NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 3

MATHEMATICS 9820/01

Paper 1 **17 September 2021**

3 hours

Additional Materials: Answer Booklet
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in the brackets [ ] at the end of each question or part question.

1 Let S be a set containing $n+2$ integers. By considering the pigeonhole principle, show that there are two elements in the set S whose difference is a multiple of $2n$ or there are two elements in the set S whose sum is divisible by $2n$. [5]

2 **(i)** Given integers a, b and m such that $\gcd(b,m)=1$, show that the m numbers

$$a,\ a+b,\ a+2b,\ ...,\ a+(m-1)b$$

are all incongruent modulo m. [2]

(ii) Let $n>2$. Prove that if all the n terms of the arithmetic progression

$$p,\ p+d,\ p+2d,\ ...,\ p+(n-1)d$$

are primes, then the common difference d is divisible by every prime smaller than n. [4]

3 Prove that $a^2+b^2+c^2 \geq ab+bc+ca$ for any real numbers a, b and c. [3]

Hence prove that for any positive real numbers x, y and z satisfying $xy+yz+zx=x+y+z=S$,

(i) $S \geq 3$, and [2]

(ii) $\dfrac{1}{x^2+y+1}+\dfrac{1}{y^2+z+1}+\dfrac{1}{z^2+x+1} \leq 1$. [4]

4 **(a)** Show that the sequence

99, 999, 9999, 99999, …

does not contain a number which is a sum of two squares. [2]

(b) *Dirichlet's Theorem* states the following:

If a and b are coprime positive integers, then the arithmetic progression

$$a,\ a+b,\ a+2b,\ a+3b,\ ...$$

contains infinitely many primes.

(i) Show that for any positive integer n, there exists a prime ending with n consecutive 9's. [3]

(ii) Prove that the arithmetic progression above contains infinitely many composite numbers for any pair of coprime positive integers a and b. [4]

5 Use the substitution $u^2 = \tan x$ to find

$$\int \sqrt{\tan x}\, \mathrm{d}x,$$

showing all your working clearly. Express all values in exact form. [11]

6 **(a)** Show that $\dfrac{x+14}{(x-1)(x-2)(x+4)}$ can be expressed as $\dfrac{Ax+B}{(x-1)(x-2)} + \dfrac{C}{x+4}$, where A, B and C are constants to be determined. [2]

(b) Let $\dfrac{p(x)}{q(x)}$ be a rational function such that the degree of the numerator $p(x)$ is less than the degree of the denominator $q(x)$. Furthermore, assume that $q(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$, where x_i are all distinct. Show, by mathematical induction, that for any $n \geq 1$, $\dfrac{p(x)}{q(x)}$ can be written in the form

$$\sum_{i=1}^{n} \frac{c_i}{x-x_i}$$

for suitable constants c_i. [8]

7 In basketball, any single shot is worth either one, two or three points. Mid-way through a basketball game, the scoreboard shows 6 points.

(a) Find the number of ways to score 6 points in basketball if we are interested in the order of each type of shot made. [2]

Suppose that now we want to count the number of ways to score 6 points but we are only interested in the number of each type of shot made and not the order in which they were made. The following approach will help with the counting systematically. In scoring 6 points, the contribution from one-point shots to that score is

0 point $\oplus$ 1 point $\oplus$ 2 points $\oplus$ 3 points $\oplus$ 4 points $\oplus$ 5 points $\oplus$ 6 points,

where $\oplus$ means the exclusive-or.

We symbolise this algebraically as

$$x^0 + x^1 + x^2 + x^3 + x^4 + x^5 + x^6 \text{ ---- (1)},$$

where $\oplus$ signs have been replaced by ordinary addition and where the total contribution appears in the exponents.

Similarly, we can algebraically symbolise the contribution from two-point shots and three-point shots to the score of 6 points respectively as

$$x^0 + x^2 + x^4 + x^6 \text{ --- (2)} \qquad \text{and} \qquad x^0 + x^3 + x^6 \text{ --- (3)}.$$

Proceed to multiply the expressions in (1), (2) and (3) to each other. This product is called a generating function. Lastly, identify the required coefficient of x^r, where $0 \le r \le 18$, which in this case, is the coefficient of x^6.

You may consider using the above approach to answer the following questions.

(b) Suppose that we are only interested in the number of each type of shot made and not the order,

(i) find the number of ways to score 6 points. [1]

(ii) write down the generating function to find the number of ways to score 10 points. [2]

(c) **(i)** Find the coefficient of x^{24} in the product

$$(1+x^2)(1+x^3)(1+x^5)(1+x^7)(1+x^{11})(1+x^{13})(1+x^{17})(1+x^{19}).$$ [1]

(ii) Interpret the answer in part **(c) (i)** in terms of the partitions of 24. [2]

(d) **(i)** Given that $(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots = \mathrm{P}(x)$ where $\mathrm{P}(x)$ is a power series, find $\mathrm{P}(x)$. [1]

(ii) What does the equation in part **(d) (i)** say about all positive integers? [2]

8 The Fundamental Theorem of Algebra states that every polynomial function $\mathrm{p}(x)$ of degree n can be expressed as a product of linear factors

$$\mathrm{p}(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

where $\alpha_1,\ \alpha_2,\ldots,\alpha_n$ are the roots of $\mathrm{p}(x)=0$ in $\mathbb{C}$, not necessarily all distinct.

Subject to certain conditions, this result can be extended to non-polynomial functions. In particular,

$$\sin x = x\prod_{\substack{k\in\mathbb{Z},\\ k\neq 0}}\left(1-\frac{x}{\beta_k}\right),$$

where β_k's are the roots of the equation $\sin x=0,$ and $\prod u_k$ denotes the product of the terms $u_1,\ u_2,\ldots,u_n,\ldots$ of a sequence.

(i) State an expression for β_k in exact form. [1]

(ii) Express $\dfrac{\sin x}{x}$ in the form $\displaystyle\prod_{k=1}^{\infty}\left(1-\frac{x^2}{\gamma_k}\right)$, showing your working clearly. [2]

(iii) By considering the coefficient of a suitable term in the expansion of $\dfrac{\sin x}{x}$, find exactly the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. [3]

(iv) Obtain a formula for $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{2m}}$ in terms of m for any positive integer m. [3]

Prove that if $p > 1$, then the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges. [3]

9 A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. Let g_n be the number of calves born in the n^{th} year. It is given that $g_1 = g_2 = g_3 = 1$.

(i) Explain why $g_n = g_{n-1} + g_{n-3}$ for $n \geq 4$. [3]

(ii) Find g_{10}. [3]

An **ordered composition** of a positive integer *n* is a sequence $(a_1, a_2, a_3, ..., a_k)$ of positive integers such that $\sum a_i = n$. For example, all ordered compositions of 4 are

$$(4), (3,1), (2,2), (1,3), (2,1,1), (1,2,1), (1,1,2) \text{ and } (1,1,1,1).$$

Given that $n \in \mathbb{Z}^+$, we let h_n be the number of ordered compositions of *n*, consisting of positive integers 3 or 1 (or both). It is given that $h_1 = 1$.

(iii) Find h_2 and h_3. [2]

(iv) Explain why $h_n = h_{n-1} + h_{n-3}$ for $n \geq 4$. [2]

(v) Express h_{2n} as a sum of $\binom{p}{q}$, where $\binom{p}{q}$ counts the number of ways to choose *q* objects from *p* distinct objects. [3]

10 A **lattice point** is a point in a Cartesian coordinate system such that both its *x*- and *y*-coordinates are integers.

Pick's Theorem

> Suppose that all the vertices of a polygon are lattice points, then the area, *A*, of the polygon can be expressed as
>
> $$A = I + \frac{B}{2} - 1,$$
>
> where *I* is the number of lattice points that are interior to the polygon and *B* is the number of lattice points on its boundary.

Prove *Pick's Theorem* when the polygon is

(i) any rectangle with all the sides parallel to axes, [4]

(ii) any right-angled triangle with two of its sides parallel to the *x*- and *y*-axes respectively, [3]

(iii) any triangle. [3]

Hence, prove *Pick's Theorem* for any *n*-sided polygon. [4]

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