RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2019 Year 6

Chapter 15C: Numerical Methods of First Order Differential Equations

SYLLABUS INCLUDES

H2 Further Mathematics:

- Approximation of solutions of first order differential equations using Euler's method (including the use of the improved Euler formula)

PRE-REQUISITES

- Differentiation & Integration Techniques
- Differential Equations

CONTENT

1 Introduction

In this chapter we would introduce numerical methods to approximate the solutions of such differential equations, especially for those where a solution cannot be obtained using calculus, i.e. the differential equation cannot be solved exactly and explicitly using methods discussed in Chapters 14A and 14B. For example $\frac{dy}{dx}+y^2=x$ is such a differential equation. Let us start with an example to illustrate a simple approximation.

Suppose the balance y in a bank account is related to time t via the differential equation

$$\frac{dy}{dt}=0.01y,$$

the interpretation would then be that interest is compounded at a rate of 1%. However, since t is continuous, the interest is compounded continuously, instead of at fixed discrete intervals. Hence to get the balance at t, $y(t)$, the solution is not $y(t)=y_0(1+0.1)^t$ where y_0 represents the initial amount in the bank account. To calculate values y_n which are intended to approximate the true values $y(t_n)$ where t_n represents the balance at the n^{th} period, we use the definition of the derivative

$$0.01y(t)=\lim_{\Delta t\to 0}\frac{y(t+\Delta t)-y(t)}{\Delta t},$$

to make an approximation

$$0.01y_n\approx\frac{y_{n+1}-y_n}{\Delta t},$$

which can be rearranged into $y_{n+1}=y_n+\Delta t(0.01y_n)$.

If we are interested in the balance at yearly intervals, set $\Delta t=1$, we obtain

$$y_{n+1}=y_n+0.01y_n=1.01y_n,$$

hence by treating the continuous interest rate as an annual interest rate, we can approximate the balance in the account at the n^{th} year using $y_n=y_0(1+0.1)^n$.

The above illustrates an instance where the solution of a differential equation is approximated by a numerical approach.

In this chapter, we have the following:

- Let $t,y\in\mathbb{R}$, then $f(t,y)$ is a function in two variables such that $f:\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$.

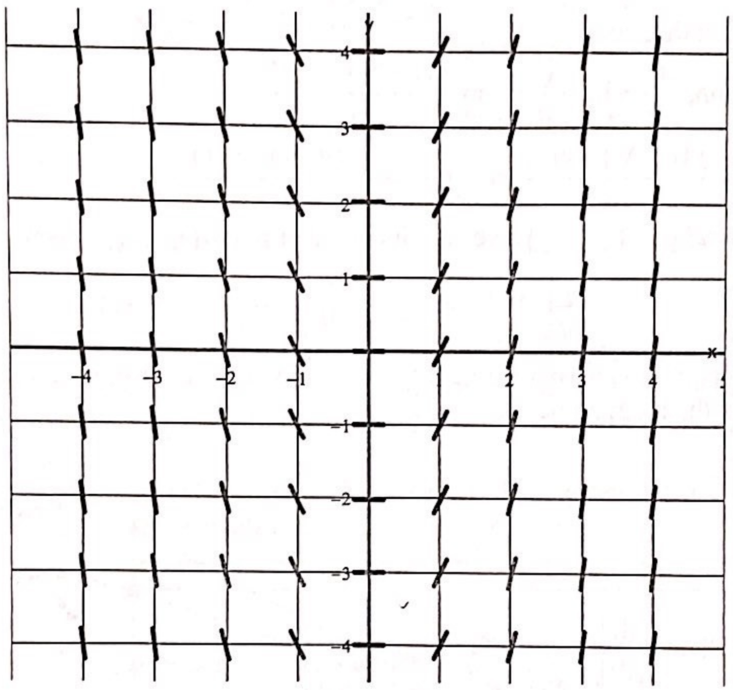
  (E.g. $f(t,y)=t^2-y$, then $f(2,1)=2^2-1=3$, i.e. given any pair of real values of t and y, the function maps it to a real constant)
- Define time steps as a sequence of real numbers such that $t_n=t_{n-1}+\Delta t$, where Δt is a constant, known as the **step size**.

Before we look at the two key numerical methods you will be need to know, let us first explore how we can use the idea of a slope field to visually have an idea on how the solution curve might look.

1.1 Slope Field

Suppose $y(x)$ is a particular solution to the differential equation $\frac{dy}{dx} = f(x, y)$. Then the tangent at each point of the graph of $y(x)$ has a slope given by the value of $\frac{dy}{dx} = f(x, y)$ at that point. If we systematically evaluate f over a rectangular grid of points in the xy-plane and draw a mini line segment, known as a '**slope mark**' at each point of the grid with slope defined by f, then the resulting diagram consisting of all these slope marks is called a **slope field** or a **tangent field** or a **direction field** of the differential equation $\frac{dy}{dx} = f(x, y)$.

The diagram below shows the slope field diagram for $\frac{dy}{dx} = 2x$ with a 9×9 grid.



1.2 Applications of Slope Field

There are two pieces of qualitative information that can be readily found from the slope field for a differential equation.

(a) Sketch of solutions.

Since the slope marks in the slope field are in fact tangents to the actual solutions to the differential equations, the slope field provides a visual appearance or shape of a family of solution curves of the differential equation. Particular solutions for the differential equation can be sketched by following the slope marks in such a way that the solution curves are tangent to each of the slope marks they meet.

(b) Long Term Behaviour.
In many cases we are less interested in the actual solutions to the differential equations as we are in how the solutions behave as x increases. Slope fields can be used to find information about this long-term behaviour of the solution. This is particular useful if one is interested in how a physical quantity(y) behaves with respect to time(t) in the long run.

2 Euler's Method

The approach illustrated in the introduction is an example of Euler's method which key principle is the use of a linear approximation for the tangent to the solution curve.
Given an initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{f}(t, y), \ y(t_0) = y_0$$

we can evaluate f for any given pair of values (t, y), and hence obtain the slope of the solution curve passing through that point.

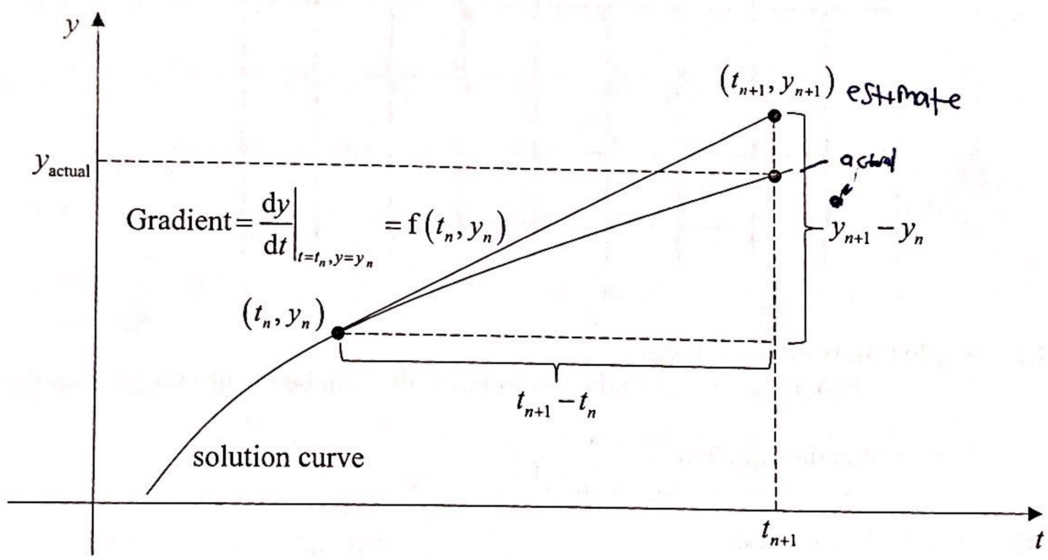
Recall from definition, $\frac{\mathrm{d}y}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}$.

For small Δt, $\frac{\mathrm{d}y}{\mathrm{d}t} \approx \frac{y(t+\Delta t) - y(t)}{\Delta t} \Rightarrow \mathrm{f}(t, y) \approx \frac{y(t+\Delta t) - y(t)}{\Delta t}$.

Hence, given a general point (t_n, y_n), we can find another point (t_{n+1}, y_{n+1}) using

$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=t_n, y=y_n} = \mathrm{f}(t_n, y_n) \approx \frac{y_{n+1} - y_n}{t_{n+1} - t_n} \qquad (1)$$

and repeat this procedure to approximate the solution curves via a series of linear approximations. This is illustrated in the diagram below.



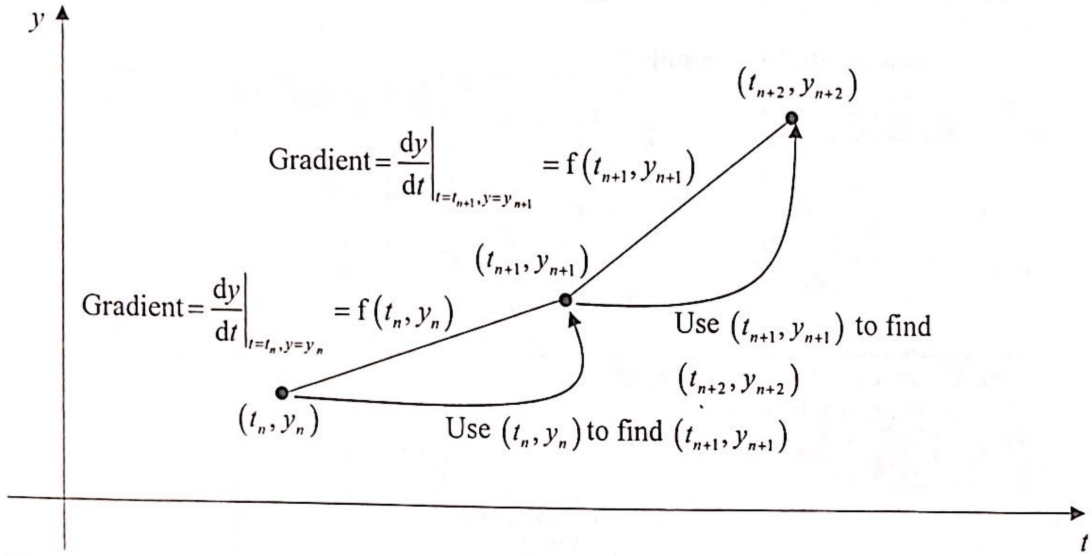
In general, we choose a fixed value for Δt (time step) and set $t_{n+1} = t_n + \Delta t$. Thus (1) becomes

$$\boxed{\frac{y_{n+1} - y_n}{\Delta t} \approx \mathrm{f}(t_n, y_n)},$$

or equivalently,

$$\boxed{y_{n+1} = y_n + \Delta t \cdot \mathrm{f}(t_n, y_n)}.$$

Suppose that the value of y_n is known or already determined at t_n, we can find the value of y at t_{n+1}. Repeating this process, we would be able to obtain an approximation for the value of y for any desired value of t. This is illustrated in the diagram below.



Example 1

Consider the initial value problem

$$\frac{dy}{dt} = 2y - 1, \quad y(0) = 1.$$

(i) Apply Euler's method to obtain an approximation for $y(1)$ using step size 0.2.

(ii) Plot all intermediate approximations on a graph and compare them with the exact solution.

(iii) Repeat (i) and (ii) with step size $\Delta t = 0.1$ instead. Comment on the effects of reducing the step size.

(iv) Explain without reference to (ii), whether the answers to (i) would likely be under-estimates or over-estimates.

Solution

Here $f(t, y) = 2y - 1$, so Euler's formula will take the form

$y_{n+1} = y_n + \Delta t \cdot (2y_n - 1)$

(i) Beginning with $n = 0$, $t_0 = 0$, $y_0 = 1$ and setting $\Delta t = 0.2$:

we first compute $f(t_n, y_n)$: $f(t_0, y_0) = 2y_0 - 1 = 2(1) - 1 = 1$

then $\Delta t \cdot f(t_n, y_n)$: $\Delta t \cdot f(t_0, y_0) = 0.2(1) = 0.2$

then using Euler's formula to compute y_{n+1}: $y_1 = y_0 + \Delta t\,(2y_0 - 1) = 1.2$

Repeating the same procedure for $n = 1, 2, 3$ and 4:

Observe that $y_n = y_{n-1} + 0.2(2y_{n-1} - 1)$ is equivalently $y_{n+1} = 1.4y_n - 0.2$, therefore, similar to how we compute successive terms in a recurrence relation, G.C. can be used to calculate the approximations quickly.

G.C. Commands for Example 1

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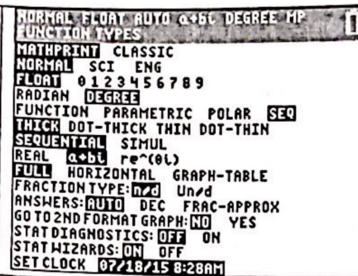
MODE ▼ ▼ ▼ ▶ ▶ ▶ ENTER</td><td>
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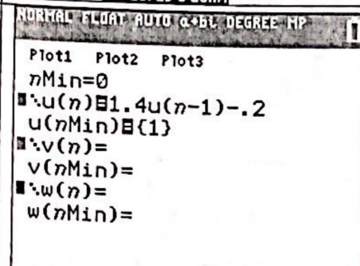
Y= 1 . 4 2nd 7 (X,T,Θ,n − 1) − 0 . 2 ▼ 1 ENTER</td><td>
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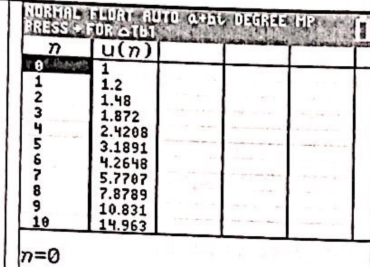
<tr><td>Copy values from table.</td><td>
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It is often useful to construct a table for the computation using Euler's method,

t	y	$2y-1$	$\Delta t(2y-1)$
0	1.0000	1.0000	
0.2	1.2000		
0.4	1.4800	1.9600	0.3920
0.6	1.8720	2.7440	0.5488
0.8	2.4208	3.8416	0.7683
1.0	3.1891	-	-

(ii) Solving $\frac{dy}{dt} = 2y - 1$ with initial condition $y(0) = 1$, we obtain the solution

$$y = \frac{1}{2}\left(e^{2t} + 1\right) \quad (2)$$

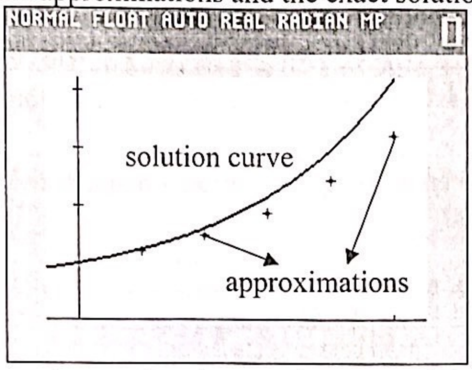
We can then compute the actual value y_{actual} using (2) and thus, the absolute error given by $|y_{\text{actual}} - y_{\text{approx}}|$ at each step and the percentage error

$$\frac{|y_{\text{actual}} - y_{\text{approx}}|}{y_{\text{actual}}} \times 100\%$$

of each approximate value.

t_n	y_n	$f(t_n, y_n)$	$\Delta t \cdot f(t_n, y_n)$	y_{actual}	Error $\lvert y_{\text{actual}} - y_n \rvert$	Percentage error (%)
0	1.0000	1.0000	0.2000	1.000	0.0000	0.000
0.2	1.2000	1.4000	0.2800	1.2459	~~0.049~~	3.6850
0.4	1.4800	1.9600	0.3920	1.6128	0.1328	8.2324
0.6	1.8720	2.7440	0.5488	2.1601	0.2881	13.3357
0.8	2.4208	3.8416	0.7683	2.9765	0.5557	18.6700
1.0	3.1891	-	-	4.1945	1.0054	23.9695

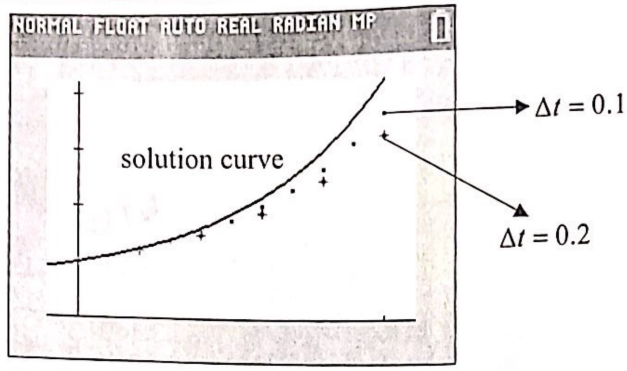
Plotting the intermediate approximations and the exact solution on the same axes:



From the plot above, we observe that the error $|y_{\text{actual}} - y_n|$ increases as n increases, i.e. as t_n gets further away from t_0.

(iii) Repeating with step size $\Delta t = 0.1$, we obtain the table below,

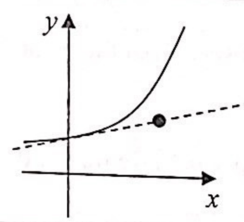
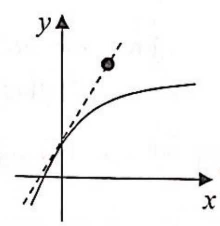
t_n	y_n	$f(t_n, y_n)$	$\Delta t \cdot f(t_n, y_n)$	y_{actual}	Error $\lvert y_{\text{actual}} - y_n \rvert$	Percentage error (%)
0	1.0000	1.0000	0.1000	1.0000	0.0000	0.0000
0.1	1.1000	1.2000	0.1200			0.9835
0.2	1.2200	1.4400	0.1440			2.0798
0.3	1.3640	1.7280	0.1728	1.4111	0.0471	3.3350
0.4	1.5368	2.0736	0.2074	1.6128	0.0760	4.7106
0.5	1.7442	2.4883	0.2488	1.8591	0.1150	6.1846
0.6	1.9930	2.9860	0.2986	2.1601	0.1671	7.7343
0.7	2.2916	3.5832	0.3583	2.5276	0.2360	9.3373
0.8	2.6499	4.2998	0.4300	2.9765	0.3266	10.9728
0.9	3.0799	5.1598	0.5160	3.5248	0.4449	12.6229
1.0	3.5959	-	-	4.1945	0.5987	14.2724



↓step size → ↓% error

(iv) From the table in **(i)**, we observe that grads seem to be ↑ as t ↑, hence the approximation wld be an underestimate

Note: In general, if $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing or decreasing in the interval investigated, i.e. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \geq 0$ or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \leq 0$ respectively, then the approximation obtained would be under- and over-estimates respectively.

Nature of First Derivative	Increasing	Decreasing
Graph		
Nature of Estimate	Under-estimate	Over-estimate

The reality however is far from this…

Whether an estimate is an under-estimate or over-estimate very much depends on the **step size.**

Consider the following example:

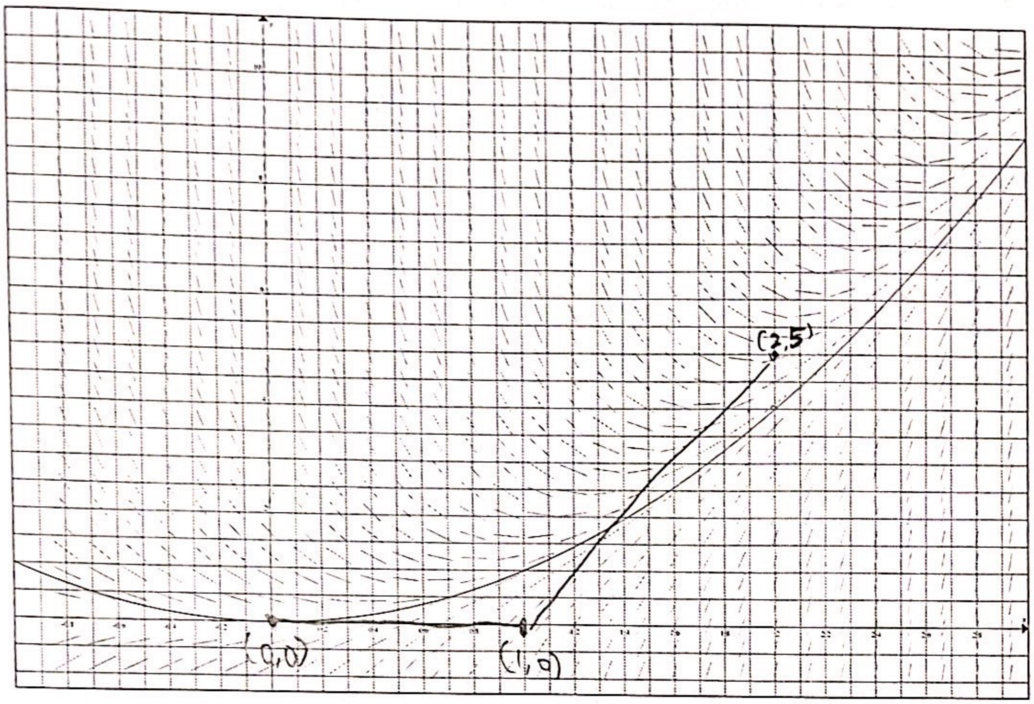
A particular solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 3(y - x^2)$ has $y = 0$ when $x = 0$. Use the Euler method with step size 1 to estimate y at $x = 2$.

Constructing a table for computation using Euler method with $\Delta x = 1$:

x	y	$2x-3(y-x^2)$	$\Delta x(2x-3(y-x^2))$
0	0	0	0
1	0	5	5
2	5	-	-

Therefore, at $x = 2$, the value of y is 5. So the increasing gradient suggests this is an under-estimate.

However, it is easy to check the exact solution to the above initial value problem is $y = x^2$. But this means the actual value when $x = 2$ is 4. Which means 5 is an over-estimate. WHAT HAS HAPPENED??



To understand what has happened, let us look at the slope field and the solution curve.

First notice that no matter the choice of the step size, the first approximation is always 0, since the gradient of the slope field at the origin is 0.

Then as x increases (when $y = 0$), we see that the gradient of the slope field increases, so it should not be surprising that for a sufficiently large step size, the subsequent estimate will be an over-estimate, instead of an under-estimate.

For different differential equations, the step-size required to ensure that our estimates are as we expect, over/under-estimates according to the second derivative, is clearly going to be different.

To further illustrate this, consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - n(y - x^2)$ that has $y = 0$ when $x = 0$. Let h be the step size. As discussed above, Euler's method gives $y(h) = 0$. Calculating $y(2h)$, we have $y(2h) = y(h) + h(2h + nh^2) = h^2(2 + nh)$. The exact solution is again $y = x^2$. So comparing against the actual value of $(2h)^2 = 4h^2$, we see that $h^2(2 + nh) > 4h^2 \Leftrightarrow nh > 2 \Leftrightarrow n > \frac{2}{h}$. What this means is that given any step size h, it suffices to choose an n satisfying the inequality above for the Euler Method to produce an overestimate of the actual value at $y(2h)$.

So whenever this question is posed at the A-level, what you should do is as we have shown you

1)	Discuss whether $\frac{dy}{dx}$ is increasing/decreasing at the 2/3 points you are using the Euler method and conclude that you have an under-estimate/over-estimate.
If you want to feel safer,	
2)	Show that $\frac{d^2y}{dx^2}>0$ or $\frac{d^2y}{dx^2}<0$ for all x, y In that ~~integral~~ interval and conclude again you have an under-estimate/over-estimate.
If you want to feel even safer,	
3)	Calculate the exact value and make the 100% correct conclusion.
To show off without wasting time doing 3),	
4)	Say that whether it is an under/over-estimate depends on the step size.

Example 2 [9824/2009/1(i)]

A particular solution of the differential equation $\frac{dy}{dx}=\frac{3x^4+y^2}{2xy}$ has $y=1$ when $x=1$.

Use the Euler method with step size 1 to estimate y at $x=3$. State with a reason whether this value of y is an under-estimate or an over-estimate.

Solution

Constructing a table for computation using Euler method with $\Delta x=1$:

x	y	$\frac{3x^4+y^2}{2xy}$	$\Delta t\left(\frac{3x^4+y^2}{2xy}\right)$
1	1	2	2
2	3	4.75	4.75
3	7.75	-	-

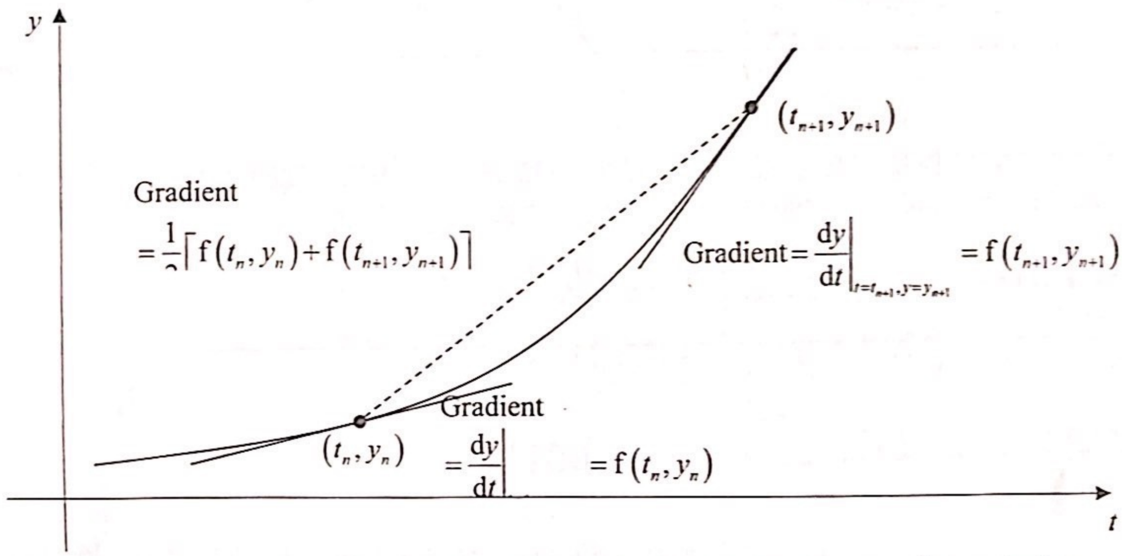
Therefore, at $x=3$, the value of y is 7.75.

The value is an under-estimate as $\frac{dy}{dx}=\frac{3x^4+y^2}{2xy}$ seems to be increasing as x increases.

3 Improved Euler's Method

The Euler formula is based on the assumption that the gradient of the line segment joining any two successive points, (t_n, y_n) and (t_{n+1}, y_{n+1}), on the graph of y is equal to the value of $\frac{\mathrm{d}y}{\mathrm{d}t}$ at (t_n, y_n). To improve the approximation, one may instead assume that the gradient of the line segment is equal to the average of the values of $\frac{\mathrm{d}y}{\mathrm{d}t}$ at (t_n, y_n) and (t_{n+1}, y_{n+1}):

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{1}{2}\left[\mathrm{f}(t_n, y_n) + \mathrm{f}(t_{n+1}, y_{n+1})\right] \quad (3)$$



However, the formula cannot be used directly to calculate y_{n+1} because y_{n+1} is required to evaluate $\mathrm{f}(t_{n+1}, y_{n+1})$ on the right-hand side.

To overcome this problem, y_{n+1} on the right-hand side is replaced by a first estimate, denoted by $\tilde{y}_{n+1}$, **obtained using Euler's method,**

$$\frac{\tilde{y}_{n+1} - y_n}{\Delta t} = \mathrm{f}(t_n, y_n),$$

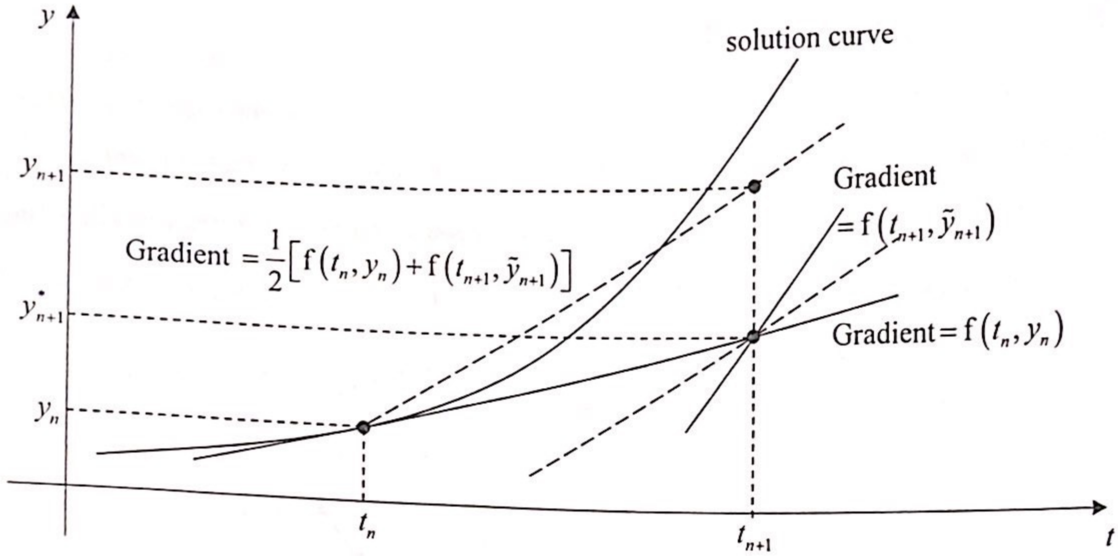
and this estimate is substituted into (3):

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{1}{2}\left[\mathrm{f}(t_n, y_n) + \mathrm{f}(t_{n+1}, \tilde{y}_{n+1})\right]$$

or equivalently

euler

$$\boxed{y_{n+1} = y_n + \Delta t\left[\frac{\mathrm{f}(t_n, y(t_n)) + \mathrm{f}(t_{n+1}, \tilde{y}_{n+1})}{2}\right]}.$$



Therefore, the Improved Euler's method (also known as the Heun formula) consists of two steps:

Step 1: Apply Euler's method to find $\tilde{y}_{n+1}$: $\tilde{y}_{n+1} = y_n + \Delta t \cdot \mathrm{f}(t_n, y_n)$

Step 2: Find y_{n+1}: $y_{n+1} = y_n + \Delta t\left[\dfrac{\mathrm{f}(t_n, y(t_n)) + \mathrm{f}(t_{n+1}, \tilde{y}_{n+1})}{2}\right]$.

Example 3 (Continuation of Example 1)

Consider the initial value problem

$$\frac{dy}{dt} = 2y - 1, \qquad y(0) = 1.$$

Apply the Improved Euler's method to obtain an approximation for $y(1)$ using step $\Delta t = 0.1$. Compare with results obtained using Euler's method.

Solution

With $\Delta t = 0.1$,

$$\tilde{y}_{n+1} = y_n + \Delta t \cdot (2y_n - 1)$$

$$y_{n+1} = y_n + \Delta t\left(\frac{(2y_n - 1) + (2\tilde{y}_{n+1} - 1)}{2}\right)$$

Beginning with $t_0 = 0, y_0 = 1$, $\Delta t = 0.1$, the first two approximate values are

$$\tilde{y}_1 = 1.0000 + (0.1)\cdot(2(1.0000) - 1) = 1.1,$$

$$y_1 = 1.0000 + 0.1\left(\frac{2(1.0000) - 1 + (2(1.1) - 1)}{2}\right) = 1.1100$$

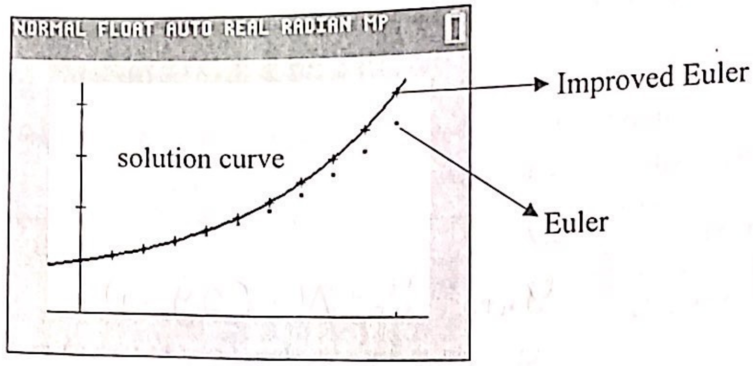
$$\tilde{y}_2 = 1.1100 + (0.1)\cdot(2(1.1100) - 1) = 1.2320$$

$$y_2 = 1.1100 + 0.1\left(\frac{(2(1.1100) - 1) + (2(1.2320) - 1)}{2}\right) = 1.2442$$

The table below shows the approximate values obtained in all ten steps.

t_n	y_n	$f(t_n, y_n) = 2y_n - 1$	$\tilde{y}_{n+1}$	$f(t_{n+1}, \tilde{y}_{n+1}) = 2\tilde{y}_{n+1} - 1$	$\Delta t \cdot \left(\frac{f(t_n, y_n) + f(t_{n+1}, \tilde{y}_{n+1})}{2}\right)$
0.0	1.0000	1.0000	1.1000	1.2000	0.1100
0.1	1.1100	1.2200	1.2320	1.4640	0.1342
0.2	1.2442	1.4884	1.3930	1.7861	0.1637
0.3	1.4079	1.8158	1.5895	2.1790	0.1997
0.4	1.6077	2.2153	1.8292	2.6584	0.2437
0.5	1.8514	2.7027	2.1216	3.2432	0.2973
0.6	2.1487	3.2973	2.4784	3.9568	0.3627
0.7	2.5114	4.0227	2.9136	4.8273	0.4425
0.8	2.9539	4.9077	3.4446	5.8892	0.5398
0.9	3.4937	5.9874	4.0924	7.1849	0.6586
1.0	4.1523	-	-	-	-

Comparing the two methods,



y_{actual}	Improved Euler y_n	Error for Improved Euler $\lvert y_{\text{actual}} - y_n \rvert$	Percentage error of Improved Euler	Euler y_n	Error for Euler $\lvert y_{\text{actual}} - y_n \rvert$	Percentage error of Euler
1.0000	1.0000	0.0000	0.00	1.0000	0.0000	0.00
1.1107	1.1100	0.0007	0.06	1.1000	0.0107	0.96
1.2459	1.2442	0.0017	0.14	1.2200	0.0259	2.08
1.4111	1.4079	0.0032	0.23	1.3640	0.0471	4.71
1.6128	1.6076	0.0052	0.32	1.5368	0.0760	4.71
1.8591	1.8514	0.0077	0.41	1.7442	0.1149	6.18
2.1601	2.1487	0.0114	0.53	1.9930	0.1671	7.74
2.5276	2.5114	0.0162	0.64	2.2916	0.2360	9.34
2.9765	2.9539	0.0226	0.76	2.6499	0.3266	10.97
3.5248	3.4937	0.0311	0.88	3.0799	0.4449	12.62
4.1945	4.1523	0.0422	1.01	3.5959	0.5986	14.27

From the above plot and table, the percentage error by the improved Euler's method is superior to that of the Euler's method.

Example 4 [9824/2013/7(ii)]

A solution of the differential equation $\frac{dy}{dx} = y - x$ has $y = 2$ at $x = 0$.

Copy and complete the table showing the use of the improved Euler method with step size 0.5 to estimate y at $x = 1$.

x	y	$y-x$	$\tilde{y}$	$\frac{\Delta y}{\Delta x}$
0	2	2	3	$\frac{2+2.5}{2}$
0.5	3.125	2.625	4.438	$\frac{2.625+3.438}{2}$
1	4.64075			

$=3.0315$

Solution

x	y	$y-x$	$\tilde{y}$	$\frac{\Delta y}{\Delta x}$
0	2	2	3	$\frac{2+2.5}{2}$
0.5	3.125	2.625	4.438	$\frac{2.625+3.438}{2}$
1	4.64075			

3 Miscellaneous Examples

Example 5

Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t}+2y=2-\mathrm{e}^{-4t}, \qquad y(0)=1.$$

Give the approximations at $t=1$, $t=2$, $t=3$, $t=4$, and $t=5$.

Use step size $\Delta t=0.1$, $\Delta t=0.05$, $\Delta t=0.01$, $\Delta t=0.005$, and $\Delta t=0.001$ for the approximations using Euler's method.

How does changing the values of Δt affect the accuracy of the approximations?

Solution

Here $\mathrm{f}(t,y)=2-\mathrm{e}^{-4t}-2y$, so Euler's formula takes the form

$$y_{n+1}=y_n+\Delta t\,(2-e^{-4t_n}-2y_n)$$

With the integrating factor e^{2t}, the differential equation can be solved to obtain $y=1+\frac{1}{2}\mathrm{e}^{-4t}-\frac{1}{2}\mathrm{e}^{-2t}$.

Below are two tables, one gives approximations to the solution and the other gives the errors for each approximation.

		Approximate y_n				
t	y_{actual}	$\Delta t=0.1$	$\Delta t=0.05$	$\Delta t=0.01$	$\Delta t=0.005$	$\Delta t=0.001$
1	0.9414902	0.9313244	0.9364698	0.9404994	0.9409957	0.9413914
2	0.9910099	0.9913681	0.9911126	0.9910193	0.9910139	0.9910106
3	0.9987637	0.9990501	0.9988982	0.9987890	0.9987763	0.9987662
4	0.9998323	0.9998976	0.9998657	0.9998390	0.9998357	0.9998330
5	0.9999773	0.9999890	0.9999837	0.9999786	0.9999780	0.9999774

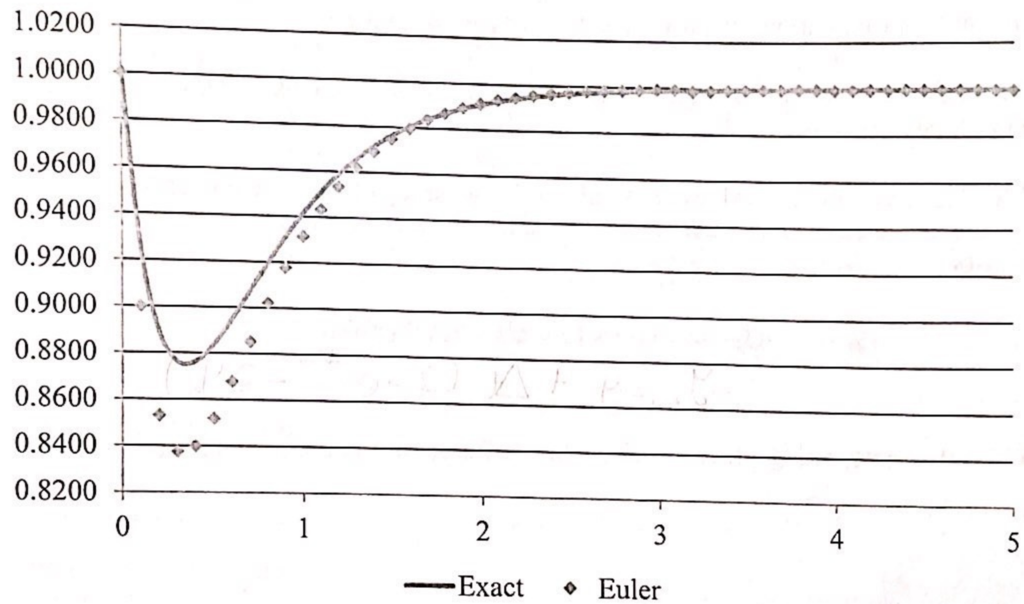
	Percentage Errors (%)				
t	$\Delta t = 0.1$	$\Delta t = 0.05$	$\Delta t = 0.01$	$\Delta t = 0.005$	$\Delta t = 0.001$
1	1.0800	0.53000	0.10500	0.053000	0.010500
2	0.0360	0.01000	0.00094	0.000410	0.000070
3	0.0290	0.01300	0.00250	0.001300	0.000250
4	0.0065	0.00330	0.00067	0.000340	0.000067
5	0.0012	0.00064	0.00013	0.000068	0.000014

We observe that decreasing Δt improves the accuracy of the approximation.

In general, decreasing the step size Δt, by factor of 10 also decreased the % error by a factor of 10 as well

Remarks:

- As t increases, the approximation actually tends to get better. This is not the case completely as we can see that in all but the first case, the error at $t = 3$ is worse than the error at $t = 2$, but after that point, it only gets better. This should not be expected in general. In this case, this is more a function of the shape of the solution. Below is a graph of the solution as well as the approximations for $\Delta t = 0.1$.



- The approximation is worst where the function is changing rapidly. This should not be too surprising. Recall that we are using tangent lines to get the approximations and so the value of the tangent line at a given t will often be significantly different than the function due to the rapidly changing function at that point.
- In this case, because the function ends up fairly flat as t increases, the tangents start looking like the function itself and so the approximations are very accurate. This will not always be the case of course.

Example 6 [JJC/2012/4(modified)]

A solution of the differential equation

$$\frac{dy}{dx} = \left(1+y^2\right)\tan x,$$

where x is in radians, has $y = 1$ at $x = 0$.

(i) Use the Euler method with step size 0.5 to estimate y at $x = 1$.

(ii) Copy and complete the table showing the use of the improved Euler method with step size 0.5 to estimate y at $x = 1$.

x	y	$\left(1+y^2\right)\tan x$	$\tilde{y}$	$\frac{\Delta y}{\Delta x}$
0	1	0	1	0.5463
0.5	1.2732	1.4318		
1				

(iii) State, with a reason, which of these two estimates for y at $x = 1$ is likely to be more accurate.

(iv) Explain whether these estimates are likely to be over-estimates or under-estimates.

Solution

(i) $y_1 = 1 + (0.5)[(1+1^2)\tan 0] = 1$

$y_2 = 1 + (0.5)[(1+1^2)\tan 0.5] = 1.5463$

The estimate of y at $x = 1$ is 1.55 (to 3 s.f.).

In table form:

x	y	$\left(1+y^2\right)\tan x$	$\Delta x\left[\left(1+y^2\right)\tan x\right]$
0	1	0	0
0.5	1	1.0926	0.5463
1	1.5463		

The estimate of y at $x = 1$ is 1.55 (to 3 s.f.).

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(ii) When $x = 0.5$, $\tilde{y} = 1.2732 + (0.5)(1.4318) = 1.9891$

$$\frac{\Delta y}{\Delta x} = \frac{1.4318 + [(1+1.9891^2)\tan 1]}{2} \approx 4.5756$$

When $x = 1$, $y = 1.2732 + (0.5)(4.575560) \approx 3.5610$

Filling up the table:

x	y	$(1+y^2)\tan x$	$\tilde{y}$	$\frac{\Delta y}{\Delta x}$
0	1	0	1	0.5463
0.5	1.2732	1.4318	1.9891	4.5756
1	3.5610			

(iii) The improved Euler estimate, 3.56 (to 3 s.f.), is likely to be more accurate than the Euler estimate, 1.55.

At each iteration, the improved Euler method makes a predictn first b4 gg on 2 correct this predictn to give an estimate. ∴ for the same step size, the improved Euler method is more accurate

(iv) From the tables, we observe that

the gradients seem to be ↑ as t↑, ∴ approximatn is likely to be an underestimate

Example 7 [RI/2014/7 (modified)]

A solution to the differential equation

$$\frac{dy}{dx} = (y-1)^2$$

has $y = -1.5$ at $x = 0$.

(i) Show that the exact value of y at $x = 1$ is $\frac{2}{7}$.

(ii) Use the Euler method with step size 0.5 to estimate y at $x = 1$.

(iii) Use a sketch to explain why the Euler method with step size chosen in (ii) does not provide a good approximation to the exact solution.

(iv) Copy and complete the table showing the use of the improved Euler method with step size 0.5 to estimate y at $x = 1$.

x	y	$(y-1)^2$	$\tilde{y}$	$\dfrac{\Delta y}{\Delta x}$
0	-1.5	6.25	1.625	3.3203
0.5	0.16015	0.70535		
1				

(v) Give a reason why the Improved Euler method is preferred to using smaller step sizes in the Euler Method.

Solution

(i) $\dfrac{dy}{dx} = (y-1)^2 \Rightarrow \displaystyle\int \frac{1}{(y-1)^2}\,dy = \int dx$

$$\therefore -\frac{1}{y-1} = x + c$$

When $x = 0$, $y = -1.5$: $c = -\dfrac{1}{-2.5} = \dfrac{2}{5}$

$$\therefore y - 1 = -\frac{1}{x + \frac{2}{5}} = -\frac{5}{5x+2}$$

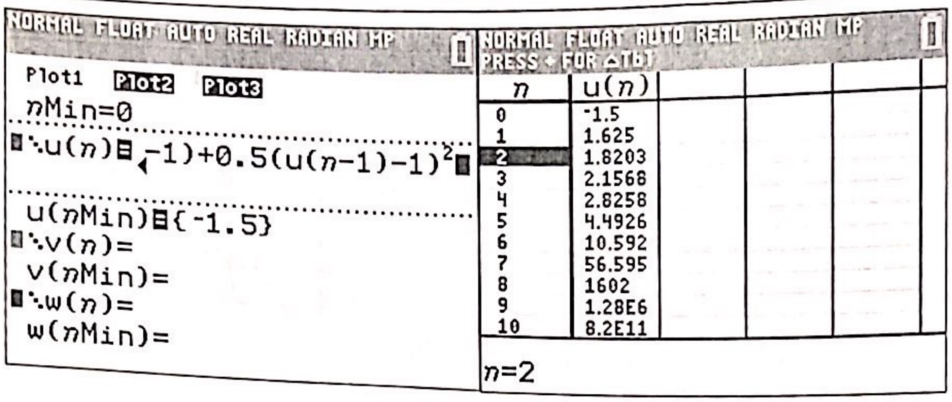
When $x = 1$: $y = 1 - \dfrac{5}{7} = \dfrac{2}{7}$ (shown)

(ii) Using the Euler Method,

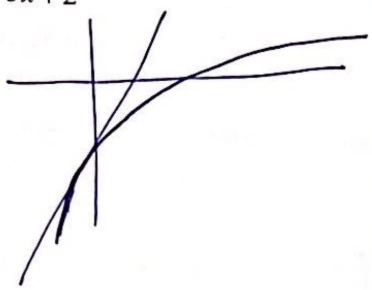
$y(0.5) \approx y(0) + 0.5\,(y(0)-1)^2 = -1.5 + 0.5(6.25) = 1.625$

$y(1) \approx y(0.5)\, 0.5\,(y(0.5)-1)^2 \approx 1.82$

Alternative Method use GC



(iii) Plotting $y = 1 - \dfrac{5}{5x+2}$ and the tangent at $x = 0$:



As seen from the sketch, the initial tangent to the curve is steep, which results in the first approx to be erroneous

(iv)

x	y	$(y-1)^2$	$\tilde{y}$	$\dfrac{\Delta y}{\Delta x}$
0	−1.5	6.25	1.625	3.3203
0.5	0.16015	0.70535	0.51282	0.47135
1	0.39582			

(v) The additional computional cost required for smaller sizes is more significant compared to using the Improved Euler method.

Summary