Chapter 3 (Pure Mathematics): Exponential and Logarithmic Functions

Objectives

At the end of the chapter, you should be able to:

(a)	understand the characteristics of the indices;
(b)	understand the characteristics of the exponential and logarithmic functions, e^x and $\ln x$
(c)	understand that $y = \mathrm{e}^x$ and $x = \ln y$ are equivalent statements;
(d)	understand the nature of exponential growth and decay;
(e)	solve simple exponential and logarithmic equations;

Content

1.1 Indices

1.1.1 Laws of Indices

1.1.2 Exponential Equations

1.2 A Particular Exponential Functions

1.3 Logarithmic functions

1.3.1 Definition of a Logarithm

1.3.2 Laws of Logarithm

1.4 Solving Miscellaneous Equations

References

New Syllabus Additional Mathematics (8th Edition), Shinglee Publishers Pte Ltd.

Introduction

There are many real life applications that use logarithm, from finding the time to reach an investment goal to modeling many natural processes, particularly in living systems. We perceive loudness of sound as the logarithm of the actual sound intensity, and dB (decibels) is a logarithmic scale. We also perceive brightness of light as the logarithm of the actual light energy, and star magnitudes are measured on a logarithmic scale.

Most interesting of all could be the measurement of earthquake intensity on the Richter scale. The Richter magnitudes are based on a logarithmic scale (base 10). What this means is that for each whole number you go up on the Richter scale, the amplitude of the ground motion recorded by a seismograph goes up ten times. Using this scale, a magnitude 5 earthquake would result in ten times the level of ground shaking as a magnitude 4 earthquake.

Magnitude of an earthquake is $$M = \log\left(\frac{I}{S}\right)$$

where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a "standard earthquake" (whose amplitude is 1 micron $=10^{-4}$ cm).

1.1 Indices

Background: Let us consider this example:

$$3^2 = 3 \times 3$$

where the **base** is 3 and the **index** is 2. Index is also called the **exponent**.

Laws of Indices

If bases a and b are **positive** real numbers, and indices m and n are real numbers, then

1st Law $a^m \times a^n = a^{m+n}$

Example $2^3 \times 2^5 = 2^8$

2nd Law $\dfrac{a^m}{a^n} = a^{m-n}$

Example $\dfrac{2^5}{2^3} = 2^{5-3} = 2^2$

3rd Law $\left(a^m\right)^n = a^{mn}$

Examples $\left(2^3\right)^5 = 2^{15}$ and $\left(2^5\right)^3 = 2^{15}$

4th Law $a^n \times b^n = (ab)^n$

Example $2^5 \times 3^5 = (2 \times 3)^5 = (6)^5$

5th Law $a^0 = 1$

Example $3^0 = 1$

6th Law $a^{-n} = \dfrac{1}{a^n}$ (n is a ***positive*** integer)

Example $2^{-3} = \dfrac{1}{2^3}$

7th Law $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ (m and n are ***positive*** integers)

Example $2^{\frac{5}{3}} = \sqrt[3]{2^5}$

8th Law $a^{\frac{1}{n}} = \sqrt[n]{a}$

Example $2^{\frac{1}{3}} = \sqrt[3]{2}$

1.1.2 Exponential Equations

To solve an exponential equation where both sides can be converted to the same base, use the following **Equality of Indices** where we equate the unknown index to the known index *n*.

If $a^x = a^n$, then $x = n$, where $a \neq -1, 0, 1$.

In general, if base $a > 0$, then $a^n > 0$ for all real values of *x*.

Consider the equation: $3^x = 81$.

This equation is called an exponential equation, which also involves an *unknown exponent or index* (in this case is known as *x*).

Example 1

Solve the following equations: (a) $3^x = 81$ (b) $3^{2x} \times 2^x = 18^{-1}$

Solution:

(a)

$$3^x = 81$$
$$3^x = 3^4$$
$$x = 4$$

(b)

$$3^{2x} \times 2^x = 18^{-1}$$
$$9^x \times 2^x = 18^{-1}$$
$$(9 \times 2)^x = 18^{-1}$$
$$x = -1$$

Example 2

Solve the equation $9^{x+1} = 1 - 8(3^x)$

Solution:

$$9^{x+1} = 1 - 8(3^x)$$
$$9^x \times 9^1 + 8(3^x) - 1 = 0$$
$$9(3^2)^x + 8(3^x) - 1 = 0$$
$$9(3^x)^2 + 8(3^x) - 1 = 0$$

Let $y = 3^x$. Then $9y^2 + 8y - 1 = 0$.

$$9y^2 + 8y - 1 = 0$$
$$(9y - 1)(y + 1) = 0$$

$9y-1=0$ or $y+1=0$

$y=\frac{1}{9}$ or $y=-1$

$3^x=\frac{1}{9}$ or $3^x=-1$ (no solution as $3^x>0$)

$3^x=\frac{1}{3^2}=3^{-2}$

$\therefore x=-2$

Example 3

Solve the following simultaneous equations:

$$2^x \times 4^y = \frac{1}{8} \quad \text{and} \quad \frac{9^x}{3^{y+1}}=27.$$

Solution:

$2^x \times 4^y = \frac{1}{8}$ and $\frac{9^x}{3^{y+1}}=27$

$2^x \times 2^{2y} = \frac{1}{2^3}$ $\qquad \frac{3^{2x}}{3^{y+1}}=3^3$

$x+2y=-3$ ....(1) $\qquad 3^{2x-(y+1)}=3^3$

$2x-y-1=3$

$2x-y=4$ ....(2)

$x+2y=-3$ ....(1)

$4x-2y=8$ ....(3) $\qquad 2\times(2)$

$5x=5$

$\therefore \quad x=1$

Substitute $x=1$ into (1) :

$1+2y=-3$

$2y=-4$

$y=-2$

The solution is $x=1$ and $y=-2$.

Exercise 1:

1. Solve the following simultaneous equations:

$$3^x \times 9^y = \frac{1}{3} \quad \text{and} \quad \frac{8^x}{2^{y+1}} = 8.$$

Solution:

$3^x \times 9^y = \frac{1}{3}$ and $\frac{8^x}{2^{y+1}} = 8$

$3^x \times 3^{2y} = 3^{-1}$ $\qquad \frac{2^{3x}}{2^{y+1}} = 8$

$x + 2y = -1$ ....(1) $\qquad 2^{3x-(y+1)} = 2^3$

$3x - y - 1 = 3$

$3x - y = 4$ ....(2)

$x + 2y = -1$ ....(1)

$6x - 2y = 8$ ....(3) $\quad 2\times(2)$

$7x = 7$

$\therefore \quad x = 1$

Substitute $x = 1$ into (1) :

$1 + 2y = -1$

$2y = -2$

$y = -1$

The solution is $x = 1$ and $y = -1$.

2. The equation of a curve is given by $y = ka^x$, where a and k are constants. Given that the curve passes through (2, 16), (3, 32) and (5, p), find the values of a, k and p.

Solution: $y = ka^x$

At point (2,16), $\quad x = 2, y = 16$

$y = ka^x$

$16 = ka^2$

$k = \frac{16}{a^2}$ ....(1)

At point $(3, 32)$, $x = 3, y = 32$

$$y = ka^x$$

$$32 = \left(\frac{16}{a^2}\right)a^3 \qquad \text{from (1)}$$

$$a = 2 \quad \text{..........................(2)}$$

$$\therefore \quad k = \frac{16}{2^2} = 4$$

At point $(5, p)$ $x = 5, y = p$

$$y = ka^x$$

$$p = \left(\frac{16}{2^2}\right)2^5 \qquad \text{from (1) and (2)}$$

$$p = 128$$

1.2 A Particular Exponential Function:

$$y = \mathrm{e}^x, \qquad x \in \mathbb{R}$$

Background

$y = \mathrm{e}^x$ is also known as the exponential function The expression e^x is a function with a number of applications. The value of e is $2.718281\ldots$.

It is a value arising from an attempt to perform a Calculus operation known as differentiation of the logarithmic function (see **1.3** below).

1.3 Logarithmic Functions

Background: If we need to solve the equation $10^x = 100$ we need only to rewrite the equation as $10^x = 10^2$, and arrive at the solution $x = 2$.

What do we do if we need to solve an equation like this: $10^x = 7$?
To solve for *x*, we **convert** the exponential equation to a logarithmic equation.

1.3.1 Definition of a Logarithm

For $x > 0$ and $b > 0$, $b \neq 1$,

$$y = \log_b x \text{ is equivalent to } b^y = x.$$

Where *b* is the **base** and *y* is the **index** or **exponent**.

Note

(i) Logarithms with base 10 is known as common log: $y = \log_{10} x = \lg x$

In particular,

$$y = \log_{10} 10 \Leftrightarrow 10^y = 10$$
$$\Rightarrow y = 1$$

(ii) Logarithms with base e is known as natural log: $y = \log_e x = \ln x$

In particular,

$$y = \log_e \mathrm{e} \Leftrightarrow \mathrm{e}^y = \mathrm{e}$$
$$\Rightarrow y = 1$$

Example 4

1. Convert the following from exponential equation to logarithmic equation.

(i) $4^3 = 64$ (ii) $5^4 = 625$ (Verify the answer with your calculator!)

2. Solve the equation $\lg 2x = 3$.

Solution:

1i) $4^3 = 64 \Leftrightarrow 3 = \log_4 64$ (ii) $5^4 = 625 \Leftrightarrow 4 = \log_5 625$

2. $\lg 2x = 3 \Rightarrow \log_{10}(2x) = 3$

$$10^3 = 2x$$
$$x = \frac{10^3}{2} = 500$$

Example 5

Solve the equation $\ln 2x = 5$.

Solution:

$$\ln 2x = 5$$
$$\log_e 2x = 5$$
$$2x = \mathrm{e}^5$$
$$x = \frac{\mathrm{e}^5}{2}$$

Laws of Logarithm

Assume $X, Y, a, b > 0$

1st Law $\log_a X + \log_a Y = \log_a XY$

Example $\log_2 3 + \log_2 5 = \log_2 15$

$\lg 3 + \lg 4 = \lg 12$

2nd Law $\log_a X - \log_a Y = \log_a \frac{X}{Y}$

Example $\log_2 20 - \log_2 5 = \log_2 \frac{20}{5} = \log_2 4 = \log_2 2^2 = 2\log_2 2 = 2$

3rd Law $\log_a X^n = n\log_a X$

Note that the 3rd law can be derived from the 1st law as illustrated in the example below:

Example $\log_a X^3 = \log_a (X \times X \times X) = \log_a X + \log_a X + \log_a X = 3\log_a X$

Exercise 2

Fill in the blanks.

1. $\log_3 4 + \log_3 13 =$ $\boxed{\log_3 52}$
2. $\lg 7 + \lg 4 =$ $\boxed{\lg 28}$
3. $\log_3 5 +$ $\boxed{\log_3 4}$ $= \log_3 20$
4. $\lg 6 +$ $\boxed{\lg 5}$ $= \lg 30$
5. $\log_3 \frac{13}{7} = \log_3 \boxed{13} - \log_3 \boxed{7}$
6. $\log_3 4x + \log_3 3x - \log_3 x^2 = \log_3 \boxed{12}$

1.4 Solving Miscellaneous Equations

$$\log_a x = \log_a y$$
$$\Leftrightarrow \quad x = y$$

In particular,

$\log_e x = \log_e y$

$\ln x = \ln y$

$\Leftrightarrow x = y$

Important Property:

$a^{\log_a x} = x$

$e^{\log_e x} = e^{\ln x} = x$

Important: The laws of logarithm are applicable to natural logarithm.

<u>Example 6</u>

1 Solve the following equations:

(a) $e^{3x-1} = 148$ (b) $e^{2x} + 3e^{-2x} = 4$

Solution:

(a) $e^{3x-1} = 148$

$\Rightarrow$ Apply ln to both sides: $\ln e^{3x-1} = \ln 148$

$(3x-1)\ln e = \ln 148$ (3rd Law)

$3x - 1 = \ln 148$ (Recall ln e is 1)

$\Rightarrow \quad x = \frac{1}{3}(1 + \ln 148)$

(b) $e^{2x} + 3e^{-2x} = 4$

Multiply e^{2x} throughout,

$(e^{2x})^2 + 3 = 4e^{2x}$

$\left(e^{2x}\right)^2 - 4\left(e^{2x}\right) + 3 = 0$

$\left(e^{2x} - 1\right)\left(e^{2x} - 3\right) = 0$

$e^{2x} = 1$ or $e^{2x} = 3$

Apply ln to both sides:

$\ln(e^{2x}) = \ln(1)$ or $\ln(e^{2x}) = 3$

$2x \ln e = \ln 1 = 0$ or $2x \ln e = \ln 3$ $(\because \ln 1 = 0 \text{ and } \ln e = 1)$

$x = 0$ or $x = \frac{1}{2}\ln 3$

2 Evaluate x if $\log_2(1+x) + \log_2(5-x) - \log_2(x-2) = 3\log_2 2$.

$\log_2 \frac{(1+x)(5-x)}{x-2} = 3\log_2 2$ (Law 1st and 2nd)

$\frac{(1+x)(5-x)}{x-2} = 2^3$

$5 + 4x - x^2 = 8x - 16$

$x^2 + 4x - 21 = 0$

$(x-3)(x+7) = 0$

$\Rightarrow \quad x = 3$ or $x = -7$ [rejected as $\log_2(1-7)$ or $\log_2(-7-2)$ is not defined.]

$\therefore \quad x = 3$

3 Solve the equation $\lg x = 10^{\lg 3}$ where $\lg x$ represents $\log_{10} x$.

$\lg x = 3 \Rightarrow x = 10^3$

4 Given that $u = \log_9 X$, find, in terms of u, (a) x, (b) $\log_9(3x)$.

(a) $x = 9^u$	(b) $\log_9(3x) = \log_9 3 + \log_9 x = \log_9 9^{\frac{1}{2}} + u = \frac{1}{2} + u$

Exercise 3

1 Solve the equation $4^x.32^x = 6$.

$4^x.32^x = 6 \Rightarrow 128^x = 6 \Rightarrow x\ln 128 = \ln 6$

$\Rightarrow x = \dfrac{\ln 6}{\ln 128}$

[or $x\lg 128 = \lg 6 \Rightarrow x = \dfrac{\lg 6}{\lg 128}$]

2 By means of the substitution $3^x = y$, solve the equation $3^{2x} - 3^{x+2} + 8 = 0$.

$3^{2x} - 3^{x+2} + 8 = 0 \Rightarrow (3^x)^2 - 3^2.3^x + 8 = 0 \Rightarrow y^2 - 9y + 8 = 0$

$\Rightarrow (y-1)(y-8) = 0 \Rightarrow y = 1$ or 8

$3^x = 1$ or $3^x = 8 \Rightarrow x = 0$ or $x = \dfrac{\ln 8}{\ln 3}$

3 Solve the simultaneous equations $\log_2(x-14y) = 3$, $\lg x - \lg(y+1) = 1$.

$\log_2(x-14y) = 3 \Rightarrow x - 14y = 2^3 = 8 \Rightarrow x = 14y + 8$ .... (1)

$\lg x - \lg(y+1) = 1 \Rightarrow \lg\dfrac{x}{y+1} = \lg 10 \Rightarrow x = 10y + 10$ .....(2)

$(1) - (2)$, $0 = 4y - 2 \Rightarrow y = \dfrac{1}{2}$, $x = 15$

4 Given that $\log_4 x = a$ and $\log_2 y = b$, express xy and $\dfrac{x}{y}$ as powers of 2. If $xy = 128$ and $\dfrac{x}{y} = 4$ calculate the values of a and b.

> $\log_4 x = a \Rightarrow x = 4^a = 2^{2a}$, $\log_2 y = b \Rightarrow y = 2^b$
>
> $xy = 2^{2a+b}$, $\dfrac{x}{y} = 2^{2a-b}$
>
> $xy = 128 = 2^7 \Rightarrow 2a + b = 7$ …. (1)
>
> $\dfrac{x}{y} = 4 = 2^2 \Rightarrow 2a - b = 2$ …. (2)
>
> (1) + (2), $4a = 9 \Rightarrow a = \dfrac{9}{4}$, $b = \dfrac{5}{2}$

5 Given that $e^x.e^y = e^5$ and that $\ln(2x + y) = \ln 3 + \ln 4$, calculate the value of x and of y.

> $e^x.e^y = e^5 \Rightarrow x + y = 5$ …. (1)
>
> $\ln(2x + y) = \ln 3 + \ln 4 = \ln 12 \Rightarrow 2x + y = 12$ ….(2)
>
> $(2) - (1)$, $x = 7$, $y = -2$

6 Solve the inequality $\lg x + \lg(2 - x) < 1$

$\lg x + \lg(2-x) < 1$ $\Rightarrow \lg x(2-x) < 1$ $\Rightarrow x(2-x) < 10$ $\Rightarrow -x^2 + 2x - 10 < 0$ $\Rightarrow x^2 - 2x + 10 > 0$ But since $x^2 - 2x + 10 = (x-1)^2 + 9 > 0$ Hence $x \in \mathbb{R}$	But for the logarithms to be defined, $x > 0$ and $2 - x > 0$ $\Rightarrow x > 0$ and $x < 2$ $\Rightarrow 0 < x < 2$

7 **2017/MJC/BT1/Q5**

After a nearby factory has commenced operation, the population P, of fish in a lake, after t months, can be modelled by

$$P = 600\left(2 + e^{-0.2t}\right).$$

(i) Find the initial population of fish in the lake just before the factory commences operation. [2]

(ii) Find, to the nearest whole number, the population of fish after one year. [2]

(iii) Explain in simple terms what will eventually happen to the population of fish in the lake using this model. [2]

(iv) Determine the number of complete months for the population of fish to first drop below 1218. [3]

(i)	$P = 600\left(2 + e^{-0.2t}\right)$ When $t = 0$, $P = 600(2+1)$ $P = 1800$
(ii)	$P = 600\left(2 + e^{-0.2(12)}\right)$ $P = 1254$
(iii)	As t increase to infinity, $e^{-0.2t}$ will tend to 0. Hence, P will tend to $600(2+0)$ which is 1200. The population of fish will decrease and approach 1200.
(iv)	$600\left(2 + e^{-0.2t}\right) < 1218$ Using GC, When $t = 17$, $P = 1220 > 1218$ When $t = 18$, $P = 1216.4 < 1218$ The population of fish will first drop below 1218 after 18 complete months. Alternatively, $600\left(2 + e^{-0.2t}\right) < 1218$ $2 + e^{-0.2t} < 2.03$ $e^{-0.2t} < 0.03$ $-0.2t < \ln(0.03)$ $t > \dfrac{\ln(0.03)}{-0.2}$ $t > 17.533$ The population of fish will first drop below 1218 after 18 complete months.

End answers

1. $x = \dfrac{\ln 6}{\ln 128}$	2. $x = 0$ or $x = \dfrac{\ln 8}{\ln 3}$	3. $y = \dfrac{1}{2},\ x = 15$
4. $a = \dfrac{9}{4},\ b = \dfrac{5}{2}$	5. $x = 7$, $y = -2$	6. $x \in \mathbb{R}$
7. (i) 1800 (ii) 1254 (iv) after 18 completed months.		

Summary

Laws of Indices

If bases a and b are **positive** real numbers, and indices m and n are real numbers, then

1st Law $a^m \times a^n = a^{m+n}$

2nd Law $\dfrac{a^m}{a^n} = a^{m-n}$

3rd Law $\left(a^m\right)^n = a^{mn}$

4th Law $a^n \times b^n = (ab)^n$

If base $a > 0$, and indices m and n are ***positive*** integers, then

5th Law $a^0 = 1$

6th Law $a^{-n} = \dfrac{1}{a^n}$

7th Law $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

8th Law $a^{\frac{1}{n}} = \sqrt[n]{a}$

Exponential functions

$y = e^x \Rightarrow \ln y = x,\ x \in \mathbb{R}, y > 0$

Note : $\log_e x$ is written as $\ln x$ (Natural logarithm)

Logarithmic functions

Assume $X, Y, a, b > 0$

1st Law: $\log_a X + \log_a Y = \log_a XY$

2nd Law: $\log_a X - \log_a Y = \log_a \dfrac{X}{Y}$

3rd Law: $\log_a X^n = n \log_a X$

Note: $\log_a a = 1$ and $\log_a 1 = 0$

Checklist

I am able to:

- ☐ understand the concept of indices
- ☐ solve exponential and logarithmic equations
- ☐ understand the nature of exponential growth and decay