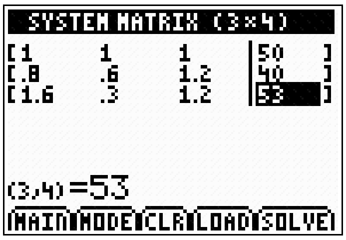
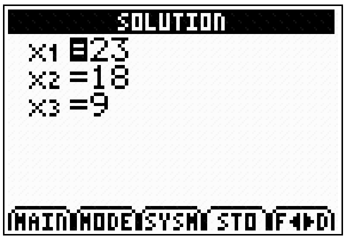
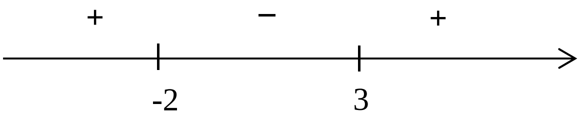
Prelims 2 Paper 1 Suggested Solutions

Qn	Solution
1	Let x, y, z be the number of apples, oranges and pineapples respectively. $x+y+z=50$ $0.8x+0.6y+1.2z=40$ $1.6x+0.3y+1.2z=53$  SYSTEM MATRIX (3×4) [1 1 1 | 50] [.8 .6 1.2 | 40] [1.6 .3 1.2 | 53] (3,4) =53  SOLUTION x1 =23 x2 =18 x3 =9 From GC, Adam bought 23 apples, 18 oranges and 9 pineapples.
2(i)	Since x, y, z are the first three terms of a geometric progression, $\frac{y}{x}=\frac{z}{y}$ $y^2=xz$ $x=\frac{y^2}{z}$ Since z, x, y are three consecutive terms of an arithmetic progression, $x-z=y-x$ $2x=y+z$ Solving the 2 above equations, $2\left(\frac{y^2}{z}\right)=y+z$ $2y^2=yz+z^2$ Dividing throughout by y^2 : $2=\left(\frac{z}{y}\right)+\left(\frac{z^2}{y^2}\right)$ $\therefore\left(\frac{z}{y}\right)^2+\left(\frac{z}{y}\right)-2=0$ (shown)
(ii)	Geometric progression has common ratio $r=\frac{z}{y}$.

	Solving $\left(\frac{z}{y}\right)^2+\left(\frac{z}{y}\right)-2=0$ gives $r=1$ or -2. Since $\lvert r\rvert \not< 1$, so sum to infinity of geometric progression does not exist.
3(i)	$x^2-2x+4=\left(x^2-2x+1^2\right)-1^2+3$ $=\left(x-1\right)^2+3$ Since $\left(x-1\right)^2\geq 0$, $\left(x-1\right)^2+3>0$ for all real values of *x*. Since $x^2-2x+4>0$ for all real values of *x*, $\frac{\left(x^2-2x+4\right)\left(x-3\right)}{\left(x+2\right)}\geq 0$ $\frac{\left(x-3\right)}{\left(x+2\right)}\geq 0$  $x<-2$ or $x\geq 3$
(ii)	Using (ii), $\frac{\left(x^2-2\lvert x\rvert+4\right)\left(\lvert x\rvert-3\right)}{\left(\lvert x\rvert+2\right)}\geq 0$ $\Rightarrow \lvert x\rvert<-2$ or $\lvert x\rvert\geq 3$, Since $\lvert x\rvert\geq 0$, reject $\lvert x\rvert<-2$ $\lvert x\rvert\geq 3 \Rightarrow \quad x\leq -3$ or $x\geq 3$
4(i)	$z^4=\sqrt{3}-\mathrm{i}$ $\left\lvert\sqrt{3}-\mathrm{i}\right\rvert=2$ $\arg\left(\sqrt{3}-\mathrm{i}\right)=-\frac{\pi}{6}$

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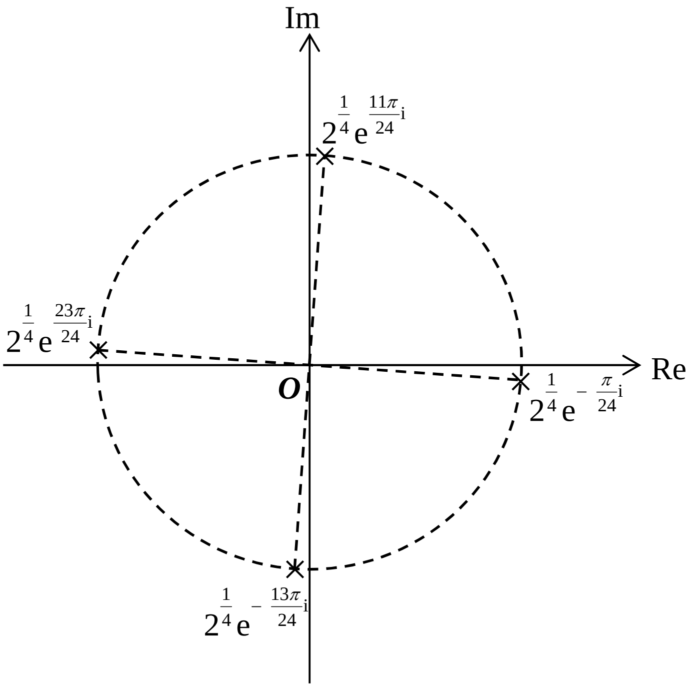
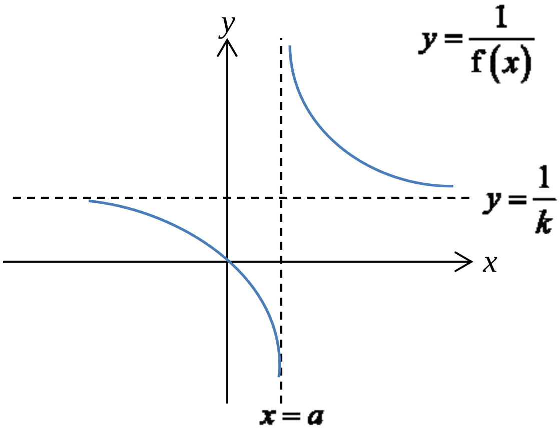
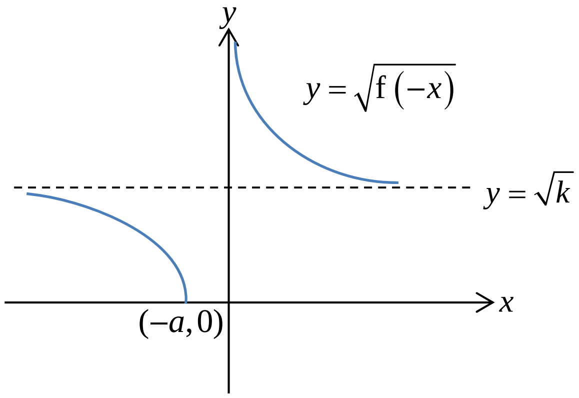
	$z^4 = 2\mathrm{e}^{-\frac{\pi}{6}\mathrm{i}}$ $z^4 = 2\mathrm{e}^{\left(2k\pi-\frac{\pi}{6}\right)\mathrm{i}}$ $z = 2^{\frac{1}{4}}\mathrm{e}^{\frac{1}{4}\left(\frac{12k-1}{6}\right)\pi\mathrm{i}} \quad , \ k = 0\,,\pm1\,,2$ $z = 2^{\frac{1}{4}}\mathrm{e}^{-\frac{13\pi}{24}\mathrm{i}}\ ,\ 2^{\frac{1}{4}}\mathrm{e}^{-\frac{\pi}{24}\mathrm{i}}\ ,\ 2^{\frac{1}{4}}\mathrm{e}^{\frac{11\pi}{24}\mathrm{i}}\ ,\ 2^{\frac{1}{4}}\mathrm{e}^{\frac{23\pi}{24}\mathrm{i}}$
(ii)	 The cartesian equation is $x^2 + y^2 = \sqrt{2}$
5(i)	Let $\dfrac{5+x^2}{(2+x)(1-x)^2} = \dfrac{A}{(2+x)} + \dfrac{B}{(1-x)^2}$. Hence $5+x^2 = A(1-x)^2 + B(2+x)$. Let $x=1$, $\quad 5+1^2 = B(2+1)$ $\therefore B = 2$ Let $x=-2$, $\quad 5+2^2 = A(1+2)^2$ $\therefore A = 1$ $\dfrac{5+x^2}{(2+x)(1-x)^2} = \dfrac{1}{(2+x)} + \dfrac{2}{(1-x)^2}$

(ii)	$\frac{5+x^2}{(2+x)(1-x)^2}$ $=\frac{1}{(2+x)}+\frac{2}{(1-x)^2}$ $=(2+x)^{-1}+2(1-x)^{-2}$ $=2^{-1}\left(1+\frac{x}{2}\right)^{-1}+2\left[1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2+\dots\right]$ $=\frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}+\dots\right)+2\left[1+2x+3x^2+\dots\right]$ $=\frac{5}{2}+\frac{15}{4}x+\frac{49}{8}x^2+\dots$
(iii)	For expansion of $(1-x)^{-2}$ to be valid, $\lvert -x\rvert<1$ $\lvert x\rvert<1$ For expansion of $\left(1+\frac{x}{2}\right)^{-1}$ to be valid, $\left\lvert\frac{x}{2}\right\rvert<1$ $\lvert x\rvert<2$ Hence for the expansion of $\frac{5+x^2}{(2+x)(1-x)^2}$ to be valid, $\lvert x\rvert<1$, $-1<x<1$.
6 **(a)** **(i)**	

(ii)	
(b)	Method 1 After 1st transformation: $\mathrm{g}\left(x\right)\rightarrow\mathrm{g}\left(\frac{1}{2}x\right)$ After 2nd transformation: $\mathrm{g}\left(\frac{1}{2}x\right)\rightarrow\mathrm{g}\left(-\frac{1}{2}x\right)$ After final transformation: $\mathrm{g}\left(-\frac{1}{2}x\right)\rightarrow 1+\mathrm{g}\left(-\frac{1}{2}x\right)$ $1+\mathrm{g}\left(-\frac{1}{2}x\right)=1-\frac{1}{x}$ $\mathrm{g}\left(-\frac{x}{2}\right)=-\frac{1}{x}\quad\Rightarrow\quad\mathrm{g}\left(\frac{x}{2}\right)=\frac{1}{2}\left(\frac{1}{\frac{x}{2}}\right)$ $\mathrm{g}\left(x\right)=\frac{1}{2x}$

	Method 2 Let $\mathrm{h}(x)$ be the expression after the final transformation. (a) Before final transformation: $1-\frac{1}{x}-1=-\frac{1}{x}$ (b) Before 2nd transformation: $-\left(-\frac{1}{x}\right)=\frac{1}{x}$ Before 1st transformation (original expression) $\frac{1}{2(x)}=\frac{1}{2x}$ $\mathrm{g}(x)=\frac{1}{2x}$
7(i)	$u_2=\frac{15}{16}$ $\quad u_3=\frac{63}{64}$ $\quad u_4=\frac{255}{256}$
(ii)	Considering $1-u_2=\frac{1}{16}$ $\quad 1-u_3=\frac{1}{64}$ $\quad 1-u_4=\frac{1}{256}$ $\therefore 1-u_n=\left(\frac{1}{2}\right)^{2n}$ Hence, the conjecture is $u_n=1-\left(\frac{1}{2}\right)^{2n}$. Let P_n be the statement $u_n=1-\left(\frac{1}{2}\right)^{2n}$ for $n\in\square^{+}$. When n = 1, LHS = u_1 = $\frac{3}{4}$ RHS $=1-\left(\frac{1}{2}\right)^{2(1)}=\frac{3}{4}$ Since LHS = RHS, P_1 is true. Assume that P_k is true for **some** $k\in\mathbf{Z}^{+}$.

i.e. assume $u_k = 1 - \left(\frac{1}{2}\right)^{2k}$, for some $k \in \mathbf{Z}^+$.

To prove that P_{k+1} is true,

i.e. prove $u_{k+1} = 1 - \left(\frac{1}{2}\right)^{2(k+1)}$.

$$\begin{aligned}
u_{k+1} &= u_k + \frac{3}{4}\left(\frac{1}{2}\right)^{2k} \\
&= 1 - \left(\frac{1}{2}\right)^{2k} + \frac{3}{4}\left(\frac{1}{2}\right)^{2k} \\
&= 1 - \left(\frac{1}{2}\right)^{2k}\left(1 - \frac{3}{4}\right) \\
&= 1 - \left(\frac{1}{2}\right)^{2k}\left(\frac{1}{4}\right) \\
&= 1 - \left(\frac{1}{2}\right)^{2k+2}
\end{aligned}$$

Hence, P_k is true $\Rightarrow P_{k+1}$ is true.

Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for **all** $n \in \mathbf{Z}^+$.

(iii)

$$\begin{aligned}\sum_{r=2}^{N}\frac{3}{4}\left(\frac{1}{2}\right)^{2r} &= \sum_{r=2}^{N}\left(u_{n+1}-u_n\right)\\ &= u_3-u_2\\ &\quad +u_4-u_3\\ &\quad +u_5-u_4\\ &\quad \vdots\\ &\quad +u_N-u_{N-1}\\ &\quad +u_{N+1}-u_N\\ &= u_{N+1}-u_2\\ &= \left(1-\left(\frac{1}{2}\right)^{2(N+1)}\right)-\left(1-\left(\frac{1}{2}\right)^{2(2)}\right)\\ &= \left(\frac{1}{2}\right)^{4}-\left(\frac{1}{2}\right)^{2N+2}\\ &= \left(\frac{1}{4}\right)\left(\frac{1}{4}-\left(\frac{1}{2}\right)^{2N}\right)\end{aligned}$$

(iv)

$$\left(\frac{1}{4}\right)\left(\frac{1}{4}-\left(\frac{1}{2}\right)^{2N}\right)>\frac{3}{50}$$

Using GC Table,

N	$\left(\frac{1}{4}\right)\left(\frac{1}{4}-\left(\frac{1}{2}\right)^{2N}\right)$
3	0.05859
4	0.06152
5	0.06226

Therefore, the smallest integer value of N is 4.

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8(a)	$\int_5^{p+9} \frac{1}{\sqrt{9-x}}\,\mathrm{d}x = \int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x$ $\int_5^{p+9} (9-x)^{-\frac{1}{2}}\,\mathrm{d}x = \frac{1}{2}\int_0^{\frac{1}{4}} \frac{2}{\sqrt{1-(2x)^2}}\,\mathrm{d}x$ $\left[-2(9-x)^{\frac{1}{2}}\right]_5^{p+9} = \frac{1}{2}\left[\sin^{-1}2x\right]_0^{\frac{1}{4}}$ $\left[-2(-p)^{\frac{1}{2}}+2(9-5)^{\frac{1}{2}}\right] = \frac{1}{2}\sin^{-1}\frac{1}{2}$ $4-2(-p)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{\pi}{6}\right)$ $2(-p)^{\frac{1}{2}} = 4-\left(\frac{\pi}{12}\right)$ $p = -\left[2-\left(\frac{\pi}{24}\right)\right]^2$ $= -\left(\frac{48-\pi}{24}\right)^2$
(b)	$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,\mathrm{d}x$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\left(-2\cos\theta\sin\theta\right)\mathrm{d}\theta$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos\theta}{\sqrt{\sin^2\theta}}\left(-2\cos\theta\sin\theta\right)\mathrm{d}\theta$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} -2\cos^2\theta\,\mathrm{d}\theta$ $= -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \left(\cos 2\theta+1\right)\mathrm{d}\theta$ $= -\left[\frac{1}{2}\sin 2\theta+\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}$ $= -\frac{1}{2}\left[\left(\sin\frac{\pi}{2}+\frac{\pi}{4}\right)-\left(\sin\pi+\frac{\pi}{2}\right)\right]$ $= -\frac{1}{2}\left(1+\frac{\pi}{4}-\frac{\pi}{2}\right)$ $= -\frac{1}{2}+\frac{\pi}{4}$

9(i)	$\overrightarrow{OP}=\frac{2}{5}\mathbf{a}$ Since *OPQB* is a parallelogram, $\overrightarrow{OB}=\overrightarrow{PQ}$ $\mathbf{b}=\overrightarrow{OQ}-\overrightarrow{OP}$ $\mathbf{b}=\overrightarrow{OQ}-\frac{2}{5}\mathbf{a}$ $\overrightarrow{OQ}=\frac{2}{5}\mathbf{a}+\mathbf{b}$
(ii)	Area of triangle *OAQ* $=\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OQ}\right\|$ $=\frac{1}{2}\left\|\mathbf{a}\times\left(\frac{2}{5}\mathbf{a}+\mathbf{b}\right)\right\|$ $=\frac{1}{2}\left\|\mathbf{a}\times\frac{2}{5}\mathbf{a}+\mathbf{a}\times\mathbf{b}\right\|$ $=\frac{1}{2}\left\|\mathbf{a}\times\mathbf{b}\right\|$ Therefore, k is $\frac{1}{2}$.
(iii)	*OPB* : *OAB* 2 : 5
(iv)	Since $\mathbf{a}\times\mathbf{b}$ is a unit vector, $\left\|\mathbf{a}\times\mathbf{b}\right\|=\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|\sin\theta$ $1=\left\|\mathbf{a}\right\|\left\|\mathbf{b}\right\|\sin\theta$ $1=2\left\|\mathbf{b}\right\|\sin 60^\circ$ Therefore, $\left\|\mathbf{b}\right\|=\frac{1}{\sqrt{3}}$

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<td>10(i)</td>
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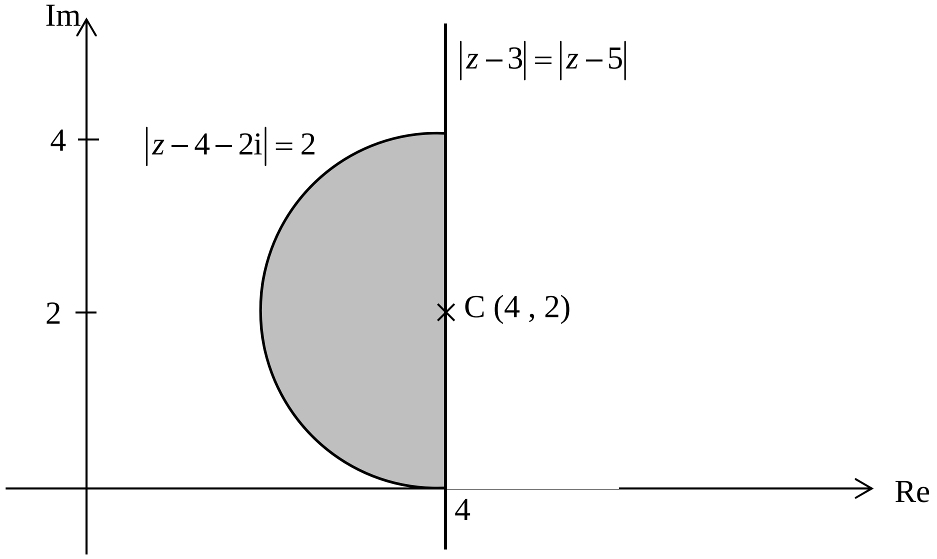

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<td>(ii)
(a)</td>
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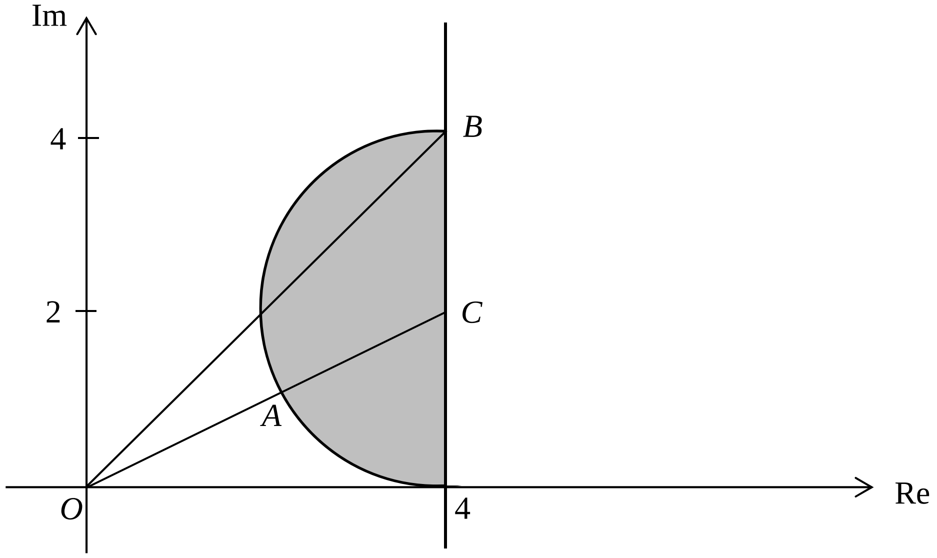

Greatest value of $\lvert z\rvert = OB$

$= 4\sqrt{2}$

Least value of $\lvert z\rvert = OA$

$= \sqrt{4^2+2^2} - 2$

$= \sqrt{20} - 2$

$= 2\sqrt{5} - 2$
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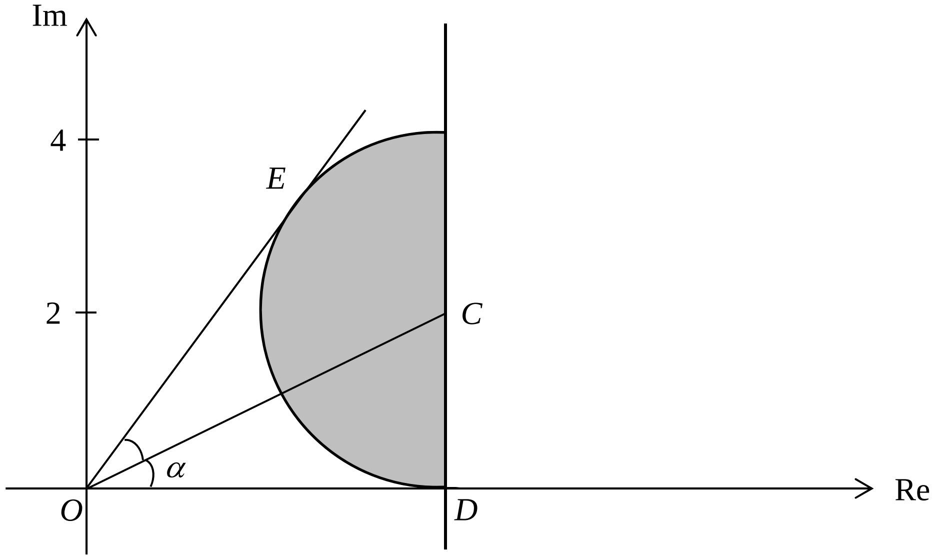


Least value of $\arg(z) = 0$

$\tan\alpha = \dfrac{2}{4}$

$\alpha = 0.46364$

Note that $\angle COE = \alpha$

Hence, greatest value of $\arg(z) = 2\alpha$

$= 0.927$

11
(i)

$x\cos 2x = 0$

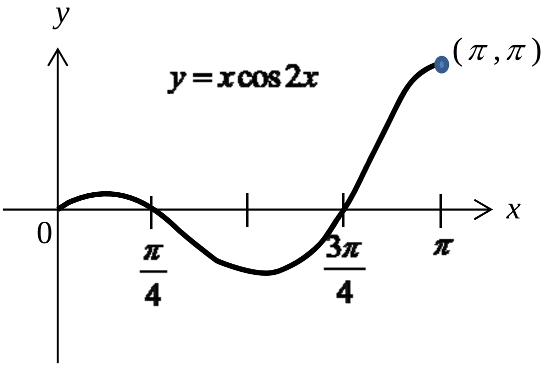
$x = 0 \quad \text{or} \quad \cos 2x = 0$

$2x = \cos^{-1} 0$

$= \dfrac{\pi}{2} \quad \text{or} \quad \dfrac{3\pi}{2} \quad \text{for} \quad 0 \le 2x \le 2\pi$

$x = \dfrac{\pi}{4} \quad \text{or} \quad \dfrac{3\pi}{4} \quad \text{for} \quad 0 \le x \le \pi$

Therefor the x-intercepts are $x = 0, \quad x = \dfrac{\pi}{4}, \quad x = \dfrac{3\pi}{4}$

(ii)	
(iii)	$\int x\cos 2x\ \mathrm{d}x$ $=\frac{1}{2}x\sin 2x-\int \frac{1}{2}\sin 2x\ \mathrm{d}x$ $=\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$ $\int_{\frac{\pi}{4}}^{\pi}\lvert x\cos 2x\rvert\ \mathrm{d}x$ $=-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\cos 2x\ \mathrm{d}x+\int_{\frac{3\pi}{4}}^{\pi} x\cos 2x\ \mathrm{d}x$ $=-\left[\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}+\left[\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x\right]_{\frac{3\pi}{4}}^{\pi}$ $=\frac{7\pi}{8}+\frac{1}{4}$
(iv)	$\int_{\frac{3\pi}{4}}^{\pi}\pi\left(x\cos 2x\right)^2\ \mathrm{d}x$ ≈ 10.465 $=10.5$ (3 s.f.)

12 (a)

$$\frac{1}{3}\pi r^2 h = 50\pi \text{ cm}^3$$

$$r^2 h = 150$$

$$h = \frac{150}{r^2}$$

$$h^2 + r^2 = l^2$$

$$A = \pi r l$$

$$\begin{aligned} A^2 &= \pi^2 r^2 l^2 \\ &= \pi^2 r^2 (h^2 + r^2) \\ &= \pi^2 r^2 h^2 + \pi^2 r^4 \\ &= \pi^2 r^2 \left(\frac{150}{r^2}\right)^2 + \pi^2 r^4 \\ &= \frac{22500\pi^2}{r^2} + \pi^2 r^4 \end{aligned}$$

Differentiating with respect to x:

$$2A\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{2(22500)\pi^2}{r^3} + 4\pi^2 r^3$$

Since $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$,

$$0 = -\frac{2(22500)\pi^2}{r^3} + 4\pi^2 r^3$$

$$\frac{2(22500)\pi^2}{r^3} = 4\pi^2 r^3$$

$$r^6 = 11250$$

$$r = 4.7336 \approx 4.73$$

$$h = 6.694 \approx 6.69$$

r	4.73^-	4.73	4.73^+
$\frac{\mathrm{d}A}{\mathrm{d}r}$	-ve	0	+ve
Slope	╲	—	╱

Therefore, r = 4.73 (3s.f) and h = 6.69 (3 s.f) require the least amount of material.

(bi) Given $V=\frac{\pi h^3}{12}$,

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{3\pi h^2}{12}=\frac{\pi h^2}{4}$$

When $h=3$, $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{4}{\pi(2)^2}\times -3=-\frac{3}{\pi}$(or -0.955)

The rate at which the depth is decreasing at the instant when the depth is 2 cm is $\frac{3}{\pi}$ cms^{-1}.

Alternative Method

Given $V=\frac{\pi h^3}{12}$,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}&=\frac{3\pi h^2}{12}\frac{\mathrm{d}h}{\mathrm{d}t}\\&=\frac{\pi(2)^2}{4}\left(-3\right)\\&=-\frac{3}{\pi}\end{aligned}$$

The rate at which the depth is decreasing at the instant when the depth is 2 cm is $\frac{3}{\pi}$ cms^{-1}.

(ii) Change in volume $=\frac{\pi\left(6^3-3^3\right)}{12}=\frac{189\pi}{12}$ cm^3

Time taken $=\frac{189\pi}{12}\div 3=\frac{189\pi}{36}$ s (or 16.5s (3s.f))