that $z_1 z_2 = r^2(\cos 2\alpha + i\sin 2\alpha)$. [1]

Let R be the point that represents the complex number $z_1 z_2$. Given that $\alpha = \frac{\pi}{4}$, write down the cartesian equation of the locus of R as z_1 varies. [2]

16. **2017/SRJC/I/3**

For α, $\beta \in \mathbb{R}$ such that $2\alpha < \beta$, the complex numbers $z_1 = e^{i\alpha}$ and $z_2 = 2e^{i\beta}$ are represented by the points P and Q respectively in the Argand diagram below.



(i) Find the modulus and argument of the complex numbers given by $\frac{i}{2}z_2$ and $\frac{z_1^2}{z_2}$. [4]

(ii) Copy the given Argand diagram onto your answer script and indicate clearly the following points representing the corresponding complex numbers on your diagram.

(a) A: $\frac{i}{2}z_2$, [1]

(b) B: $\frac{z_1^2}{z_2}$. [1]

You are expected to indicate clearly the relevant moduli and arguments for parts **(i)** and **(ii)** on your Argand diagram.

(iii) If $\beta = \frac{11}{12}\pi$, find the smallest positive integer n such that the point representing the complex number $(z_2)^n$ lies on the negative real axis. [3]

Topic 11: Permutations & Combinations

1. **2011/YJC/II/5**

Given the word "SUCCESS", find the number of arrangements of these letters such that

(i) there is no restriction, [1]

(ii) the three letters 'S' are separated from one another, [2]

(iii) the two letters 'C' are together and exactly two letters 'S' are together. [2]

2. **2008/CJC/Prelim**

(a) Find the number of different 4-digit numbers that can be formed from the digits 2, 4, 5, 6 if no digit is repeated and the numbers must be divisible by 2. [2]

(b) Find how many 4-letter code words can be formed from the letters of the word CUCUMBER. [4]

3. **2011/PJC/II/10**

Joshua tries to recall the 6-digit pin number for his ATM card.
(Assuming that the number 000 000 is even and valid.)
How many possible numbers can there be if he remembers that:

(i) the number starts and ends with the digit 5? [1]

(ii) the number is odd and the digits do not repeat? [2]

(iii) the digits do not repeat and there are exactly 3 odd digits? [2]

4. **2009/MJC/ Prelim (modified)**

(a) Find the number of different arrangements of the name "BENNETT TAN"

(i) beginning with B and ending with N, [1]

(ii) with all the T's separated. [3]

(b) Bennett wants to create an 8-character alpha-numeric password containing 4 letters from his name "BENNETT TAN" and 4 numbers (which can be repeated) from the digits 0-9. How many different passwords can he choose from if

(i) all the letters and numbers are distinct, [2]

(ii) only 2 of the letters are identical and only 2 of the numbers are identical. [2]

5. 2009/TJC/Prelim

A 10-digit ternary sequence is a sequence using the digits 0, 1 or 2.

Some examples of such sequences are 0101111001 and 2211001122.

Find the number of possible 10-digit ternary sequences that can be formed when

(i) there is no restriction, [1]

(ii) there are exactly three 0s, [2]

(iii) there are exactly four 0s and five 1s, [2]

(iv) there is at least 1 pair of consecutive digits that are the same. [3]

6. 2008/AJC/Prelim

(a) A party of 12 people is to travel in 3 cars, with 4 people in each car. Each car is driven by its owner. How many ways can the party be divided if 2 of the remaining 9 people refuse to travel in the same car.
[The arrangement of people within each car is not relevant.] [2]

(b) In how many ways can 4 copies of a book be distributed among 10 people, if each person can get any number of books? [4]

7. 2010/DHS/II/8

A test consists of five Pure Mathematics questions, A, B, C, D and E, and six Statistics questions, F, G, H, I, J and K.

(i) The examiner plans to arrange all eleven questions in a random order, regardless of topic. Find the number of ways to arrange all eleven questions such that

- **(a)** the last question is a Pure Mathematics question, [2]
- **(b)** a Pure Mathematics question must be separated from another with exactly one Statistics question. [2]

(ii) Later, the examiner decides that the questions should be arranged in two sections, Pure Mathematics followed by Statistics. Find the number of ways to arrange all eleven questions such that

- **(a)** question A is followed by question F, [2]
- **(b)** questions B and K are separated by more than seven questions. [3]

8. 2011/NYJC/II/6

In each of the following cases, find the number of ways of forming a group of 10 people from 15 couples if each group contains

(i) exactly 5 couples, [1]

(ii) no couples, [2]

(iii) exactly 2 couples. [3]

9. 2008/PJC/Prelim

Two families are invited to have a focus group discussion with three male and three female teachers in a school. Each family consists of one student and both his parents. The group takes their places at a round conference table.

Find the number of possible arrangements if

(i) each student must be seated between his parents and the three female teachers must not be seated together, [3]

(ii) the seats are numbered and the three male teachers must sit together. [2]

10. 2015/HCI/II/7

Three women and nine men are waiting for a job interview and they are graduates from Universities *A*, *B* and *C*. The table below shows the number of graduates from each university.

University	Number of Graduates
A	4 men
B	3 women, 2 men
C	3 men

(i) The twelve graduates sit in a row. Find the number of possible seating arrangements in which no two women sit next to each other. [2]

(ii) The interviewer randomly selects six out of the twelve graduates. Find the number of possible selections that include graduates from all universities. [3]

(iii) The twelve graduates are divided into three groups and each group consists of four graduates. Find the probability that each group has at most two graduates from University *A*. [3]

11. 2008/JJC/ Prelim

A group of 10 people consists of 4 single men, 4 single women and 1 married couple.

(a) The group sit in a row. In how many ways can they be seated if

(i) all the men are together, [1]

(ii) there are exactly 2 people between the married couple. [2]

(b) The group sit in a round table. In how many ways can they be seated if

(i) men and women alternate, [2]

(ii) the couple must be separated and men and women alternate. [2]

12. 2010/RI (JC)/II/5

(a) Find the number of three-letter code-words that can be formed from the letters of the word WHYOGEE. [3]

(b) A country is invited to send a delegation of six youths selected from six badminton players, six tennis players and five football players to participate in the opening ceremony of the Singapore 2010 Youth Olympic Games. No youth plays more than one game. The delegation is to consist of at least one, and not more than three players selected from each sport.

(i) Find the number of ways in which the delegation can be selected. [2]

During the ceremony, the youths from the delegation are to be seated in six out of ten chairs which are arranged in a row.

(ii) Find the number of ways this can be done if no two empty chairs are adjacent. [3]

13. 2010/MI/II/7

Two families are invited to a party. The first family consists of a man and both his parents while the second family consists of a woman and both her parents. The two families sit at a round table with two other men and two other women.

Find the number of possible arrangements if

(i) there is no restriction, [1]

(ii) the men and women are seated alternately, [2]

(iii) members of the same family are seated together and the two other women must be seated separately, [3]

(iv) members of the same family are seated together and the seats are numbered. [2]

14. 2011/JJC/II/5

An office has 8 staff consisting of 5 employees, 1 translator, 1 secretary and 1 manager.

(i) The group stands in a line.

(a) Find the number of different possible arrangements. [1]

(b) Find the number of different possible arrangements if the secretary and the manager do not stand together. [2]

(ii) The group sits in a circle where the seats are indistinguishable.

(a) Find the number of different possible arrangements. [1]

(b) Find the number of different possible arrangements if the secretary sits beside the manager, and the translator sits opposite the manager. [2]

15. 2011/MI/II/8

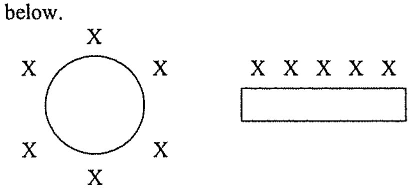
(a) Find the number of ways which the letters of the word POSITIVE can be arranged if

(i) all vowels must be next to each other, [2]

(ii) the vowels are separated and the arrangement begins and ends with the letter I. [2]

(b) The Mathematics Department of M Institute consists of 11 teachers which includes Mr. Tan and Mr. Lin. At a meeting, all 11 teachers sit at a round table with numbered seats. How many ways are there of arranging the department if Mr. Tan and Mr. Lin must sit next to each other? [3]

After the meeting, the department heads to a nearby food court for lunch. Due to the lunch crowd they only manage to find a circular table for 6 and a long table with a row of 5 seats as shown below.



Find the number of arrangements of seating all 11 teachers at the two tables such that Mr. Tan is seated next to Mr. Lin. [4]

16. **2011/IJC/II/8**

Tom, Dick and Jerry met up for dinner with their best friends, Jack and Jane at a restaurant. Jack and Jane brought their 3 children with them. They requested to be seated at a round table. Find the number of ways that the 8 of them can be seated at the round table if

(i) the family of 5 is to be seated together, [2]

(ii) the 3 children are to be separated, [3]

(iii) the chairs are labeled. [1]

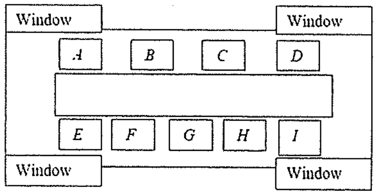
At the restaurant, there is a free flow salad bar. Jane visits the salad bar to get some salad. At the salad bar, there are altogether 10 separate bowls, each containing a different type of vegetable. At each bowl, she can choose to take some of the contents or not. Assuming that Jane takes some of the contents from at least one bowl, find how many different selections she can make. [2]

17. **2009/SRJC/ Prelim**

(a) In how many ways can 7 packs of chocolates be distributed among 20 children, if no child can get more than one pack? [1]

(b) In how many ways can 4 different gifts be distributed among 20 children if each child can get any number of gifts? [1]

(c) 9 people go to a restaurant with the layout given by the diagram below. In how many ways can the 9 people be seated on the chairs marked $A, B, C, \ldots, H, I$? [1]



Find the number of ways in which the 9 people can be seated if

(i) 3 particular people do not want to be seated on any of the 4 chairs A, D, E and I, in front of windows. [2]

(ii) 2 particular people do not want to be seated next to each other on the same side of the table. [3]

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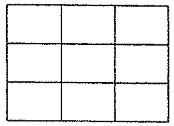
18. **2011/RJC/II/5**

Three families comprising of six adults and three children go on an outing to a theme park. The three families decide to travel together for the day.

(i) There are a total of 15 amusement rides in the theme park. A child insists on trying out at least 2 of the 15 amusement rides. Calculate the number of ways in which this can be done. [2]

(ii) One of the highlights in the theme park is the "Mummy Surprise" rollercoaster ride, where the passengers are seated in a passenger car with seats arranged in 3 rows and 3 columns as shown in the diagram.



(a) Calculate the number of ways in which the three families can be seated if there are no restrictions. [1]

(b) Tom, the youngest child in the group is fearful of the ride and insists on sitting next to his mother before he agrees to get onto the passenger car. Calculate the number of ways in which this can be done. [2]

(iii) The three families proceed to a restaurant for lunch. They are allocated a circular table with 9 seats. Calculate the number of possible seating arrangements for the group if none of the children sit together. [2]

Topic 12: Probability

1. 2011/SRJC/II/5

Events A and B are such that $\mathrm{P}(A)=\frac{1}{7}$, $\mathrm{P}(B\mid A)=\frac{1}{5}$ and $\mathrm{P}(A\cup B)=\frac{5}{7}$.

(i) Find $\mathrm{P}(A'\cap B')$ and $\mathrm{P}(B)$. [3]

(ii) Determine whether events A and B are independent events. [2]

2. 2011/VJC/II/6

The events A and B are such that $\mathrm{P}(A)=\frac{2}{3}$, $\mathrm{P}(A\mid B)=\frac{4}{7}$ and $\mathrm{P}(B\mid A')=\frac{1}{3}$. Find

(i) $\mathrm{P}(A\cup B)$, [3]

(ii) $\mathrm{P}(A\cap B)$. [3]

3. 2010/NJC/II/8(a)

In a certain sample space, it is known that events A and B are independent. Given that $\mathrm{P}(A\cup B)=\frac{3}{4}$ and $\mathrm{P}(A'\cap B)=\frac{2}{15}$, where A' is the complement of event A, find

(i) $\mathrm{P}(B)$, [3]

(ii) $\mathrm{P}(A\cap B\mid A\cup B)$. [2]

4. 2011/IJC/II/6

A box contains twenty chocolates which are identical apart from their flavours. Five of the chocolates are caramel-flavoured, seven are mint-flavoured and eight are strawberry-flavoured. John randomly selects three chocolates from the box. Find the probability that

(i) exactly two of the three chocolates are mint-flavoured, [2]

(ii) at least two of the three chocolates have the same flavour, [2]

(iii) exactly one chocolate is strawberry-flavoured given that at least two of the three chocolates have the same flavour. [3]

5. 2011/HCI/II/10

A box contains four red and eight blue balls of which two of the red and six of the blue balls have the number 2 printed on them. The remaining balls have the number 3 printed on them. Three balls are randomly drawn from the box without replacement. Find the probability that

(i) at least one blue ball is drawn, [1]

(ii) at least one ball of each colour is drawn, [2]

(iii) the sum of the numbers on the balls drawn is at least 8, [2]

(iv) the sum of the number on the balls drawn is at least 8 given that at least one blue ball is drawn. [3]

State, with a reason, whether the events 'the sum of the numbers on the balls drawn is at least 8' and 'at least one blue ball is drawn' are independent. [1]

6. 2011/ACJC/II/7

A survey shows that 20% of JC students sleep before 11 p.m., 70% of them sleep between 11 p.m. and 1 a.m., and the remaining sleep after 1 a.m. The probability that a student is late for school the next day is 2%, 5% and 10% if he/she slept before 11 p.m., between 11 p.m. and 1 a.m., and after 1 a.m. the night before, respectively.

(i) Show that the probability that a randomly chosen student is late for school is 0.049. [2]

(ii) Find the probability that a randomly chosen student slept before 11 p.m. the night before, given that he/she is not late for school. [3]

There are five school days in one week.

(iii) What is the probability that a randomly chosen student is on time for school every day in a given week? [2]

(iv) Given that out of n weeks, the probability that a student is on time for school every day in at most eight of the weeks is at most 0.14, find the least value of n. [4]

7. 2011/PJC/II/9

In a group of 100 students, 25 own an IPod, 40 own an IPhone, and 35 own either an IPod or an IPhone, but not both. Find the probability that a student chosen at random

(i) owns both an IPod and an IPhone, [2]

(ii) does not own an IPod or an IPhone, [2]

(iii) owns an IPod, given that he owns an IPhone. [2]

8. **2008/SAJC/Prelim**

A student has requested for a testimonial to apply for a job. She estimates that there is an 80% chance of getting the job if she receives an excellent testimonial, a 40% chance if she receives a moderately good testimonial and a 10% chance if she receives a fair testimonial. She further estimates that the probabilities that the testimonial will be excellent, moderate or fair are 0.7, 0.2 and 0.1 respectively.

(i) Draw a tree diagram to illustrate the information and calculate the exact probability of the student getting the job. [2]

(ii) Given that she receives the job offer, what is the exact probability that she received an excellent testimonial? [2]

(iii) Given that she did not receive a moderate testimonial, what is the exact probability that she did not receive a job offer? [2]

(iv) If there are n students, who have the exact same chance of obtaining the kind of testimonials as this student, what is the probability of at least 1 student obtaining an excellent testimonial? Leave your answer in terms of n. [2]

9. **2011/DH/II/7**

[In this question give all answers correct to 3 places of decimals.]

A computer game consists of at most 3 stages. The probability that a player clears the first stage (i.e. successful) is 0.7. For each subsequent stage,

- the conditional probability that a player clears that stage, given that he cleared the preceding stage, is half the probability of success at the preceding stage,
- the conditional probability that a player clears that stage, given that he failed at the preceding stage, is the same as the probability of success at the preceding stage.

The game ends prematurely if a player fails to clear 2 consecutive stages.

Draw a probability tree diagram to represent the above information. [2]

Find the probability that

(i) the game ends prematurely, [1]

(ii) a player clears exactly 2 stages, [3]

(iii) a player clears the third stage given that he cleared exactly 2 stages. [2]

10. **2011/NYJC/II/9**

Players A and B compete in a racquet match consisting of at most 3 sets. Each set is won by either Player A or B, and the match is won by the first person to win two sets. Player A has a probability of $\frac{2}{3}$ of winning each of the first two sets. If the match goes into the third set, Player A has a probability of p of winning this set.

(i) With the aid of a tree diagram, find in terms of p, the probability that player A will win the match. [3]

(ii) Deduce the range of values of the probability that player A wins the match. [2]

(iii) Show that the value of p in order for the match to be fair is $\frac{1}{8}$. [1]

(iv) Given that Player A wins the match, find the probability that he won the second set. [3]

11. **2011/NJC/II/7**

A set of 20 playing cards is made up of different coloured cards, each with a picture of an animal on it. The number of cards of each type are given in the following table:

	Panda	Tigress	Peacock	Monkey
Red	1	1	1	1
Yellow	3	2	2	1
Blue	3	2	2	1

(i) One card is drawn at random and events R, Y and T are as follows:

R : The card drawn is red

Y : The card drawn is yellow

T : The card drawn shows a Tigress

Find

(a) $\mathrm{P}(R' \cup T')$, **(b)** $\mathrm{P}(T \mid Y')$.

Determine whether T and Y' are independent events. Justify your answer. [4]

(ii) Three cards are drawn from the set, at random and without replacement. Find the probability of obtaining two pictures of a Monkey and a picture of a Panda. [2]

(iii) A game is played by drawing one card at a time from the set with replacement, until a picture of the Peacock is obtained. Find the least value of n such that

$$\mathrm{P}(\text{at most } n \text{ cards are needed to end the game}) > 0.99 .$$ [3]

12. 2008/HCI/Prelim

A bag initially contains 2 white and 6 black balls. A game is played by drawing 3 balls, one at a time, from the bag. Each time a ball is drawn, its colour is noted and it is replaced in the bag along with 2 other balls of the same colour.

To win the grand prize, a player has to draw 3 white balls but if he obtains only 2 white balls in his 3 draws, he gets a consolation prize. Otherwise, he will walk away empty handed.

(i) Show that the probability that a player will win the grand prize is $\frac{1}{20}$. [1]

Find the probability that

(ii) a player wins a consolation prize, [3]

(iii) a player wins a consolation prize, given that his first draw is a white ball. [3]

A player plays the game four times. Find the probability that he wins at least 2 grand prizes in his 4 attempts at the game. [3]

13. 2008/ACJC/Prelim

A street ball match between Singapore United and Singapore Rovers ends in a draw and the match is to be decided on the result of penalty kicks. One round of penalty kicks consists of two kicks, one kick being taken by each team. If both teams score or if both teams fail to score, then another round of penalty kicks is played. The match is won when, in a round of penalty kicks, one team scores and the other team fails to score.

The probability that Singapore United scores a goal with a penalty kick is 0.8 and independently, the probability that Singapore Rovers scores a goal with a penalty kick is 0.9.

(i) Find the probability that the result of the match is still undecided after 1 round. [2]

(ii) Find the probability that Singapore United wins the match in less than 3 rounds of penalty kicks given that Singapore United scores a goal in the first round of penalty kicks. [3]

(iii) Find the least number of rounds of penalty kicks n such that the probability that the result of the match is decided in at most n rounds of penalty kicks is greater than 0.98. [4]

14. 2010/JJC/II/6

In a badminton team of 8 players, 5 are boys and 3 are girls. Boy A and Girl B are the only 2 left-handed players in the team. In a particular practice, 4 players are chosen to play doubles. Find the probability that

(i) exactly 1 left-handed player is chosen, [2]

(ii) 2 girls are chosen given that exactly 1 left-handed player is chosen, [3]

(iii) either Boy A or Girl B is chosen (or both). [2]

15. 2010/HCI/II/10

A public opinion poll surveyed a sample of 1000 voters. The table below shows the number of males and females supporting Party A, Party B and Party C.

	Party A	Party B	Party C
Male	200	130	70
Female	250	300	50

(a) One of the voters is chosen at random. Events A, C and M are defined as follows:

A : The voter chosen supports Party A.

C : The voter chosen supports Party C.

M : The voter chosen is a male.

Find

(i) $P(A \mid M)$,

(ii) $P(M' \cap C')$.

Determine whether A and M are independent. [4]

(b) It is given that in the sample, 20% of Party A supporters, 30% of Party B supporters and 5% of Party C supporters are immigrants.

(i) One of the voters selected from the sample at random is an immigrant. What is the probability that this voter supports Party A? [2]

(ii) Three voters are chosen from the sample at random.
Find the probability that there is exactly one immigrant voter who supports Party C or exactly one female who supports Party A (or both). [4]

16. 2015/SAJC/II/10

A code consists of a letter followed by three numbers, and then, followed by another letter. The code is generated by a computer and thus, each of the two letters generated is equally likely to be any of the twenty-six letters of the alphabet A–Z. Each of the three digits generated is equally likely to be any of the ten digits 0–9.

(i) Find the probability that a randomly chosen code has three different digits and two different letters. [2]

A palindrome code is defined as a code that reads the same backward and forward. Examples of a palindrome code are A121A, B343B, G111G.

(ii) Find the probability that a randomly chosen code is a palindrome code. [2]

(iii) Find the probability that a palindrome code contains both the digits 2 and 3. [3]

(iv) Showing all necessary calculations, determine if the events of a code being palindrome and a code containing the digits 2 and 3 are independent. [3]

17. 2015/HCI/II/6

(a) Let A and B be events such that

$$\mathrm{P}(A'\cap B)=0.13,\ \mathrm{P}(A'\cap B')=0.38 \text{ and } \mathrm{P}(A\mid B)=0.675.$$

(i) $\mathrm{P}(A\cap B)$. [3]

(ii) Determine whether A and B are independent. [2]

(b) Two players, C and D, compete in a match. The probability that C wins the first set is p. For each set after the first, the conditional probability that any player wins the set, given that the player won the preceding set, is p. Each set is won either by player C or D. If there is no limit to the number of sets played, and a match is won only when a player wins two consecutive sets, show that the probability that C wins the match is $\dfrac{1-p+p^2}{2-p}$. [3]

18. 2015/MJC/II/11

(a) A bag contains 25 balls that are indistinguishable apart from their colours. 15 of the balls are red and the rest are blue. 8 balls are drawn at random from the bag, without replacement. The number of red balls drawn is denoted by R.

(i) Show that $\mathrm{P}(R=3)=0.106$ correct to 3 significant figures. [2]

(ii) The most probable number of red balls drawn is denoted by r. By using the fact that $\mathrm{P}(R=r)>\mathrm{P}(R=r+1)$, show that r satisfies the inequality

$$(r+1)!(14-r)!(7-r)!(3+r)!>r!(15-r)!(8-r)!(2+r)!$$

and use this inequality to find the value of r. [5]

(b) Kaylyn has an option of 2 routes to travel to school everyday. The probability that she chooses the first route is denoted by p. There is a 90% chance that she gets to school early using the first route. If she chooses the second route, there is a 85% chance that she will be early. For a general value of p such that $0\le p\le 1$, the probability that Kaylyn chooses the first route when she gets to school early is denoted by $\mathrm{f}(p)$.

Show that $\mathrm{f}(p)=\dfrac{18p}{17+p}$ and prove by differentiation that as p increases, $\mathrm{f}(p)$ increases at a decreasing rate. [4]

Topic 13: Discrete Random Variables

1. **N2002/II/12**

A box contains 4 red balls and 2 green balls. Three balls are taken at random from the box, without replacement. The number of green balls obtained is denoted by G. Show that $\mathrm{P}(G=2)=\frac{1}{5}$, and find the probability distribution of G. [4]

Find the variance of G. [3]

2. **N79/I/11**

A discrete random variable R takes values between 0 and 4 inclusive with probabilities given by

$$\mathrm{P}(R=r)=\begin{cases}\dfrac{r+1}{10}, & r=0,1,2,\\ \dfrac{9-2r}{10}, & r=3,4,\\ 0, & \text{otherwise.}\end{cases}$$

Find the expectation and variance of R.

3. **J80/I/11**

A discrete random variable X takes values 0, 1, 2, 4, 8 with probabilities as shown in the table.

x	0	1	2	4	8
$\mathrm{P}(X=x)$	p	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

(i) Evaluate p.

(ii) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(iii) Write down the values of $\mathrm{E}(2X+3)$ and $\mathrm{Var}(2X+3)$.

(iv) X_1 and X_2 are two independent observations of X. Find $\mathrm{E}(X_1+X_2)$ and, using the value of p found in **(i)** above, find $\mathrm{P}(X_1+X_2=2)$.

4. **J95/II/6**

An unbiased disc has a single dot marked on one side and two dots marked on the other side. The disc and an unbiased die are thrown and the random variable X is the sum of the numbers of dots showing on the disc and on the top of the die. Tabulate the probability distribution of X. [3]

Show that $\mathrm{P}(X\ge 4 \mid X\le 7)=\frac{8}{11}$. [3]

Write down $\mathrm{E}(X)$ and show that $\mathrm{Var}(X)=\frac{19}{6}$. [4]

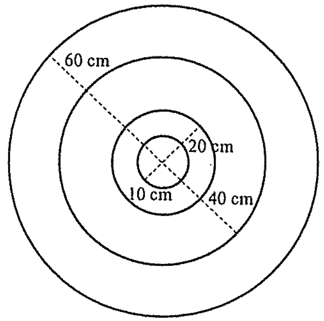
Two independent observations X_1 and X_2 are taken of X. Find

(i) $\mathrm{Var}(X_1-X_2)$, [1]

(ii) $\mathrm{P}(X_1-X_2\ge 5)$. [3]

5. **2017/IJC/II/7**

An archer shoots an arrow into a circular target board that has a radius of 60 cm. The target board further consists of three inner concentric circular sections, with radii 40 cm, 20 cm and 10 cm respectively as shown in the diagram.



The archer scores

- 50 points if the arrow lands in the centre circle of radius 10 cm,
- 20 points if the arrow lands in the ring with outer radius 20 cm,
- 10 points if the arrow lands in the ring with outer radius 40 cm,
- 0 point otherwise.

Assume that the arrow will definitely hit the target board and is equally likely to hit any portion of the target board.

(i) Let X be the number of points scored for one arrow shot. Find the expectation of X, leaving your answer in 4 significant figures. [3]

(ii) Interpret, in this context, the value obtained in part **(i)**. [1]

(iii) The archer shot at the target board forty times. Find the probability that the average score obtained by the archer is between 10 and 20 points (inclusive). [4]

6. N2000/II/6

A computer generates a random variable X whose probability distribution is given in the following table.

x	0	2	4	6
$\mathrm{P}(X=x)$	0.1	0.2	0.3	0.4

Show that $\mathrm{Var}(X)=4$. [3]

Find $\mathrm{E}(X^4)$ and $\mathrm{Var}(X^2)$. [4]

Two independent observations of X are denoted by X_1 and X_2.

Show that $\mathrm{P}(X_1+X_2=6)=0.2$, and tabulate the probability distribution of X_1+X_2. [4]

The sum of 100 independent observations of X is denoted by S. Describe fully the approximate distribute of S. [3]

7. 2017/PJC/II/6

An unbiased disc has a single dot marked on one side and two dots marked on the other side. A tetrahedral die has faces marked with score of 1, 2, 3, and 4. The probability of getting a score of 1, 2, 3, and 4 are $\frac{1}{5}, p, \frac{1}{5}$ and q respectively, where $p, q \in [0,1]$.

A game is played by throwing the disc and the die together. The random variable S is the sum of the score showing on the die and twice the number of dots showing on the disc.

(i) Find $\mathrm{P}(S=6)$. [2]

Given that $\mathrm{P}(S=4)=\frac{1}{6}$,

(ii) calculate the values of p and q. [2]

(iii) Find the probability distribution of S. [2]

8. 2013/HCI/II/6 modified

In a carton of apples, a sample of 8 apples is taken and examined for spoilt apples.

(i) State, in context, an assumption for the number of spoilt apples in the sample to be modelled by a binomial distribution. [1]

The number of spoilt apples in a random sample of size 8 may be modelled by the distribution $\mathrm{B}(8, p)$. If at least 2 apples in a sample are found to be spoilt, the carton is rejected. It is known that the probability of a carton being rejected is 0.04.

(ii) Write down an equation satisfied by p and find the value of p. [3]

(iii) 60 cartons of apples are loaded onto a lorry. Find the probability that more than 56 cartons loaded onto the lorry are not rejected. [3]

9. 2017/CJC/II/10

A box contains 2 red balls, 3 green balls and x blue balls, where $x \in \mathbb{Z}, x \geq 5$. A game is played where the contestant picks 5 balls from the box without replacement. The total score, S, for the contestant is the sum of the number of green balls chosen and thrice the number of red balls chosen. The blue balls will not contribute any points, unless all 5 balls are blue. If all the 5 balls are blue, the score will be 25 points.

(i) Show that $\mathrm{P}(S=6)=\dfrac{20x(x^2-3x+14)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$. [2]

(ii) Given that $\mathrm{P}(S=6)=\frac{5}{63}$, calculate x. [2]

(iii) Complete the probability distribution table for S. [3]

s	1	2	3	4	5	6	7	8	9	25
$\mathrm{P}(S=s)$		$\frac{5}{42}$	$\frac{5}{63}$		$\frac{5}{21}$	$\frac{5}{63}$		$\frac{5}{84}$	$\frac{1}{252}$	

(iv) Evaluate $\mathrm{E}(S)$ and find the probability that S is more than $\mathrm{E}(S)$. [2]

(v) Find the probability that there are no green balls drawn given that S is more than $\mathrm{E}(S)$. [2]

10. 2017/DHS/II/5

A new game has been designed for a particular casino using two fair dice. In each round of the game, a player places a bet of \$2 before proceeding to roll the two dice. The player's score is the sum of the results from both dice. For the scores in the following table, the player keeps his bet and receives a payout as indicated.

Score	Payout
9 or 10	\$1
2 or 4	\$5
11	\$8

For any other scores, the player loses his bet.

Let X be the random variable denoting the winnings of the casino from each round of the game.

(i) Show that $\mathrm{E}(X)=\frac{1}{12}$ and find $\mathrm{Var}(X)$. [4]

(ii) $\bar{X}$ is the mean winnings of the casino from n rounds of this game. Find $\mathrm{P}(\bar{X}>0)$ when $n=30$ and $n=50\,000$. Make a comparison of these probabilities and comment in context of the question. [3]

11. 2013/SRJC/II/8 Modified

On a random day, the Pollutants Standard Index (PSI) readings are updated every hour. The chance that a random PSI reading in a day will hit more than 200 is $\frac{1}{8}$.

A day is considered a "hazy" day if the PSI readings hit more than 200 for more than seven times in a day.

(i) Show that the probability that a day will be considered a "hazy" day is 0.00683, correct to 5 decimal places. [2]

(ii) State an assumption needed for the distribution used in part (i) to be valid. [1]

(iii) Find the least number of days such that the probability of having less than one "hazy" day is less than 0.5. [4]

(iv) Find the probability that the number of "hazy" days in a year with 365 days is not less than three. [3]

12. 2013 MJC/II/8

(a) The random variable Y has a binomial distribution with mean 1.6 and $\mathrm{P}(Y=0)=0.1296$. Find $\mathrm{P}(Y>2)$. [4]

(b) A car park has a large number of parking lots. 80 parking lots are observed and the number of occupied parking lots is denoted by X. State, in context, two assumptions needed for X to be well modelled by a binomial distribution. [2]

The probability of a parking lot being occupied is 0.95, find the probability that out of the 80 chosen parking lots, at least 90% are occupied. [4]

13. 2014/SAJC/II/12 modified

In a large consignment of eggs, p% of the eggs is damaged. Eggs are sold in trays of 30. The number of damaged eggs in a tray is denoted by X.

(i) State, in context, what must be assumed for X to be well modelled by a binomial distribution. [1]

Assume now that X has a binomial distribution.

(ii) Given that the probability of a tray containing at most 1 damaged egg is 0.87945, show that $(100-p)^{29}(100+29p)=8.7945\times10^{59}$ and hence find p. [3]

(iii) A hawker bought 40 trays of eggs. A tray of eggs is rated Grade S if it contains more than 1 damaged egg. Find the probability that the hawker bought at least 4 Grade S trays of eggs. [3]

(iv) Each week, a supermarket ordered 100 cartons, each consisting of 40 trays of eggs. A carton will be rejected if it contains at least 4 Grade S trays. Find the probability that the mean number of rejected cartons per week over a period of 52 weeks is not more than 73. [4]

14. 2014/IJC/II/9

In a standardized assessment test conducted annually, it is found that, on average, 1 in 5 students who sit for the test are awarded distinction.

(i) Out of 23 students who sit for the test, find the expected number of students who will be awarded distinction. [1]

(ii) Find the minimum number of students who must sit for the test so that the probability of having at least 9 students being awarded distinction is more than 0.7. [3]

It is further found that, on average, 93% of students who sit for the test pass the test.

(iii) 60 students from a junior college sit for the test. Find the probability that more than 55 students pass the test. [3]

15. 2015/DHS/II/7 Modified

Data is transmitted in bytes, where each byte consists of 8 bits. The probability of a bit being corrupted during its transmission is 0.03. A byte is considered 'corrupted' if it contains at least 2 corrupted bits. Assume that all bits are not corrupted prior to their transmission.

(i) Show that the probability that a randomly chosen byte is corrupted during its transmission is 0.0223. [1]

(ii) Given that a randomly chosen byte is not corrupted during its transmission, find the probability that it contains no corrupted bits. [3]

(iii) Find the probability that between 5 and 10 bytes are corrupted during the transmission of 100 bytes. State the parameter(s) of the distribution you use. [3]

16. N2015/JJC/II/10

Pauline frequently uses a social networking website to chat with her friends in the evening. On this site, she has a total of 120 friends. If she logs on to the site at 9.30 pm, on average, 35% of her friends will also be logged on to the site.

Pauline logs on to the site at 9.30 pm on a particular day. The number of her friends who are also logged on to the site is denoted by L.

(i) State, in context, two assumptions needed for L to be well modelled by a binomial distribution. [2]

(ii) Explain why one of the assumptions stated in part (i) may not be valid in this context. [1]

Assume now that these assumptions do in fact hold.

(iii) If 8 of her friends on this site are chosen at random, find the probability that more than half of them are logged on to the site. [2]

(iv) Find the greatest value of n such that the probability that at least n of her 120 friends are logged on to the site is more than 0.8. [4]

17. 2016/RJC/II/11 Modified

A chocolate shop puts gift vouchers at random into 7% of all their packets of mini chocolates produced. A customer must collect 3 vouchers to exchange for a gift.

(i) Adeline buys 8 packets of the mini chocolates. Find the probability that she gets exactly 2 gift vouchers. [2]

(ii) Aileen buys 31 packets of the mini chocolates. Find the probability that she is able to exchange for at least one gift. [2]

(iii) Ashley buys n packets of the mini chocolates. Given that she already has 2 unused vouchers from her previous purchase, find the value of n for which the probability of her being able to exchange for exactly one gift is greatest. [3]

(iv) The shopkeeper observes that the number of gifts exchanged in a day has a mean of 10 and variance of 25. Estimate the number of gifts the shop needs to stock if there is to be no more than a 5% chance of running out of gifts in a 40-day period. [3]

18. N82/I/13

The random variable X is the number of successes in n independent trials of an experiment in which the probability of success at any one trial is p.

Show that $\dfrac{P(X=k+1)}{P(X=k)}=\dfrac{(n-k)p}{(k+1)(1-p)}$, $k=0,1,2,\ldots,(n-1)$.

Find the most probable number of successes when $n=10$ and $p=\dfrac{1}{4}$.

Topic 14: Normal Distribution & Sampling

1. 2011/SRJC/II/10(a)

A random variable Z has the distribution $N(0, 1)$. Given that $P(Z>-k)=p$, where $k>0$, determine $P(-k<Z<k)$ in terms of p, leaving your answer in the simplest form. [2]

2. 2010/PJC/II/6(b)

The amount of tips a diner gives after a meal in a restaurant has mean \$5 and standard deviation \$1. Estimate the probability that the amount of tips collected from 80 randomly chosen customers is between \$350 and \$410. [3]

3. Haese Mathematics for the international student HL (Core) 3rd Edition p834 Ex 26E.2 Q8

Circular metal tokens are used to operate a washing machine in a laundromat. The diameters of the tokens are normally distributed, and only tokens with diameters between 1.94 cm and 2.06 cm will operate the machine.

(i) Find the mean and standard deviation of the distribution given that 2% of the tokens are too small and 3% are too large.

(ii) Find the probability that at most one token out of a randomly selected sample of 20 will not operate the machine.

4. 2011/MI/II/9

The masses, in kg, of a D24 durian and a Mao Shan Wang durian are independent random variables with the distributions $N(1, 0.25^2)$ and $N(1.6, 0.5^2)$ respectively. The costs, per kg, of one D24 durian and one Mao Shan Wang durian at a particular store are \$12.00 and \$22.00 respectively. Find the probability that

(i) the mass of three D24 durians is more than twice the mass of one Mao Shan Wang durian, [3]

(ii) the total cost of three D24 durians and two Mao Shan Wang durians is more than \$105. [3]

(iii) In a sample of n D24 durians, it is known that the probability that the mean mass of a D24 durian in this sample is more than 1.05 is less than 0.08. Find the least value of n. [4]

5. 2011/HCI/II/9

A cab company charges its customers according to the distance travelled and the waiting time when the cab is stationary. The distance travelled, in km, and the waiting time, in minutes, of a customer who travels from Statistics Road to Pure Math Street in a single trip, are modelled as having independent normal distributions with means and standard deviations as shown in the table below.

	Mean	Standard deviation
Distance travelled, in km	8	0.7
Waiting time, in minutes	5	2.1

(a) Find the probability that the total distance travelled in three randomly chosen trips differs from thrice the distance travelled in another randomly chosen trip by at least 3 km. [4]

Customers pay \$0.50 for every km travelled and \$0.30 for each minute of waiting time. In addition, customers have to pay a fixed flag down charge of \$2.80 per trip.

(b) **(i)** Calculate the mean and variance of the total fare paid in a randomly chosen trip.[2]

(ii) Find the probability that out of 10 randomly chosen trips, a customer is charged less than \$8 in at least 6 trips. [3]

6. 2011/IJC/II/10

The mass, in grams, of a randomly chosen jar of Tasty brand jam is a random variable with the distribution $N(300, 4^2)$. The mass, in grams, of a randomly chosen Yummy brand jam is a random variable with the distribution $N(350, 5^2)$.

(i) Find the probability that the masses of 2 randomly selected jars of Tasty brand jam differ by more than 10g. [3]

(ii) Find the probability that out of four randomly chosen jars of Yummy brand jam, exactly one weighs more than 355g and the other three weigh not more than 345g each. [3]

(iii) A crate contains ten jars of Tasty brand jam and five jars of Yummy brand jam. Find the probability that the average mass of fifteen jars of jam in a randomly chosen crate lies between 317g and 322g. [4]

7. 2011/DH/II/9

In a certain junior college, the marks (out of 100) scored by a JC 1 student in a Class Test, Common Test and Promotional Examination are denoted by C, T and S respectively. C, T and S may be modelled by normal distributions with means and standard deviations as shown in the table below.

Type of assessment	Mean	Standard deviation
Class Test, C	68	α
Common Test, T	65	8
Promotional Examination, S	70	10

(i) Given that $P(C > 85) = 0.05$, determine the value of α. [2]

In a particular year, a student sits for five Class Tests, a Common Test and a Promotional Examination. The average mark of the five Class Tests constitutes 20% of the overall assessment mark for the year. The Common Test and Promotional Examination constitute 20% and 60% of the overall assessment mark for the year respectively.

For the following parts, assume $\alpha = 10$.

(ii) Find the probability that the average mark scored by a student in the five Class Tests is more than 75. [3]

(iii) Find the probability that a student scores an overall mark of more than 80 for the year. [4]

(iv) State an assumption used in your calculations for (iii). [1]

8. 2011/RJC/II/12(i)-(ii)

The weight, x kg, of each student in a random sample of 120 students from a secondary school is measured, and the results are summarized by

$$\sum(x-50) = -100, \quad \sum(x-50)^2 = 1158.$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Another random sample of n students ($n \geq 50$) is taken from the school. Given that the probability of the sample mean weight exceeding 49.9 kg is at most 0.01, find the least value of n. [4]

9. **2011/YJC/II/8**

A fruit seller grades apples according to their mass. It is given that the mass of a randomly chosen apple follows a normal distribution with mean μ g and standard deviation 30 g. Apples with a mass exceeding 150 g are graded as 'large' while apples with a mass less than 70 g are graded as 'small'. The proportion of 'large' apples is the same as the proportion of 'small apples'.

(i) Explain why μ is 110 g. [1]

(ii) Find the probability that the total mass of two randomly chosen apples exceeds 230g. [2]

The fruit seller also grades oranges according to their mass. It is given that the mass of a randomly chosen orange has an independent normal distribution with mean 190 g and standard deviation 24 g. The fruit seller sells the apples at \$0.20 per 100 g and the oranges at \$0.15 per 100 g.

(iii) Find the probability that the average cost of an apple and two oranges exceeds \$0.25. [3]

10. **2011/TPJC/II/9 (modified)**

In an office building in Shenton Way, there are 640 male employees and 560 female employees. The weights, in kg, of male employees and female employees are modelled as having independent normal distributions with mean and standard deviations as shown in the table.

	Mean weight	Standard deviation
Male	68	2
Female	50	2

(i) Calculate the probability that the total weight of 4 female employees is less than three times the weight of a male employee. [3]

(ii) Calculate the probability that the mean weight of a random sample of 80 male employees differs from their mean by at most 0.5 kg. [3]

(iii) 25 employees are randomly chosen of which k of them are male. If the probability that the total weight of these 25 employees exceeding 1500 kg is approximately 0.987, find the value of k. [5]

11. **2010/CJC/II/8**

The masses of snapper fish and pomfret fish sold by a fishmonger are normally distributed and independent of each other. The mean mass, standard deviation and selling price of snapper fish and pomfret fish are given in the following table:

	Snapper fish	Pomfret fish
Mean mass in kg	1	0.6
Standard deviation in kg	0.1	0.05
Selling price per kg in \$	12	7

Find the probability that the

(i) total mass of 3 snapper fish and 2 pomfret fish is more than 4.5 kg. [2]

(ii) mass of 3 snapper fish exceeds twice the mass of a pomfret fish by more than 1.85 kg. [2]

(iii) total selling price of a snapper fish and 2 pomfret fish is more than \$21. [2]

A customer buys 15 fish, out of which n are snapper fish and the rest are pomfret fish. The probability that the customer pays more than \$150 is less than 0.7. Find the largest value of n. [4]

12. **2010/DHS/II/11**

(a) The distance, X km, covered by a school athlete during a regular training session has mean 4 km. During competition season, training increases in intensity and the distance, Y km, covered during the training session increases to a mean of 6 km. Given that X and Y are independent normal distributions with same variance σ^2, and $\mathrm{P}(Y - X > 3) = 0.4$, find the probability that the athlete covers a total distance between 8 km and 12 km in two randomly chosen regular training sessions. [5]

(b) The amount of time that a student spends online each day has mean 120 minutes and standard deviation 45 minutes. A random sample of 60 students is taken and they are surveyed on the amount of time that they spend online each day. Find the probability

(i) that the total time spent online each day by the 60 students is at least 7000 minutes, [3]

(ii) that the sample mean time spent online each day by the 60 students is within 5 minutes of the population mean time of 120 minutes. [3]

Explain whether you need to assume that the amount of time spent online by a student each day follows a normal distribution in your calculations above. [1]

13. 2010/MI/II/9

Durians and mangoes are sold by mass. The masses, in kg, of durians and mangoes are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard Deviation
Durians	1.6	0.2
Mangoes	0.3	0.05

Durians are sold at \$8 per kg and mangoes at \$3 per kg.

(i) Find the mass, m that will be exceeded by 80% of the durians. [1]

(ii) Find the probability that the total mass of 3 randomly chosen durians and 4 randomly chosen mangoes exceeds 6.5 kg. [3]

(iii) The mean mass of n randomly chosen durians is $\overline{D}$ kg. Given that $P(\overline{D} < 1.45) = 0.0122$, find the value of n. [3]

(iv) Find the probability that the total selling price of 3 randomly chosen durians and 4 randomly chosen mangoes is less than \$45. [3]

14. 2011/SRJC/II/12

An ornithologist, who studies the behavior of birds, captures one male and one female hornbill from a forest in Osaka, Japan. The masses of hornbills in that forest are assumed to follow normal distributions with male hornbills having mean 3500g and standard deviation 150g while female hornbills having mean 3000g and standard deviation σ.

(i) It is found from research that 5% of the female hornbills from the forest have masses exceeding 3.2kg. Show that $\sigma = 122$. [2]

(ii) Find the probability that the difference in mass between two randomly chosen male hornbills is at least 0.1kg. [3]

(iii) Find the probability that the mass of 5 randomly chosen female hornbills exceeds twice the mass of 2 randomly chosen male hornbills. [3]

(iv) Five male hornbills are randomly chosen. Find the probability that the fifth male hornbill is the third hornbill with mass exceeding 3.6kg. [3]

15. 2010/JJC/II/8

The random variable X has a normal distribution with mean 15 and variance 5. The random variable T is the sum of 2 independent observations of X.

(i) Find $P(T > 2 + 3X)$. [3]

(ii) Three independent observations of X are obtained. Find the probability that exactly two of the observations have value less than 20. [3]

The random variable Y has a normal distribution with mean μ and variance σ^2.

(iii) If $\sigma = 22.5$, find the greatest probability of $P(15.1 < Y < 29.9)$, stating the value of μ.[2]

(iv) If $\mu = 10$ and $P(X + Y > 27) = 0.25$, calculate the value of σ and state an assumption needed to carry out the calculation. [4]

16. 2011/NJC/II/10

The national average for monthly electricity usage measured in kilowatts hour (kWh), of Housing Development Authority (HDA) units is 380 kWh. The monthly electricity usage of 3-room units follows a normal distribution with mean of 290 kWh and variance, σ^2, whereas the monthly electricity usage of 5-room units follows an independent normal distribution with mean of 450 kWh and variance 105 kWh^2.

(i) Calculate the probability that the monthly electricity usage of two randomly chosen 3-room units exceeds 290 kWh each and a randomly chosen 5-room unit's monthly. [2]

(ii) Given that the probability of the total monthly electricity usage of four randomly chosen 3-room units exceeds thrice the national average is 0.868, find σ^2, correct to the nearest integer. [3]

With effect from 1 July 2010, the monthly electricity bill is charged at 24 cents per kilowatts hour.

(iii) Determine the value of a, correct to 2 decimal places, such that the probability of the monthly electricity bill of a randomly chosen 5-room unit exceeding \$$a$ is 0.9. [2]

Topic 15: Hypothesis Testing

1. 2010/MI/II/11

A beverage producer claims that each packet of soya bean milk he produces contains 250 ml of the drink. A consumer group took a sample of 50 packets and recorded the volume, x in ml of each packet. The results are summarized by: $\sum x = 12349$, $\sum x^2 = 3054283$.

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) Test, at 5% significance level, whether the producer is overstating the mean volume. [4]

(iii) In conducting the test in part **(ii)**, explain if there is a need to assume that the volume of packet of soya bean milk follows a normal distribution. [1]

(iv) In conducting the test in part **(ii)**, explain the meaning of 'a significance level of 5%' in the context of the question. [1]

(v) Given that the population variance of the volumes of packets of soya bean milk is 85, find the range of values of the sample size for which the producer's claim will not be rejected at a significance level of 5%. [3]

2. 2010/NJC/II/11(a)

The drying time, X minutes, of Noppin brand paint under specified test conditions is known to have mean value 75 minutes.

On one occasion, in the manufacture of a large batch of Noppin paint, it was suspected that an accidental chemical contamination had resulted in a change in the drying time of the paint. Due to the high costs involved in discarding the entire batch of paint, the manufacturer decided he will only do so if there was strong evidence from a test at 5% level of significance to suggest that the drying time has changed.

50 random and independent specimens of paint samples were taken from the batch and the drying time is summarised as follows: $\sum x = 3791$, $\sum x^2 = 287959$.

(i) Determine whether the manufacturer will discard the entire affected batch of paint.[6]

(ii) Assuming that the unbiased estimate of the population variance is the same as the one found in the above sample, find the probability such that the total drying time of another 60 randomly selected specimens of paint samples obtained from the same batch is between 72 hours and 76 hours. [3]

3. 2011/DHS/II/10(i)(ii)

Based on past data, the mean number of dengue cases per day was reported as 30. The authorities suspect that the recent hot and wet weather has increased the mean number of dengue cases per day.

A random sample of 60 days is taken and the number of dengue cases per day, X, is summarised by $\sum(x-8) = 1400$, $\sum(x-8)^2 = 34800$.

(i) Calculate unbiased estimates of the mean and variance of X. [2]

(ii) Test, at the 5% significance level, whether the mean number of dengue cases per day has increased. [4]

Explain, in the context of the question, the meaning of the p-value. [1]

4. 2014/YJC/II/8

An owner of a small factory uses a machine to dispense milk into bottles. The volume of milk dispensed is assumed to be a normal variable with a standard deviation of 8.5 ml.

(i) The mean volume, k ml, of milk dispensed into a bottle can be altered on the machine. Regulations stipulate that more than 99% of the bottles must contain at least 750 ml of milk. Find the least value of k. [3]

(ii) A random sample of 100 bottles of milk is chosen and a test is carried out at the 5% significance level, to determine whether the mean volume of milk dispensed is less than 750 ml. Find the range of values of the mean volume, $\bar{x}$ ml, of the 100 bottles of milk for which the null hypothesis would be rejected. [4]

5. 2016/SRJC/II/10 Parts

In the latest Pokkinon Roll game, players go to a battle arena to use their Pokkinon character to battle against each other. Alvin and Billy are interested to know how long it takes before someone wins a battle. The time taken by a randomly chosen player to win a game follows a normal distribution.

Alvin claims that on average, it will take at most 190.0 seconds to win a battle. To verify his belief, he surveyed a randomly chosen sample of 45 Pokkinon Roll gamers and found out that the mean is 195.0 seconds with a variance of 206.0 seconds^2.

Carry out an appropriate test at 1% level of significance whether there is any evidence to doubt Alvin's claim. [5]

6. 2010/RJC/II/8

The mass of a male student in Aishan Secondary School is denoted by X kilograms. The masses of a random sample of 150 male students are summarized by

$$\sum x = 8400 \quad \text{and} \quad \sum (x-56)^2 = 5555.$$

(i) Calculate unbiased estimates of the mean and variance of X. [2]

Aishan Secondary School claimed that the mean mass of a male student in the school is 55 kilograms.

(ii) Test, at the 3% significance level, whether the school is understating the mean mass of a male student. Does the Central Limit Theorem apply in this context? [5]

(iii) State what you understand by the expression 'at the 3% significance level' in the context of the question in **(ii)**. [1]

7. 2015/DHS/H1/11

It is known that the mean height of a population of sunflower seedlings is 15.7 cm. Two botanists, Abbey and Betty are conducting a research on the treatment of sunflower seedlings with an extract made from *Vinca minor* roots.

To determine whether the extract results in a change in mean height, Abbey obtained a random sample of n seedlings that were treated with the extract. The height, x cm, of each plant is measured and the data are summarised by

$$n = 60, \quad \sum x = 1008.23, \quad \sum (x-\bar{x})^2 = 802.24.$$

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) What do you understand by the term 'unbiased estimate'? [1]

(iii) Test, at the 5% significance level, whether the population mean height has changed. [4]

(iv) State, giving a valid reason, whether it is necessary to assume that the heights of the sunflower seedlings are normally distributed. [1]

Betty decided to collect her own sample of n sunflower seedlings to test if the mean height of the seedlings has increased. Her sample yielded a sample mean of 16.12 cm and a sample variance of 6.46 cm^2.

(v) Given that $n > 50$, obtain an inequality involving n such that there is sufficient evidence at the 5% level of significance that there is an increase in the mean height of the seedlings. Hence find the set of values of n for this stated result. [5]

8. 2015/AJC/II/10 Modified

A farmer claims that the mean weight of the melons grown in his farm is at least m kg. A random sample of 10 melons is chosen and the weight, x kg, of each melon is recorded. The results are as follows:

1.037	0.914	1.019	1.234	1.110
1.417	1.008	0.846	1.105	1.331

Another random sample of 40 melons is weighed and the results are as follows:

$$\sum x = 45.738 \quad \text{and} \quad \sum x^2 = 72.576.$$

(i) By combining the 2 sets of data, find unbiased estimates of the population mean and population variance of the weight of a melon grown by this farmer. [2]

(ii) Based on the combined sample of size 50, find the set of values of m for which the farmer's claim is not rejected at the 5% significance level. [3]

9. 2015/HCI/II/11

In a school, the time (in seconds) for a boy to complete a 4×10 m shuttle run is denoted by X. The expected value of X is taken to be 10.8 seconds.

(a) A teacher from the school tries a new teaching strategy on a random sample of 50 boys to help them improve on their 4×10 m shuttle run timings. The timings, y (in seconds), for each of these 50 boys to complete a 4×10 m shuttle run after the implementation of the new teaching strategy are summarised by

$$\sum y = 517.49, \qquad \sum (y-\bar{y})^2 = 122.32.$$

A test at α % level of significance using this sample results in sufficient evidence to conclude that the new teaching strategy is effective. Find the minimum value of α correct to 2 decimal places. State, giving a reason, whether it is necessary to assume that the population is normally distributed. [4]

(b) The population standard deviation for the boys in the school to complete a 4×10 m shuttle run is now assumed to be 2.8 seconds. A large random sample of boys of size n is taken from the school. The probability that their mean time to complete a 4×10 m shuttle run exceeds the actual population mean by at least 0.8 seconds is not more than 0.001. Form an inequality involving n and hence find the least possible value of n. [4]

10. 2015/AJC/H1/8(b)

In a certain housing estate, the amount of electricity consumed by a household in a month, X kilowatt hours (kWh), is normally distributed. The town council manager of the housing estate claims that the amount of electricity consumed in a month by a household in the estate is 120 kWh. To test his claim, a group of students took a random sample of 55 households from the estate, and the result is summarised by

$$\sum(x-120)=725 \quad \text{and} \quad \sum(x-120)^2=99801.$$

(i) Calculate the unbiased estimates of the population mean and the population variance, giving your answers to 1 decimal place. [2]

(ii) The test conducted at the α % significance level shows that there is sufficient evidence to support the town council manager's claim. Find the set of values of α, giving your answer to one decimal place. [3]

(iii) In their report, the students concluded that based on the sample studied, the average amount of electricity consumed in a month by a household in Singapore is 120kWh. Explain with reasons, whether you agree with their conclusion. [1]

11. 2016/NYJC/II/10

At an early stage in analysing the marks, x, scored by a large number of candidates in an examination paper, the Examination Board takes the scores from a random sample of 250 candidates. The results are summarised as follows:

$$\sum x=11872 \quad \text{and} \quad \sum x^2=646193.$$

(i) Calculate unbiased estimates of the population mean and variance to 3 decimal places. [2]

(ii) In a 1-tail test of the null hypothesis $\mu=49.5$, the alternative hypothesis is accepted. State the alternative hypothesis and find an inequality satisfied by the significance level of the test. [4]

(iii) It is subsequently found that the population mean and standard deviation for the examination paper are 45.292 and 18.761 respectively. Find the probability that in a random sample of size 250, the sample mean is at least as high as the one found in the sample above. [2]

2. 2012/MJC/II/7

(a) Describe the difference in the hypothesis between a one-tailed test and a two-tailed test. [1]

(b) The mean weight of boys in a particular school is known to be m kg. A new weight gain programme was tried out on a random sample of 50 boys in this school. The weight of these 50 students, x, after the implementation of the new weight gain programme gave the following data:

$$\sum(x-30)=1279 \quad \text{and} \quad \sum x^2=155233.$$

A hypothesis test is carried out at the 5% significance level and it is found that the weight gain programme has been effective. Find the range of values of m. [5]

13. 2016/SRJC/H1/12(b)

A new car company "Telstar" claims that their latest car model is more economical in terms of its fuel consumption as compared to its previous models. The company claims that the car consumes at most 9.5 litres for a 100 km drive. A test is conducted on 60 such cars and their average fuel consumptions are recorded as shown in the following table.

Average fuel consumption in litres, for 100 km.	8.5	8.7	8.8	9.0	9.4	9.7	10.0	10.3	10.5
Number of cars	1	1	3	6	13	12	10	9	5

(i) Calculate the mean of this sample and the unbiased estimate of the population variance. [2]

(ii) Test at 5% level of significance whether the company's claim is justified. [3]

(iii) Is it necessary to assume that the fuel consumption of a car follows normal distribution? [1]

(iv) Explain the meaning of p-value in this context. [1]

(v) Another test is conducted on another 50 such cars. It is given that the population variance is 2.5 litres^2, find the range of values of the sample mean if the company's claim was rejected at 5% level of significance. [2]

14. 2017/HCI/II/11

Yummy Berries Farm produces blueberries and raspberries packed in boxes.

(a) Yummy Berries Farm claims that the mass, x grams, of each box of blueberries is no less than 125 grams. After receiving a complaints from consumers, the Consumers Association of Singapore (CASE) took a random sample of 50 boxes of blueberries from Yummy Berries Farm and the mass of each box was recorded. The data obtained are summarised in the table.

x (grams)	120	121	122	123	124	125	126	127	128	129	130
No. of boxes	3	6	6	6	3	10	3	4	6	2	1

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 10% level of significance, whether Yummy Berries Farm has overstated its claim.
State, giving a reason, whether any assumptions about the masses of boxes of blueberries are needed in order for the test to be valid. [6]

(b) The masses of boxes of raspberries, each of y grams, are assumed to have a mean of 170 grams with standard deviation 15 grams. CASE took a random sample of n boxes of raspberries and the mean mass of boxes of raspberries from the sample is found to be 165 grams. A test is to be carried out at the 5% level of significance to determine if the mean mass of the boxes of raspberries is not 170 grams. Find the minimum number of boxes of raspberries to be taken for which the result of the test would be to reject the null hypothesis. [4]

15. 2017/NJC/II/8

Heart rate, also known as pulse, is the number of times a person's heart beats per minute. Obesity is a leading preventable cause of death worldwide. It is most commonly caused by a combination of excessive food intake, lack of physical activity and genetic susceptibility. To examine the effect of obesity on heart rate, 70 obese teenagers are randomly selected and their heart rates h are measured in a resting state. The results are summarised as follows.

$n = 70 \qquad \sum h = 5411 \qquad \sum h^2 = 426433.$

The Health Promotion Board (HPB) wishes to test whether the mean heart rate for obese teenagers differs from the normal heart rate by carrying out a hypothesis test.

(i) Explain whether HPB should use a 1-tail test or a 2-tail test. [1]

(ii) Explain why HPB is able to carry out a hypothesis test without knowing anything about the distribution and variance of the heart rates. [2]

(iii) Find the unbiased estimates of the population mean and variance, and carry out the test at the 10% level of significance for the HPB. [6]

A researcher wishes to test whether obese teenagers have a **higher** mean heart rate. He finds that the mean heart rate for 80 randomly obese teenagers is 79.4, then carries out a hypothesis test at the 10% level of significance.

(iv) Explain, with justification, how the population variance of the heart rates will affect the conclusion made by the researcher. [3]

(v) Show that the probability of any normal variable lying within one standard deviation from its mean is approximately 0.683. [1]

By considering **(iv)** and **(v)**, explain why it is likely for the researcher to reject the null hypothesis in his test if it is assumed that heart rates follow a normal distribution at the resting state. [1]

Topic 16: Correlation Coefficient and Linear Regression

1. **2012 Jan/AQA MS-SS1B/2**

Dr Hanna has a special clinic for her older patients. She asked a medical student, Lenny, to select a random sample of 25 of her male patients, aged between 55 and 65 years, and, from their clinical records, to list their heights, weights and waist measurements.

Lenny was then asked to calculate three values of the product moment correlation coefficient based upon his collected data. His results were:

(a) 0.365 between height and waist measurement;
(b) 1.16 between height and weight;
(c) −0.583 between weight and waist measurement.

For each of Lenny's three calculated values, state whether the value is definitely correct, probably correct, probably incorrect or definitely incorrect. [3]

2. **2010/MJC/II/12**

The table below shows the number of monthly new car licences, x, issued by the government and the selling price of a car, \$$y$ in year 2009. A student wants to investigate how the selling price of a car depends on the number of monthly new car licences issued by the government.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
X	1432	1339	1664	1774	2055	2275	2221	2173	1749	1854	1360	2012
Y	42000	48000	37000	36000	32850	31800	31800	32050	36000	34500	46000	33000

(i) Plot a scatter diagram for the data and explain, in the context of the question, if a linear model is appropriate in the long run. [3]

(ii) State, with a reason, which of the following model is most appropriate to fit the data points. [2]

(A) $y = ae^x + b$

(B) $y = ax^2 + b$

(C) $y = a + \frac{b}{x}$

(iii) For the model chosen in **(ii)**, calculate the product moment correlation and comment on its value. [2]

(iv) Use an appropriate regression line to estimate the selling price of the car when the monthly number of new car licences issued is **(a)** 1300 and **(b)** 2000. Comment on the reliability of your answers. [4]

(v) The student concluded that the decrease in the number of new car licences issued causes the selling price of the car to rise. State, with a reason, whether you agree with this conclusion. [1]

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3. **2011 Jan/OCR/4732/3**

A firm wishes to assess whether there is a linear relationship between the annual amount spent on advertising, £x thousand, and the annual profit, £y thousand. A summary of the figures for 12 years is as follows:

$n = 12$, $\sum x = 86.6$, $\sum y = 943.8$, $\sum x^2 = 658.76$, $\sum y^2 = 83663.00$, $\sum xy = 7351.12$.

(i) Calculate the product moment correlation coefficient, showing that it is greater than 0.9. [3]

(ii) Comment briefly on this value in this context. [1]

(iii) A manager claims that this result shows that spending more money on advertising in the future will result in greater profits. Make two criticisms of this claim. [2]

(iv) Calculate the equation of the regression line of y on x. [4]

(v) Estimate the annual profit during a year when £7400 was spent on advertising. [2]

4. **2017/RI/II/9**

(i) Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A) and (B) below. In each case, your diagram should include 6 points, approximately equally spaced with respect to x, and with all x- and y-values positive. The letters a, b, c and d represent constants.

(A) $y = a + bx^2$, where a is positive and b is negative,

(B) $y = c + d\ln x$, where c is positive and d is negative. [2]

The following table shows the Gross Domestic Product (GDP) per capita, \$$x$, and infant mortality rate, y, for a sample of 9 countries.

x (\$)	1375	2502	10569	2966	11539	2036	4260	1433	7427
y	115	69	18	65	17	83	44	112	27

(ii) Draw a scatter diagram for these values, labelling the axes clearly. [1]

(iii) Calculate the product moment correlation coefficient, and explain why its value does not necessarily mean that a linear model is the best model for the relationship between x and y. [2]

(iv) State which of the two cases in part **(i)** is more appropriate for modelling the relationship between x and y. Calculate the product moment correlation coefficient and the equation of the appropriate regression line for this case. [3]

(v) Use the regression line in part **(iv)** to find an estimate of the infant mortality rate for a country with GDP per capita of \$723. Comment on the reliability of your estimate.[3]

5. 2010/RVHS/II/11

An experiment was conducted to verify how the radiation intensity x from a radioactive source varies with time t. The following data were obtained from a particular source.

x	1	4	6	8	9	10	12	15
t	15.3	6.9	4.4	2.7	2.1	1.7	1.0	0.49

(i) Plot a scatter diagram for the above data. [1]

(ii) Calculate the product moment correlation coefficient for the above data and comment on it, in relation to your scatter diagram in (i). [2]

(iii) Calculate the equation of the least square regression line of x on t. Give a practical interpretation of the coefficient of t. [2]

(iv) The variable y is defined by $y = \ln x$. For the variables y and t, calculate the product moment correlation coefficient and hence compare the suitability of this logarithmic model and the linear model in (iii). [2]

(v) Use an appropriate regression line to give the best estimate that you can of the time when $x = 10$. [2]

6. 2014/AJC/II/11

(a) The linear product correlation coefficient between 2 variables X and Y is denoted by r. A set of 6 bivariate data yields $r = -0.9$ and a second set of 6 different bivariate data also yields $r = -0.9$. Explain, with the aid of a diagram, whether this implies that r is also negative for the combined set of 12 bivariate data. [2]

(b) It is observed in a one-year study that the linear correlation coefficient between the weight gain and the number of sleeping hours per day is close to -1. Comment briefly upon this statement: *"Since the linear correlation coefficient is close to –1, we can therefore conclude that the weight gain is caused by insufficient amount of sleep per day."* [1]

(c) The following table shows the mass of a small animal from the time of birth:

No. of days from birth, t	1	2	3	4	5	6
Mass in grams, x	21.6	22.5	28.0	35.2	49.8	75.3

(i) Without using a scatter diagram, give a reason why t and x are not linearly related. [1]

(ii) It is suggested that a suitable model is of the form $x = Ae^{Bt}$, find, correct to 4 significant figures, the estimate values of A and B. [2]

(iii) Give an interpretation, in context, of the value of A. [1]

(iv) Use the suggested model to predict the number of days from birth if the mass is found to be 100 grams. Comment on the suitability of this model for long term prediction. [2]

7. 2010/CJC/II/11

The table below gives the proportion of people (in %) in an occupation earning more than $5000 and the proportion of university graduates (in %) in that occupation.

Occupation	Teacher	Chemist	Accountant	Lecturer	Engineer	Electrician	Police	Plumber	Taxi Driver
% of graduates, x	97	87	75	84	52	36	22	10	8
% earning more than $5000, y	66	65	62	45	53	43	33	18	10

(i) Calculate the product moment correlation coefficient between x and y. [1]

(ii) Draw a scatter diagram for the data. [1]

One of the values of y appears to be incorrect.

(iii) Indicate the corresponding point on your diagram by labelling it P and explain why the scatter diagram for the remaining points may be consistent with a model of the form $y = a + b\ln x$. [2]

(iv) Omitting P, calculate the least squares estimates of a and b for the model $y = a + b\ln x$. [2]

(v) Assume that the value of x at P is correct. Estimate the value of y for this value of x. [1]

(vi) Comment on the use of the model in (iv) in predicting the value of y when $x = 100$. [1]

8. 2011/JJC/II/10

A music teacher was asked to investigate the relation between the sight-reading skills and improvisation skills of music students. A music test was given to six students. Their scores for both the sight-reading (x) and improvisation (y) components are given in the table.

x	$k+3$	9	12	14	17	19
y	9	11	13	14	16	$20-k$

(i) Given that the regression line of y on x is $y = \frac{440}{713}x + \frac{3860}{713}$, find the value of k. [3]

(ii) Using the regression line of y on x, estimate the sight-reading score when the improvisation score is 15. Give your answer correct to the nearest integer. Comment on the reliability of your answer. [2]

Some adjustments were subsequently made to the test scores after a meeting with the moderators. The moderated marks are given in the same table shown above with $k = 4$. With the updated scores, it is believed that x and y are related by the equation $y = ax^b$, where a and b are constants.

(iii) Show that the relation between $\ln y$ and $\ln x$ is linear and find the estimated values of a and b. [4]

9. 2010/PJC/II/7

In an experiment, a new computer game and a new mathematics quiz are given to a group of teenagers. It may be assumed that the teenagers are playing the computer game and attempting the mathematics quiz for the first time. The computer game score, u, and the mathematics quiz score, v, of 8 teenagers are given in the table below.

Teenager	1	2	3	4	5	6	7	8
Computer game score, u	52	90	64	74	80	82	76	74
Mathematics quiz score, v	60	90	68	74	74	90	82	70

(i) Find the linear product moment correlation between v and u and the equation of the regression line of v on u. [2]

(ii) Comment on the suitability of using the regression line in **(i)** to predict u given v. Use the appropriate regression line to predict u given that v is 85, giving your answer to the nearest whole number. [2]

The scores are actually recorded for 9 teenagers. However, the scores for the last teenager are lost.

Teenager	1	2	3	4	5	6	7	8	9
Computer game score, u	52	90	64	74	80	82	76	74	p
Mathematics quiz score, v	60	90	68	74	74	90	82	70	q

It is known that the inclusion of the scores of the last teenager does not alter the mean computer game score and one of the regression lines is given by $u = 45.556 + 0.395v$.

(iii) Find p and q, giving your answers to the nearest whole number. [3]

10. 2011/RVHS/II/11

The amount of radioactive material x (in grams) left after time t (in seconds) is shown in the following set of data below.

t (seconds)	5	10	15	20	25	30
x (grams)	1.321	0.507	0.251	0.196	0.053	0.046

(i) Draw a scatter diagram for these values and comment on the relationship between t and x. [2]

It is thought that x and t can be modelled by one of the formulae

$\ln x = a + bt$ or $\ln x = c + dt^2$, where a, b, c and d are constants.

(ii) Find the value of the product moment correlation coefficient between

(a) t and $\ln x$, (b) t^2 and $\ln x$. [2]

(iii) Use your answer to part (ii) to explain which of $\ln x = a + bt$ or $\ln x = c + dt^2$ is the better model. [1]

By using a suitable regression line, estimate the value of t for which $x = 0.123$. Explain your choice of regression line. [3]

11. 2011/AJC/II/8

A researcher wants to find out how the ground temperature affects the rate of chirps made by a ground cricket. The data gathered is as follows:

Ground temperature, t, (degree Celsius, °C)	20.9	22.0	24.0	27.0	29.1	31.6	34.1
Rate of chirps, x, (number of chirps / min)	16.2	16.5	16.9	17.7	20.2	25.3	31.2

(i) Draw a scatter diagram to illustrate the data. [1]

The researcher wants to fit a model of the form $x = Ae^{Bt}$ where A and B are constants.

(ii) Calculate the product moment correlation coefficient, r, between $\ln x$ and t. [1]

(iii) Use your answers in part (i) & (ii) to explain why the model is suitable and find the values of the least square estimates of A and B. [3]

(iv) Estimate the percentage increase in the rate of chirps when the ground temperature is increased by 5 °C.

On a particular day, there is a change in the ground temperature from 10 °C to 15 °C. Can the researcher use the answer found above to estimate the percentage change in the rate of chirps? Justify your answer. [2]

(v) The researcher wants to convert the units for the temperature, t, from Celsius to Fahrenheit, by using the formula $c = \frac{5}{9}(f - 32)$, where c is the temperature in Celsius and f is the temperature in Fahrenheit. Explain how this affects the value of r. [1]

12. 2015/HCI/II/9

The number of reported cases, x (in hundreds) of a virus in the n^{th} month are given in the table below.

n	1	2	3	4	5	6
x	0.26	3.05	4.12	4.63	5.15	k

It is given that the least squares regression line of x on n is $x = 0.943n + 0.484$.

(i) Show that the value of k to two decimal places is 5.50. [1]

Use the value of k in part **(i)** for the rest of the question.

(ii) Draw a scatter diagram to illustrate the data. [1]

(iii) Explain why a linear model is inappropriate. [1]

(iv) Explain clearly which of the following models **(A)** or **(B)**, where a and b are positive constants, is more appropriate to model these values.

(A) $x = a + b\ln n$,

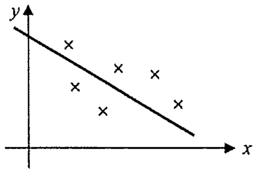
(B) $x = a + bn^2$.

Use the model that you have identified, calculate the values of a and b, and use them to predict the month in which the number of reported cases is at least 1000. [2]

(v) Give an interpretation, in context, of the value of a for the model you have identified in part **(iv)**. [1]

13. 2011/TJC/II/10

(a) The scatter diagram shows a sample of size 6 of bivariate data, together with the regression line of y on x.



State and giving a reason, whether, for the data shown, the regression line of y on x is the same as the regression line of x on y. [1]

Given that $y = -8x + 86$ is the regression line of y on x for the 6 pairs of observations with $\sum x = 60$.

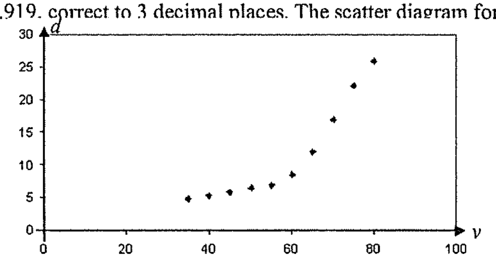
(i) Find the value of the mean of y. [2]

(ii) State, with a reason, the regression line of y on x after a new pair of observation (10, 6) is added. [2]

(b) A car is travelling along a stretch of road with speed v (km/h) when the brakes are applied. The car comes to rest after travelling a further distance of d m. The values of d for 10 different values of v are given in the table, correct to 2 decimal places.

v	35	40	45	50	55	60	65	70	75	80
d	4.90	5.30	5.93	6.45	7.00	8.50	12.00	17.00	22.10	25.90

It is given that the value of the linear product moment correlation coefficient for this data is 0.919, correct to 3 decimal places. The scatter diagram for the data is shown



(i) Calculate the product moment correlation coefficient between v and $\sqrt{d}$. What does this indicate about the scatter diagram of the points (v, $\sqrt{d}$)? [2]

(ii) Which regression line is more appropriate: $\sqrt{d}$ on v or d on v? State the reason and find the equation of the appropriate regression line. [2]

Estimate the distance a car travels after its brake is applied when the car's speed is 100km/h. Comment on the reliability of your answer. [2]

14. 2010/JJC/II/10 (modified)

Water in a reservoir undergoes a purification process before it can be consumed. The effectiveness, y %, of the process for various flow rates, $x \text{m}^3 \text{s}^{-1}$, is shown below.

x	1	2	4	6	8	10	20	30	40
y	80	60	45	40	30	25	18	15	10

The variables x and y are thought to be related by the equation $e^y = ax^b$, where a and b are constants.

(i) Give a sketch of the scatter diagram of y against $\ln x$. Comment on whether a linear model would be appropriate referring to the scatter diagram. [2]

(ii) Find the value of the product moment correlation coefficient between y and $\ln x$ and explain whether it supports your comment in part **(i)**. [2]

(iii) Find the least squares regression line of y on $\ln x$ and estimate the values of a and b. [1]

(iv) Predict the effectiveness of the process when water flows at $0.5 \text{ m}^3\text{s}^{-1}$. Comment on the reliability of your prediction. [2]

(v) Explain why in this context, the above model would not be appropriate for large values of x. [1]

(vi) Comment whether the product moment correlation between y and $\ln x$ would have been different if the flow rates were measured in cm^3s^{-1} instead. [1]

15. 2017/VJC/II/6

An experiment to determine the effect of a fertilizer on crop yield was carried out. A field was divided into eight plots of equal area and eight different amounts of fertilizer, one for each plot, were used. The table below shows the amount of fertilizer, x grams, and the crop yield, y grams, for each plot.

Amount of fertilizer (x)	15	22	37	55	62	69	78	90
Yield (y)	101	123	137	150	150	154	158	160

(i) Draw the scatter diagram for these values, labelling the axes. [1]

It is thought that the yield of a crop, y grams, can be modelled by one of the formulae

$y = a + bx$ or $y = c + d\ln x$, where a, b, c and d are constants.

(ii) Find the value of the product moment correlation coefficient between

(a) x and y,

(b) $\ln x$ and y. [2]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $y = a + bx$ or $y = c + d\ln x$ is the better model. [2]

(iv) For a plot of land, the yield of the crop was 144 grams. Using a suitable regression line estimate the amount of fertilizer used, giving your answer to the nearest gram. [2]

Comment on the reliability of the estimate when the model in part **(iv)** is used to estimate the value of y when $x = 110$. [1]

Revision Package (Topical) Answers

Topic 1: Equations & Inequalities

1. **(a)** $x > \frac{1}{4}$ **(b)** $1 < x < 3, x \neq 2$

2. $x < 0$ or $x \geq 1$; $x \geq 0$

3. $x \leq -\frac{1}{2}$ or $1 \leq x < 3$; $\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$ or $\theta = \frac{\pi}{2}$

4. $x > \frac{a}{3}$; $x < -\frac{2}{3}$

5. $x < -\frac{1}{4}$ or $x > \frac{9}{2}$

6. $T_r = \frac{1}{6}r^3 - \frac{1}{2}r^2 + \frac{4}{3}r$

7. $f(x) = -\frac{5}{2}x^3 + \frac{31}{6}x^2 + 2x - \frac{8}{3}$

8. $a = 1.265, b = -0.144, c = 2.667$

9. 24

10. 160, 160, 80

Topic 2: Graphing Techniques

1. **(iii)** $0 < k < 20$

2. $-2\sqrt{2} < y < 2\sqrt{2}$

3. **(i)** $b = 5, c = -1$ **(iv)** $\left\{x : x \in \mathbb{R},\ x < 1 \text{ or } x > \frac{4}{4+a}\right\}$

4. **(i)** 2 **(iii)** $x = 0, x = 1, y = \frac{1}{2}$ **(iv)** 0

6. $A = 2, B = 9$

7. **(i)** $y \geq -\frac{1}{8}$; $a = -\frac{2}{9}, b = 1, c = \frac{20}{9}$; $k = 6$

13. $x < a$ or $b < x < a + b$

14. $y = \frac{4}{x^2}$

15. **(a)** $k \leq -2$ or $k > 3$

16. **(b)** $a = 3, b = -4, c = 3$

Topic 3: Functions

1. **(i)** $f^{-1}(x) = \sqrt{\frac{1}{x} + 2},\ D_{f^{-1}} = \left(-\infty, -\frac{1}{2}\right) \cup (0, \infty)$

(ii) $fg : x \mapsto \frac{1}{4x^2 + 12x + 7},\ x > 0,\ R_{fg} = \left(0, \frac{1}{7}\right)$

2. **(i)** 1.5 **(ii)** $f^{-1}(x) = \frac{1}{4} + \sqrt{\frac{8x+25}{16}},\ x \geq 0$ **(iii)** $\frac{1+\sqrt{7}}{2}$

3. **(i)** $\lambda \leq -4$ **(ii)** $f^{-1} : x \to 2 - \sqrt{x+2},\ x \geq -2$

4. **(i)** $(-\infty, \ln 2]$

5. **(b)** **(i)** $f^{-1} : x \mapsto -3 + \sqrt{1-x}\ ,\ x \leq 0$ **(iii)** $-2 \leq x \leq \frac{-7+\sqrt{17}}{2}$

Topic 3: Functions (continued)

6. **(ii)** 1 **(iii)** $f^{-1}(x) = \frac{1}{2}\left(3 - \sqrt{x-14}\right),\ x \geq 15,\ R_{f^{-1}} = (-\infty, 1]$

(iv) $hf : x \mapsto -e^{4x^2-12x+23},\ x \leq 1;\ R_{hf} = \left(-\infty, -e^{15}\right]$

7. $f^{-1} : x \mapsto \sqrt{\frac{5-x^2}{a}},\quad 0 \leq x \leq \sqrt{5}$ **(ii)** $[1, 2)$

8. **(i)** $(0, 2]$ **(iii)** $\frac{1}{5}$; $f^{-1}(x) = \frac{1}{5}\left(1 + \sqrt{\frac{2}{x} - 1}\right)$

9. **(ii)** 0 **(iii)** $g^{-1}(x) = \sqrt{4 + \ln x}\ ,\ [e^{-4}, \infty)$

(iv) $gf : x \to e^{\frac{4x^2}{(x-1)^2} - 4},\ x > 1$ **(v)** $1 < x < 3$

10. **(i)** $f^{-1} : x \to 2 - \sqrt{x+1},\quad x > -1$ **(iii)** $y = x$; $\frac{5-\sqrt{13}}{2}$

(iv) $gf : x \mapsto \ln\left[\left((x-2)^2 - 1\right)^2 + 1\right],\ x < 2$

11. **(ii)** $3 - \frac{9}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}-1}{1+\sqrt{2}}\right)$ **(iii)** $-\frac{3}{2}$ **(v)** -1.781

12. **(i)** $f^{-1} : x \to 1 - x^2\ ,\ x \geq 0$ **(iii)** $x = 0,\ 1,\ \frac{-1+\sqrt{5}}{2}$

13. **(i)** $R_f = (-\infty, a) \cup (a, \infty)$ **(ii)** $\frac{1-5a}{a-5}$ **(iv)** a

14. **(i)** $k > \frac{1}{2}$ **(ii)** $[0, 1]$

15. **(i)** 26; -4 **(iii)** $[-2, -0.5]$; $\frac{\pi}{12} < x \leq \frac{\pi}{8}$

16. **(ii)** $f^{-1}(x) = e^x + a,\ x \leq \ln 2$ **(iii)** **(a)** $g(x) = e^x + a$ **(b)** $g(x) = \left(e^x + a\right)^2$

17. **(ii)** $-\frac{1}{3}$ **(iii)** $\lambda > 3$; $\left[\frac{7-\lambda}{3-\lambda},\ 1\right)$

18. **(iii)** $f^{-1} : x \mapsto \frac{2x+k}{x-2},\ x \in \mathbb{R},\ x \neq 2,;\quad R_{f^{-1}} = (-\infty, 2) \cup (2, \infty)$

(iv) $-\frac{2(1+k)}{3}$ **(v)** $a \geq 2$

Topic 4: Sequences and Series

1. (i) $8a$ (ii) 5
2. $x > 0.924$
3. (a) (ii) $-1 < x < 0$ (b) (ii) $\frac{11}{2}$
4. (a) 944 644 (b) $-\frac{1}{3}$
5. (ii) 7
6. (iii) 15 September 2019 (iv) \$1 328.78
7. (ii) 19 (iii) \$4 549
8. 10.8 cm^2; $30\pi\text{ cm}^2$
9. (iii) $\left[29+8(3^{n-3})\right](3^{n-2})$ (iv) 6
10. (i) $5x$ (ii) 2025 (iii) 0.2377
11. (ii) $(2^N-1)\left(6(2^N)-1\right)$ (iii) $6(2^N)-5$ (iv) 19
12. 574 753
13. (i) $\frac{1}{2r-1}-\frac{1}{2r+1}$ (ii) $\frac{N}{2N+1}$ (iii) $\frac{N}{3(2N+3)}$
14. $\frac{1}{4}\left(\frac{1}{4}-\frac{1}{(n+1)^2(n+2)^2}\right)$; $\frac{1}{576}$
15. (i) $1-\frac{1}{N-2}$ (ii) $1-\frac{1}{N-4}$ (iii) 1
16. (i) $A=1, B=-1$ (ii) $-2+(2n+1)\cdot 2^{n+1}$ (iii) $(2n+3)\cdot 2^{n+1}-6$
17. (i) $A=1, B=-2, C=1$ (iii) $\ln\frac{1}{2}$ (iv) $\ln\left[\frac{4N}{5(N-1)}\right]$
18. $n(n+1)$
19. (ii) $\sin^2 n\theta$
20. (i) 2; 4; 12 (ii) $S_n = n^3-2n^2+3n$

Topic 5: Techniques and Applications of Differentiation

1. (i) $x > 1$ (ii) $1 < x < 2$
2. (ii) $6(3y+1)=e^{x+y}$
3. (i) $(4, 1), (-4, 1)$
4. (i) $\frac{2y-x}{2(y-x)}$ (ii) $(-2, -1), (2, 1)$ (iii) $\frac{10}{3}$
5. (ii) $-2\tan\theta$ (iii) $(0, 2\sin t)$ (iv) $4x^2+y^2=1$
6. (i) $\operatorname{cosec} t$ (ii) $y=(\operatorname{cosec}\theta)x-\cot\theta$ (iii) $\frac{3}{4}$ (iv) $\frac{1}{4x^2}-\frac{1}{4y^2}=1$
7. $\frac{\pi}{3}$
8. $a=20.42, b=2.74$
9. (ii) 561.18 (iii) \$ 0.28
10. (i) $\frac{32}{81}\pi$ (ii) $\frac{8\sqrt{3}}{\pi}$ rad/s
11. (i) -0.0509 cm/min (ii) 0.0456 cm/min
12. $k=2$
13. (i) $PQ=QR=\sqrt{2}r$ (ii) 0.1 rad/sec
14. (ii) 14.2
15. (a) 2.32 cm/s (b) (ii) $\left(-\frac{17}{8}, -8\right)$
16. $T=\frac{2a-x}{100}+\frac{\sqrt{x^2+a^2}}{60}$; $\frac{3}{4}a$; $\frac{a}{30}$
17. (i) $h=\frac{2}{x^2}$ (iii) 0.89
18. (ii) $8\pi a^2$
19. (a) (i) $\frac{1}{4}$ cm/ s
20. (i) $\frac{v^2\sin\theta\cos\theta}{5}$

Topic 6: Techniques and Applications of Integration

1. (a) $\frac{\sqrt{3}}{3}\tan^{-1}\sqrt{3}t+C$ (b) $e^{e}-e$

2. (i) $a=\frac{1}{4},\ b=-1$ (ii) $-\frac{1}{2}\sqrt{6-4x-2x^2}-\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{2}\right)+c$

3. (a) $\frac{1}{4}\tan 2\theta+C$ (b) $-\frac{1}{10}\cos 5x-\frac{1}{2}\cos x+C$

4. (a) $\frac{1}{2}e^3+\frac{1}{2}e^5-1$ (b) $x(\ln x)^2-2x\ln x+2x+C$

5. (a) $\frac{\pi}{12}$ (b) $\ln\frac{3}{2}$

6. (a) $2\tan^{-1}(\sqrt{x}+1)+c$ (b) $x\cos x+\sin x$; $x\sin x\ln x+\cos x+c$

7. (a) $\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{4}{3}(x+1)^{\frac{3}{2}}+2\sqrt{x+1}+C$ (b) $-\frac{1}{15}\left(e^{5x}-e\right)^{-3}+C$ (c) $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$

8. (a) $-\frac{1}{4}\sqrt{1-8x-4x^2}+\frac{1}{2}\sin^{-1}\frac{2\sqrt{5}(x+1)}{5}+C$ (b) $2\left[\sqrt{3}-\frac{\pi}{3}\right]$ 9. 13.8

10. (i) $-\frac{1}{2(1+x^2)}+c$ (ii) $\frac{1}{2}\left(\frac{x}{1+x^2}+\tan^{-1}x\right)+c$
 (iii) $\frac{1}{2}\left(\tan^{-1}x-\frac{x}{1+x^2}\right)+c$ (iv) $3\tan^{-1}x+\frac{2x-1}{1+x^2}+c$

11. $2\pi^2$ 12. (i) $y=(\cot 2p)(x-r)$; $(r,0)$; $\frac{dh}{dt}=\frac{1}{10\pi}$ cm s^{-1}

13. (i) $\frac{1}{2e^{-y}(1-y)}$; $x=2e^{-1}$ (ii) $-ne^{-n}-e^{-n}+1$; Area = 2

14. (i) $\pi(\sin 2\alpha+2\alpha)$ (ii) $\frac{3}{2}\pi$

15. (ii) $\frac{1}{2}\left(1+e^{-2}\right)$ 16. (ii) $a=7, b=6$

17. $y=1+2x+2x^2$; (i) 0.408 (ii) $y=1+2x$; 1.47

18. (a) (i) $3e(2e-1)$ (ii) 2.28 (b) (ii) $\ln 2$

19. (i) $\sqrt{2}-1$ (ii) 1.674 (iii) $\pi\left(1-\frac{\pi}{4}\right)$

20. (a) $\frac{1}{6}\tan^{-1}\frac{x}{3}-\frac{x}{2(x^2+9)}+c$ (b) (i) $\frac{1}{12}\pi^2-\frac{1}{6}\pi$ (iii) 4.86

21. (iii) $2e^{-1}$; $1+e-\frac{\left(-1+e^{-1}\right)^2}{1+e}-4e^{-1}$

22. (a) (i) $k=4$ (ii) $\frac{3\pi}{4}$ (b) $\frac{29\pi}{5}-8\pi\ln 2$

Topic 7: Maclaurin's Series and Binomial Expansion

1. (ii) $y=(e-1)+ex+\frac{ex^2}{2}-\frac{ex^3}{6}+\ldots$ (iii) $a=\frac{1}{e-1},\ b=-\frac{e}{(e-1)^2}$

2. (ii) $y=e^{\frac{\pi}{2}}\left(1-x+\frac{x^2}{2}-\frac{x^3}{3}+\ldots\right)$ (iii) $e^{\frac{\pi}{2}}\left(-1+x-x^2+\ldots\right)$, $-\frac{3}{4}e^{\frac{\pi}{2}}$

3. (ii) $\frac{576}{301}$ (iii) 5

4. (i) $y=1-x+\frac{5}{2}x^2+\ldots$ (ii) $\frac{e^{-1}-1}{2}<x\le\frac{1}{2}$ (iii) $\frac{20}{3}$

5. $y=1-4x^2+12x^4-32x^6+\ldots$; $(-1)^r(r+1)2^r$; $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$

6. $a=\sqrt{5},\ b=\frac{3\sqrt{5}}{5},\ c=\frac{3\sqrt{5}}{25}$

7. (a) (ii) $a=\frac{1}{\sqrt{3}},\ b=\frac{4}{3}$ (b) (i) $y=1+x-\frac{x^2}{2}+\ldots$ (ii) $e+ex$

8. (ii) $a=\frac{1}{2^n}, b=-\frac{7n}{2^{n+1}}$ (iii) $-2<\theta<2$

9. (i) $3x-6x^2+\frac{3}{2}x^3+5x^4+\ldots$ (ii) $y=3x-6x^2+\frac{3}{2}x^3$

10. (i) $3+(3n+1)x+\left(n+\frac{3n(n+1)}{2}\right)x^2+\ldots$ (ii) 3

11. (i) $2(1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots.)$ (ii) (b) $y=-\frac{1}{2}x^2-\frac{1}{12}x^4+\ldots$
 (iii) $2+3x-4x^2+\frac{23}{3}x^3+\ldots..$

12. (i) $y=\frac{1}{2}x-\frac{3x^2}{4}+\frac{11x^3}{12}+\ldots$ (iii) 0.120 (iv) $y=\frac{1}{2}x$

13. (i) $f(x)\approx\frac{1}{2}+\frac{1}{4}x+\frac{1}{4}x^2$ (ii) $\frac{1}{2}n+\frac{1}{8}n^2+\frac{1}{12}n^3$; 0.291667 (iii) 0.293218

15. $\cos^{-1}x=\frac{\pi}{2}-x-\frac{x^3}{6}+\ldots$; $\sin^{-1}x=x+\frac{x^3}{6}+\ldots$.

16. (ii) $y\approx\frac{1}{2}x^2+\frac{1}{12}x^4$ (iii) $\sqrt{\frac{3}{7}}$

17. (ii) $2\theta-2\sqrt{3}\theta^2+\frac{20}{3}\theta^3$

18. (i) $f(0)=1,\ f'(0)=\frac{1}{3}$ (ii) $m=3,\ n=-\frac{1}{9}$.

Topic 8: Differential Equations

1. $y = Ae^{-\frac{x}{2}} + 2$; $y = -3e^{-\frac{x}{2}} + 2$

2. $y = 1 - \dfrac{1}{x+2}$

3. $y = \dfrac{\tan(x+C)}{x}$

4. (i) $y = \ln(x^2+1) + c$ (iii) $y = \ln(x^2+1)$

5. $\dfrac{dh}{dt} = 1.3\cos\left(\dfrac{\pi}{6}t\right)$

6. $y = \dfrac{2}{3}\left(x + \dfrac{1}{3}\right)\ln(1+3x) + \dfrac{4}{3}x + 1$

7. $y = x\ln x - x + C$

8. (i) $x = \dfrac{1}{\sqrt{0.813}}\tan\sqrt{0.813}t$ (ii) $x = \tan^{-1} t + d$

9. (i) $y = \dfrac{2}{1+e^{-2t}}$ (ii) 2000

10. (a) $y = \ln|\sec x + \tan x| + cx + d$; $y = \ln|\sec x + \tan x| + 2$ (b) $x = 400 - 100e^{-0.1t}$

11. $y = e^{-2x}\left(\dfrac{x^2}{2} + x + c\right)$ (i) $e^{-2x}\left(\dfrac{x^2}{2} + x + 1\right)$ (ii) $y = 1 - x + \dfrac{1}{2}x^2 - \dfrac{1}{3}x^3 + \ldots$

12. $t = -100\ln\left|\dfrac{x-4}{4}\right| + 80\ln\left|\dfrac{x-5}{5}\right|$; 28.4 months

13. $f(x) = \dfrac{1}{3}x^3$

14. (a) $y = -\dfrac{x^2}{x+C}$ (b) $v = 1 - Be^{-kt}$

15. (ii) $x = \dfrac{250}{3} + 15e^{\frac{3}{5}t}$

16. (ii) $t = \dfrac{100}{3}\ln\left(\dfrac{2x}{15-x}\right)$ (iii) 122

17. (ii) 89.0 l (iii) 141 l

18. $A = 4$; $k < 0$

19. 11.43am

20. (i) $x = \dfrac{1}{2}\left(1 + e^{-0.1t}\right)$

21. 43.9 days; $\dfrac{dN}{dt} = \dfrac{1}{40}N - r$; 300

22. (i) $P = \dfrac{10}{1+9e^{-0.64t}}$

23. (i) $x = e^{-t} + 3t$ (ii) $y = e^{-t}\left[\ln\left(\dfrac{30-t}{30}\right)\right]$

24. (ii) $0 < k < 0.03$ (iii) $t = 7.16$h

25. (i) $\dfrac{dA}{dt} = kA(100-A)$ (ii) 7.07 days (iii) $79.58 \le A \le 100$
 (iv) $A = \dfrac{100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}{4 + e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}$

Topic 9: Vectors

1. (i) 90° (ii) $\mathbf{r} = \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$; $\dfrac{x-2}{4} = \dfrac{z+1}{3}$, $y = 7$

2. (ii) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c}$

3. (a) $\dfrac{2}{9}$ (b)(i) $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ (ii) $\dfrac{3}{5}\begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$

4. (ii) $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ (iii) 169.1° (iv) $\dfrac{3\sqrt{3}}{2}$

5. (ii) $\dfrac{2\sqrt{13}}{3}$ (iii) $\dfrac{\lambda\sqrt{3}}{6}$

6. (i) $-\dfrac{1}{9}$ (ii) $\dfrac{26}{9}$ (iii) $\dfrac{1}{2}\sqrt{3226}$; $\sqrt{\dfrac{3226}{163}}$

7. (a) (i) $\mathbf{b} - \dfrac{1}{2}\mathbf{a}$; $\dfrac{1}{7}(5\mathbf{a} + 2\mathbf{b})$ (ii) $\mathbf{r} = \mathbf{a} + \mu(\mathbf{b} - \dfrac{3}{2}\mathbf{a})$ (iii) $\dfrac{1}{8}(5\mathbf{a} + 2\mathbf{b})$
 (b) (i) $\dfrac{1}{7}$ (ii) 64.8° or 115.2°

8. (i) $p = -\dfrac{1}{2}$, $q = 3$ (ii) $\dfrac{\pi}{3}$

9. (i) $2\sqrt{6}$ (ii) 41.8° (iii) $\sqrt{21}$ (v) $4\sqrt{41}$

10. (i) $2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ (ii) $6\sqrt{2}$ (iii) 64.6°

11. (i) $2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ (ii) 7 (iii) 6.06

12. (i) $\left(4, -\dfrac{1}{2}, -\dfrac{1}{2}\right)$ (ii) $\mathbf{r} = -11\mathbf{i} + 7\mathbf{j} + 12\mathbf{k} + \alpha(16\mathbf{i} - 8\mathbf{j} - 11\mathbf{k})$

13. (i) $\dfrac{7}{2}\mathbf{i} + \alpha\mathbf{j} + \dfrac{7}{2}\mathbf{k}$ (iii) $\mathbf{r} = \mathbf{i} + 6\mathbf{k} + \mu(5\mathbf{i} + 2\alpha\mathbf{j} - 5\mathbf{k})$

14. (i) 13.1° (ii) (13, 13, –5) (iii) $x - 8y - 10z = -41$

15. (ii) $\dfrac{\sqrt{140}}{7}$ (iii) $\left(\dfrac{5}{2}, \dfrac{3}{2}, \dfrac{11}{2}\right)$ (iv) 78.8° (v) $\dfrac{20}{7}\sqrt{14}$

16. (ii) $-\mathbf{i} - \mathbf{j}$ (iii) $-7\mathbf{i} + 11\mathbf{j} + 18\mathbf{k}$ (iv) $\mathbf{r}\cdot(-\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0$
 (v) $\dfrac{27\sqrt{3}}{2}$

17. (i) 39.2° (ii) 1.41 (iv) $\mathbf{r}\cdot(5\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 6$

Topic 9: Vectors (continued)

18. (i) $-4 \pm 2\sqrt{3}$ (ii) $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ (iii) 2

19. (i) $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 4$ (ii) $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ (iii) $\begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$; $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$

20. (i) $\frac{1}{10}$ (ii) $\mathbf{r} = t \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$

21. (a) $5 + 4\lambda = \mu$ (b) $\lambda = -1$, $\mu = 1$

22. (i) $\begin{pmatrix} 29 \\ 4 \\ -18 \end{pmatrix}$ (ii) $\mathbf{r} = \begin{pmatrix} 29 \\ 4 \\ -18 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ (iii) 5.5° (iv) $\alpha = 6$, $\beta \neq 6$

23. (i) $a = 2$, $b = 1$, $c = 3$ (ii) 1

24. $p = 9$, $q = -3$; $\mathbf{r} = \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 15 \\ 10 \\ -13 \end{pmatrix}$

25. (i) $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 2$ (ii) $(-2, 1, -2)$ (iii) $\frac{1}{5}\begin{pmatrix} 2 \\ -5 \\ -4 \end{pmatrix}$; $\left(-\frac{6}{5}, -1, \frac{12}{5}\right)$

(iv) $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

Topic 10: Complex Numbers

1. $-15 - 8i$; $2 + 2i$, $-14 - 2i$

2. (a) $1 + i$, $\frac{1}{4}(1 \pm \sqrt{15})i$ (b)(i) $\sqrt{2}$; $\frac{7\pi}{12}$ (ii) 3

3. $1 - 2i$, $\pm\sqrt{5}$; $\pm 2i$, $1 \pm \sqrt{5}$

4. (a) $a = 5$, $b = \frac{7}{13}$ (b) $\frac{1}{8}$; $\frac{5\pi}{6}$; $-\frac{1}{262144}$

5. (a) (i) 1 (ii) 2.82 (b) $a = -\tan^{-1}\frac{1}{2}$; $b = \frac{1}{2}\ln 5$

6. (i) $\sqrt{2}e^{-\frac{\pi}{4}i}$ (ii) $32 - 33i$

7. (i) $\frac{2\pi}{3}$ (ii) 4

8. $\sqrt{2}\left[\cos\left(\frac{7\pi}{12}\right) - i\sin\left(\frac{7\pi}{12}\right)\right]$; $\frac{1}{2}\left[(1-\sqrt{3}) - (1+\sqrt{3})i\right]$; $\frac{1+\sqrt{3}}{2\sqrt{2}}$

9. (a) $\frac{1}{4} - i$ (b) 1

10. $\frac{1}{\sqrt{2}}$; $-\frac{\pi}{4}$ (i) $2\sqrt{2}$; $-\frac{\pi}{12}$ (ii) $c = 3, d = 1$; $\sqrt{3} + 1$; $1 - \sqrt{3}$

11. (i) $re^{-i\theta}$ (ii) 2; $\frac{5\pi}{6}$ (iii) $\left(z^2 - 2z + 4\right)\left(z^2 + 2\sqrt{3}z + 4\right)$

12. (b) $\frac{1}{2} - i$; $4 - 3i$, $-1 - \frac{1}{2}i$ (c) (ii) $\frac{\sqrt{3}}{8}$; $-\frac{\pi}{6}$

13. (i) $w = 6 + 2i$, $z = 1 - 3i$

14. (a) $w = -1 - i$, $z = 7 + 2i$ (b) (i) $\frac{1}{8}$; $\frac{\pi}{3}$ (ii) $n = \frac{3(2m+1)}{2}$, $m \in \mathbb{Z}$

15. $x = 0, y > 0$

16. (i) $1, \beta - \frac{3\pi}{2}$; $\frac{1}{2}, 2\alpha - \beta$ (iii) 12

Topic 11: Permutations & Combinations

1.	**(i)**	420	**(ii)**	120	**(iii)**	72			
2.	**(a)**	18	**(b)**	606					
3.	**(i)**	10 000	**(ii)**	75 600	**(iii)**	72 000			
4.	**(a)**	**(i)** 1680	**(ii)**	23 520					
	(b)	**(i)** 42 336 000	**(ii)**	65 318 400					
5.	**(i)**	59 049	**(ii)**	15 360	**(iii)**	1 260	**(iv)**	57 513	
6.	**(a)**	1260	**(b)**	715					
7.	**(i)**	**(a)** 18 144 000	**(b)**	259 200					
	(ii)	**(a)** 2 880	**(b)**	8 640					
8.	**(i)**	3 003	**(ii)**	3 075 072	**(iii)**	11 531 520			
9.	**(i)**	5 760	**(ii)**	26 127 360					
10.	**(a)**	261 273 600	**(b)**	805	**(c)**	0.8			
11.	**(a)**	**(i)** 86 400	**(ii)**	564 480					
	(b)	**(i)** 2 880	**(ii)**	1 728					
12.	**(a)**	135	**(b)**	**(i)** 9 450	**(ii)**	25200			
13.	**(i)**	362 880	**(ii)**	2880	**(iii)**	2 592	**(iv)**	43 200	
14.	**(i)**	**(a)** 40 320	**(b)**	30 240					
	(ii)	**(a)** 5 040	**(b)**	240					
15.	**(a)**	**(i)** 1 440	**(ii)**	144	**(b)**	7 983 360; 1 209 600			
16.	**(i)**	720	**(ii)**	1 440	**(iii)**	40 320; 1023			
17.	**(a)**	77 520	**(b)**	160 000	**(c)**	**(i)** 43 200	**(ii)**	292 320	
18.	**(i)**	32 752	**(ii)**	**(a)** 362 880	**(b)**	60 480	**(iii)**	14 400	

Topic 12: Probability

1.	**(i)**	$\frac{2}{7}$; $\frac{3}{5}$	**2.**	**(i)** $\frac{7}{9}$	**(ii)**	$\frac{4}{27}$		
3.	**(i)**	$\frac{8}{23}$	**(ii)**	$\frac{296}{1035}$				
4.	**(i)**	$\frac{91}{380}$	**(ii)**	$\frac{43}{57}$	**(iii)**	$\frac{62}{215}$		
5.	**(i)**	$\frac{54}{55}$	**(ii)**	$\frac{8}{11}$	**(iii)**	$\frac{13}{55}$	**(iv)**	$\frac{25}{108}$
6.	**(i)**	0.049	**(ii)**	0.20610	**(iii)**	0.778	**(iv)**	14
7.	**(i)**	$\frac{3}{20}$	**(ii)**	$\frac{1}{2}$	**(iii)**	$\frac{3}{8}$		
8.	**(i)**	$\frac{13}{20}$	**(ii)**	$\frac{56}{65}$	**(iii)**	$\frac{23}{80}$	**(iv)**	$1-(0.3)^n$
9.	**(i)**	0.090	**(ii)**	0.435	**(iii)**	0.535		
10.	**(i)**	$\frac{4}{9}(p+1)$	**(ii)**	$\frac{4}{9}<p<\frac{8}{9}$	**(iv)**	$\frac{2+p}{2+2p}$		
11.	**(i)**	**(a)** $\frac{19}{20}$	**(b)**	$\frac{1}{4}$	**(ii)**	$\frac{7}{380}$	**(iii)**	17
12.	**(ii)**	$\frac{3}{20}$	**(iii)**	$\frac{2}{5}$	**(iii)**	0.140		
13.	**(i)**	0.74	**(ii)**	0.172	**(iii)**	13		
14.	**(i)**	$\frac{4}{7}$	**(ii)**	$\frac{2}{5}$	**(iii)**	$\frac{11}{14}$		
15.	**(a)**	**(i)** $\frac{1}{2}$	**(ii)**	$\frac{11}{20}$	**(b)**	**(i)** $\frac{2}{5}$	**(ii)**	0.434
16.	**(i)**	$\frac{9}{13}$	**(ii)**	$\frac{1}{260}$	**(iii)**	$\frac{1}{50}$		
17.	**(a)**	**(i)** 0.27	**18.**	**(a)**	**(ii)**	5		

Topic 13: Discrete Random Variables

1. $\frac{2}{5}$ **2.** 2.1; 1.29

3. (i) $\frac{1}{16}$ (ii) 2; 3.5 (iii) 7; 14 (iv) 4; $\frac{9}{32}$

4. 5 (i) $\frac{19}{3}$ (ii) $\frac{5}{144}$ **5.** (i) 6.389 (iii) 0.00965

6. 598.4; 198.4; $S \sim \mathrm{N}(400, 400)$ approximately

7. (i) $\frac{3}{10}$ (ii) $p=\frac{1}{3}$, $q=\frac{4}{15}$

8. (ii) $(1-p)^8 + 8p(1-p)^7 = 0.96$; 0.0410 (iii) 0.781

9. (ii) 5 (iii) $\frac{5}{84};\frac{5}{21};\frac{5}{42};\frac{1}{252}$ (iv) $\frac{1159}{252};\frac{127}{252}$ (v) $\frac{11}{127}$

10. (i) $\frac{1307}{144}$ (ii) 0.560, 1.00 **11.** (iii) 102 (iv) 0.455

12. (a) 0.179 (b) 0.982

13. (ii) 2.00 (iii) 0.727 (iv) 0.686

14. (i) 4.6 (ii) 51 (iii) 0.588

15. (ii) 0.802 (iii) 0.0249 **16.** (iii) 0.106 (iv) 38

17. (i) 0.0888 (ii) 0.371 (iii) 26 (iv) at least 453

18. 2

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Topic 14: Normal Distribution & Sampling

1. $2p-1$ **2.** 0.868 **3.** (i) 2.00 cm; 0.0305 cm (ii) 0.736

4. (i) 0.427 (ii) 0.534 (iii) 50

5. (a) 0.216 (b) (i) 8.3; 0.5194 (ii) 0.0821

6. (i) 0.0771 (ii) 0.00253 (iii) 0.384

7. (i) 10.3 (ii) 0.0588 (iii) 0.0346

8. (i) 49.2; 9.03 (ii) 91 **9.** (ii) 0:407 (iii) 0.694

10. (i) 0.710 (ii) 0.975 (iii) 15

11. (i) 0.0544 (ii) 0.401 (iii) 0.322; 11

12. (a) 0.345 (b) (i) 0.717 (ii) 0.611

13. (i) 1.43 (ii) 0.0828 (iii) 9 (iv) 0.859

14. (ii) 0.363 (iii) 0.976 (iv) 0.0540

15. (i) 0.0109 (ii) 0.0371 (iii) 22.5 ; 0.258 (iv) 1.95

16. (i) 0.125 (ii) 80 (iii) 104.85

Topic 15: Hypothesis Testing

1. (i) 246.98; 88.3 (v) $n \le 25$, where $n \in \mathbb{Z}^+$

2. (ii) 0.991 **3.** (i) $\frac{94}{3}$; $\frac{6400}{177}$

4. (i) 770 (ii) $\bar{x} \le 748$ **6.** (i) 566 ; $\frac{5555}{149}$

7. (i) 16.8; 13.6 (v) $n \ge 101$ where $n \in \mathbb{Z}^+$

8. (i) 1.13518; 0.420 (ii) $m < 1.29$

9. (a) 2.20 (b) $\frac{0.8}{\left(\frac{2.8}{\sqrt{n}}\right)} \ge 3.0902$; 117

10. (i) 133.2; 1 671.2 (ii) $\alpha < 1.7$

11. (i) 47.488; 330.986 (ii) $\mathrm{H}_1 : \mu < 49.5$; $\alpha > 4.02$ (iii) 0.0321

12. (b) $m < 54.7$ **13.** (i) 9.69; 0.277 (v) $\bar{x} \ge 9.87$

14. (a) (i) 124.4; 7.43 (b) 35

15. (iii) 77.3; 118.3

Topic 16: Correlation Coefficient and Linear Regression

2.	(ii)	$y = a + \frac{b}{x}$	(iii)	0.978	(iv)	(a)	\$46 900	(b)	\$33 500		
3.	(i)	0.956	(iv)	$y = 16.0x + 36.7$	(v)	£ 81 600					
4.	(iii)	–0.898	(iv)	–0.978 ; $y = 430 - 45.0\ln x$	(v)	134					
5.	(ii)	–0.885	(iii)	$x = 11.6 - 0.806t$	(iv)	–0.995	(v)	1.84			
6.	(c)	(ii)	$A = 14.38, B = 0.2530$	(iv)	7.66						
7.	(i)	0.906	(iv)	$a = -35.4, b = 22.3$	(v)	63.5					
8.	(i)	3	(ii)	16	(iii)	2.96, 0.587					
9.	(i)	0.888; $v = 15.8 + 0.814u$	(ii)	83	(iii)	$p = 74, q = 40$					
10.	(ii)	(a)	–0.982	(b)	–0.949	(iii)	21				
11.	(ii)	0.940	(iii)	$A = 5.53, B = 0.0476$	(iv)	26.9%					
12.	(iv)	$a = 0.639,\ b = 2.87$; 27th month									
13.	(a)	(i)	6	(b)	(i)	0.947	(ii)	36.39			
14.	(ii)	–0.982	(iii)	$y = 73.3 - 18.4\ln x$; $a = 6.89 \times 10^{31}$, $b = -18.4$	(iv)	86.0					
15.	(ii)	(a)	0.936	(b)	0.988	(iv)	50				