Determinants, cross products *and* the scalar triple product

Exploring the cross product in greater detail

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Preface

> **looks at MF27* What is this crappy formula for the cross product?*
>
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Ask most H2 Mathematics students to give you the formula for a cross product. You'll be seeing one of two types of these students:

- The student can memorize and regurgitate the required answer.
- They either scavenge through their formula booklet or are unable to come up with the answer.

According to **MF27**, the official formula booklet, we have the following:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Would you look at that?! It's **hard to memorize**, hard to derive and just outright confusing!

Instead of that mess, what if I told you that there's a way to memorize it much more efficiently? Don't believe me?

Have a look below:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ c_3 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Now, you may be wondering: *what is that? What are those vertical lines?* No, they're not the absolute value; you can't take that for a matrix! Instead, we have what's called the **determinant**.

With the determinant, we can calculate something *explicitly excluded* from the syllabus: the **scalar triple product**, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. In fact, with the determinant, we can give it a geometric meaning and enhance our knowledge of vectors!

This document was written in Microsoft Word instead of the usual LaTeX, because making diagrams in LaTeX is a pain! Also, check out `https://lib.gsn.bz` for more stuff.

What is a determinant?

To put it simply, a *determinant* of a **square** matrix, denoted by $\det \mathbf{A}$, is a function of the square matrix which has some special properties. If $\mathbf{A}$ is two-dimensional, we have

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

where $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. But what is the geometric significance?

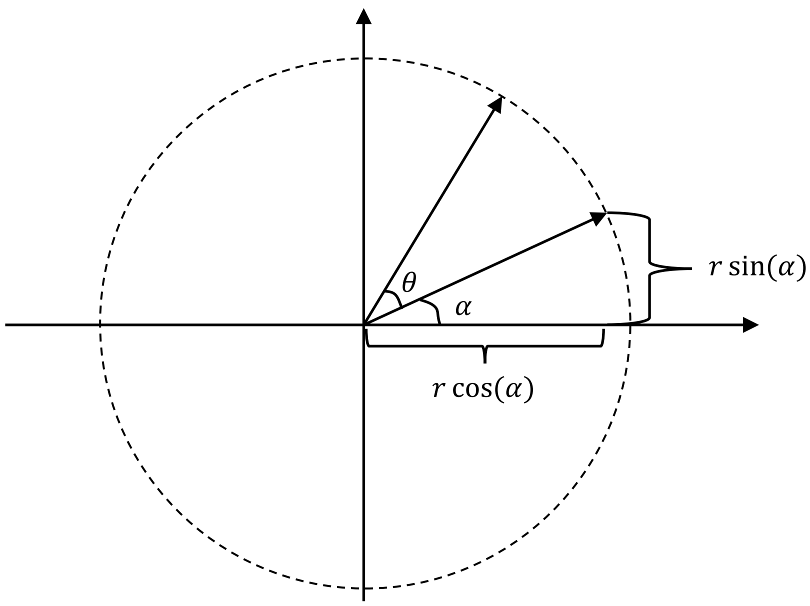
Suppose we have a vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. We define a *linear transformation* T as a function that, for any vectors $\mathbf{u}$ and $\mathbf{v}$, the properties

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \\ T(c\mathbf{u}) &= cT(\mathbf{u}) \end{aligned}$$

are true. For example, the transformation

$$\begin{aligned} x' &= x\cos(\theta) - y\sin(\theta) \\ y' &= x\sin(\theta) + y\cos(\theta) \end{aligned}$$

is linear. The above transformation corresponds to rotating a point about the origin by θ:



Note that $r = \sqrt{x^2 + y^2}$ and $\tan(\alpha) = \frac{y}{x}$. This allows us to consider the *angle* and the *length* of the line from the origin. See that

$$\begin{aligned} x' &= r\cos(\alpha + \theta) = r\cos(\alpha)\cos(\theta) - r\sin(\alpha)\sin(\theta) = x\cos(\theta) - y\sin(\theta) \\ y' &= r\sin(\alpha + \theta) = r\sin(\alpha)\cos(\theta) + r\cos(\alpha)\sin(\theta) = x\sin(\theta) + y\cos(\theta). \end{aligned}$$

Is this a linear transformation? Yes, it is.

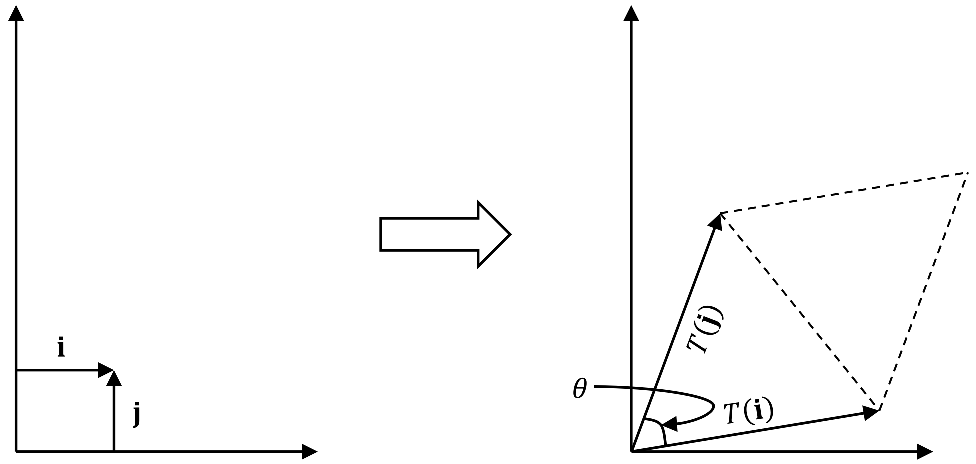
Exercise 1. Verify that this is a linear transformation.

Note that we can represent **any** linear transformation as a **matrix**:

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \Leftrightarrow R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}$$

This **key result** holds due to matrix multiplication.

Now, we are finally able to address the geometric significance of the determinant. Consider the *unit square* bounded by the vectors **i** and **j**, as shown below.



Applying a linear transformation T to this, we see a *parallelogram*. What is the area of this parallelogram? Obviously, it is

$$2 \times \frac{1}{2}|T(\mathbf{i})||T(\mathbf{j})|\sin(\theta) = |T(\mathbf{i})||T(\mathbf{j})|\sin(\theta).$$

Now, suppose that $T(\mathbf{i})$ is the position vector of the point (a, b) and $T(\mathbf{j})$ that of the point (c, d). Due to our result from earlier, we can express:

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Exercise 2. Verify that applying this to the unit vectors **i** and **j** yields the position vectors mentioned above.

Considering the perpendicular vector $\perp T(\mathbf{i}) = \begin{bmatrix} -b \\ a \end{bmatrix}$, we can express the above area using the dot product, since

$$\begin{aligned} |T(\mathbf{i})||T(\mathbf{j})|\sin(\theta) &= |\perp T(\mathbf{i})||T(\mathbf{j})|\sin(90^\circ - \theta) \\ &= |\perp T(\mathbf{i})||T(\mathbf{j})|\cos(\theta) \\ &= \big(\perp T(\mathbf{i})\big) \cdot T(\mathbf{j}) \\ &= \begin{bmatrix} -b \\ a \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} \\ &= ad - bc. \end{aligned}$$

Notice that we can now see the determinant **representing the (signed) area of a parallelogram** created by the vectors $T(\mathbf{i})$ and $T(\mathbf{j})$; furthermore, since the original area created by the two unit vectors was $|1||1|\sin(90^\circ) = 1$, the determinant can *represent the* ***change*** *in area* caused by a linear transformation.

Exercise 3. Deduce that swapping the rows of a 2×2 matrix negates its determinant.

Now, we define the determinant for a 3×3 matrix as follows:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

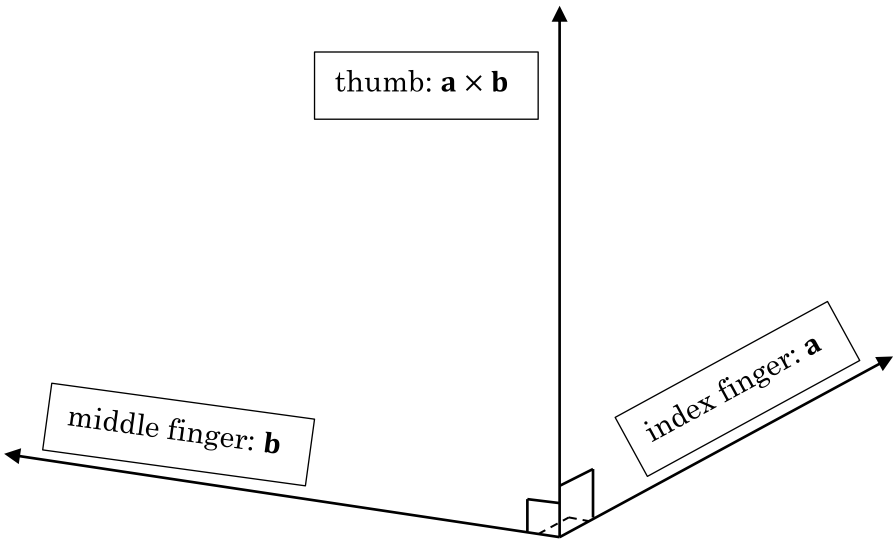
Sorry if this definition is unmotivated; you can find a derivation of this online.

The cross product with the determinant

Recall that the cross product of two three-dimensional vectors **a** and **b** is defined as the vector perpendicular to both vectors such that the vector's magnitude is the area of the parallelogram created by **a** and **b**. We'll have the following formula:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin(\theta)\, \hat{\mathbf{n}}.$$

Here, $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane containing the two vectors, which follows the right hand rule (shown below).



If we look at the unit vectors **i**, **j**, and **k**, we usually have the following:

$$\begin{gathered} \mathbf{i} \times \mathbf{j} = \mathbf{k} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j}. \end{gathered}$$

Since any vector can be defined as a linear combination (a sum) of these vectors, we'll have

$$\begin{aligned} &\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + \mathrm{a}_2\mathbf{j} + \mathrm{a}_3\mathbf{k}) \times (b_1\mathbf{i} + \mathrm{b}_2\mathbf{j} + \mathrm{b}_3\mathbf{k}) \\ &= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \end{aligned}$$

Remark. *The full working is not shown here, for brevity.*

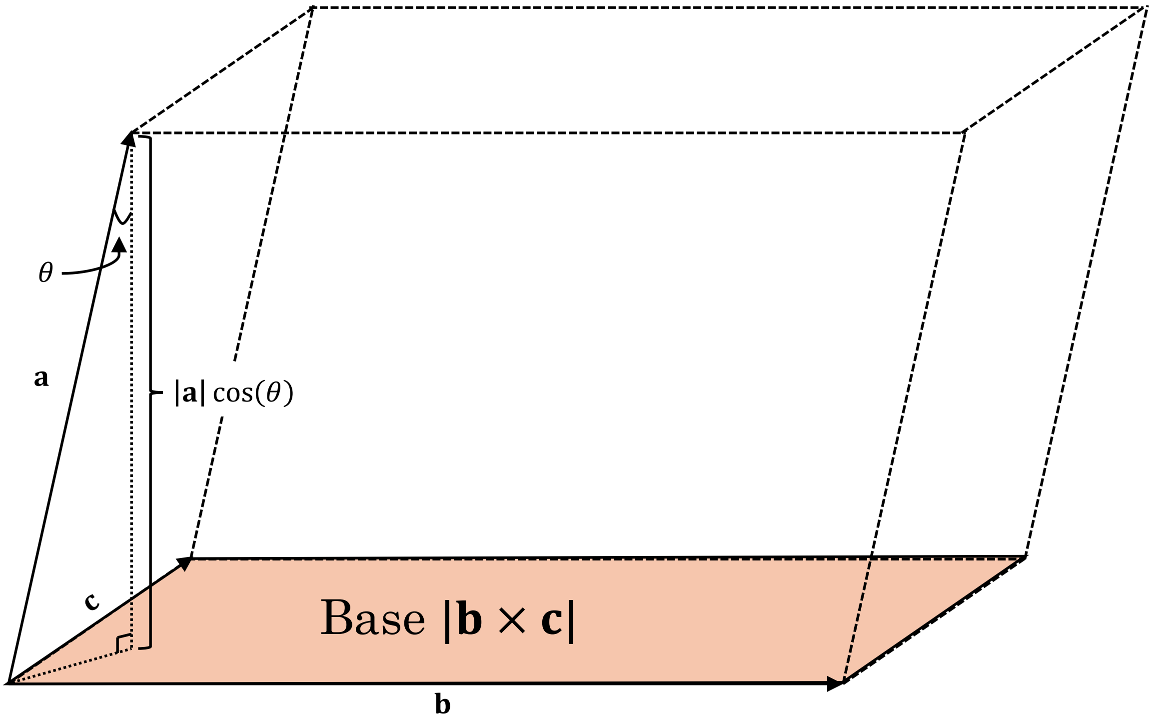
Because of Exercise 3, we can observe that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

The scalar triple product

Finally, we get to the key highlight of this document: the scalar triple product. Let's look at the *scalar triple product* first:

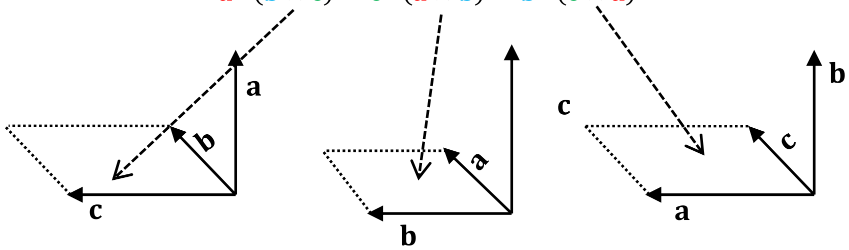
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

That's cool and all, but what exactly does this mean? Well, we know that the cross product will give us the volume of the parallelogram created by **b** and **c** and extended into a prism by the vector **a**. Such a solid is called a *parallelepiped*, which is shown below:



The vectors follow a *right-hand rule*. We have the result

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$



Using the dot (or scalar) product, see that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = |\mathbf{a}||\mathbf{b} \times \mathbf{c}|\cos(\theta)$$

where θ is the angle between **a** and the perpendicular to the plane containing **b** and **c** (that's the definition of the cross product). As the diagram shows above, see that we can *circularly* shift the vectors and get the same volume!

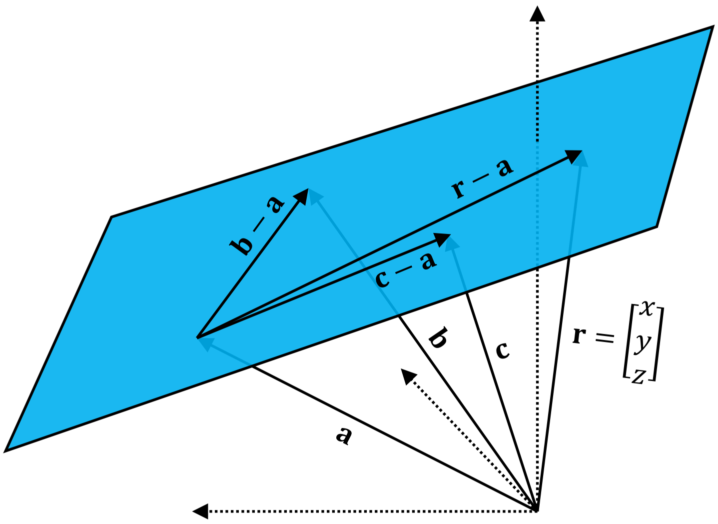
Remark. *If you swap the vectors above (e.g.* $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$*), the cross product has the property* $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$*, implying that* $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Notice that if **a**, **a**, and **a** represented **coplanar** points, the volume of the parallelepiped that the three vectors create is **zero**.

In fact, using the determinant, one has

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Using this fact, we'll be able to find the Cartesian equation for the plane containing three points (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3). Let's represent these **coplanar** points with the vectors **a**, **b**, and **c**, respectively.



See that since the three points are coplanar, and if the point represented by the position vector **r** lies on the plane, the vectors create a parallelepiped with zero volume. Particularly,

$$(\mathbf{r} - \mathbf{a}) \cdot [(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})] = 0.$$

Remark. *Another interpretation of this can be that the vector* $(\mathbf{r} - \mathbf{a})$ *is perpendicular to the cross product* $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$.

In fact, at this point, the order of the vectors above does not matter, since they will still create a volume of zero!

Therefore, the Cartesian equation of the plane containing the three points is

$$\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & b_3 - a_3 \end{vmatrix} = 0.$$

If you have a line and a point, you just need to find two points on that line and compute the determinant above.

That's it!

Conclusion

All in all, we've seen the uses of the determinant, and its geometrical meanings. With a linear transformation (expressed as a square matrix), we can see the factor by how much any area spanned (created) by two vectors changes. Also, the determinant gives us an easier way to memorize the formula for computing a cross product, so we do not need to rely primarily on the formula booklet **MF27** to calculate it!

Furthermore, by visualizing a parallelepiped – the three-dimensional version of a parallelogram, we saw that the triple scalar product can be expressed as a determinant, and this allows us to find the Cartesian equation of a plane containing any three points, with an easy geometric intuition!

I hope you enjoyed reading this document and learning some mathematics!

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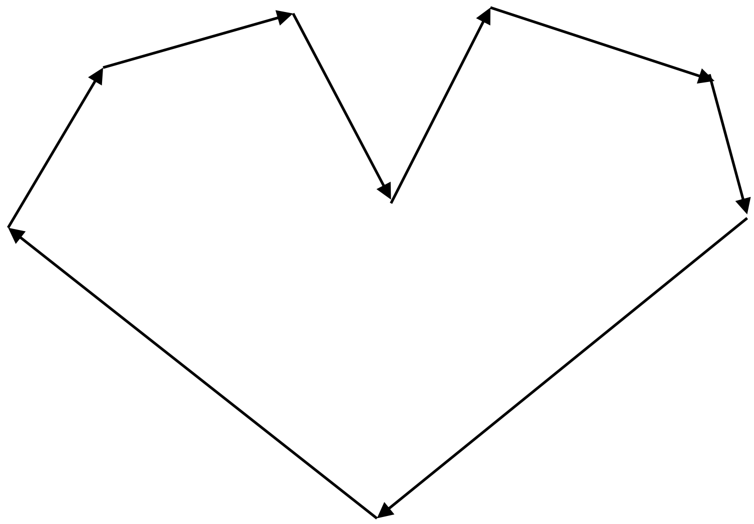

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