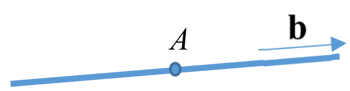
**RAFFLES INSTITUTION**
**H2 Mathematics (9758)**
**2023 Year 6**

# 2023 Year 6 H2 Mathematics Common Test: Solutions with Comments

**1** The points $A$, $B$ and $C$ have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The points $A$ and $B$ are fixed while $C$ varies.

**(a)** Given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors such that $(\mathbf{c}-\mathbf{a})\times\mathbf{b}=\mathbf{b}\times(\mathbf{c}-\mathbf{a})$, find the relationship between $(\mathbf{c}-\mathbf{a})$ and $\mathbf{b}$. [2]

**(b)** Hence describe geometrically the set of all possible positions of the point $C$. [2]

| **Solutions** | | **Comments** |
|---|---|---|
| **(a)**<br>**[2]** | Since $\mathbf{b}\times(\mathbf{c}-\mathbf{a})=-(\mathbf{c}-\mathbf{a})\times\mathbf{b}$,<br>$(\mathbf{c}-\mathbf{a})\times\mathbf{b}=\mathbf{b}\times(\mathbf{c}-\mathbf{a})$<br>$\Rightarrow(\mathbf{c}-\mathbf{a})\times\mathbf{b}=-(\mathbf{c}-\mathbf{a})\times\mathbf{b}$<br>$\Rightarrow 2\left[(\mathbf{c}-\mathbf{a})\times\mathbf{b}\right]=\mathbf{0}$<br>$\Rightarrow(\mathbf{c}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$<br>Hence, $(\mathbf{c}-\mathbf{a})$ and $\mathbf{b}$ are parallel or $\mathbf{c}=\mathbf{a}$. | $(\mathbf{c}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$ implies $\lvert\mathbf{c}-\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta=0$, where $\theta$ is the angle between $(\mathbf{c}-\mathbf{a})$ and $\mathbf{b}$#<br><br>Thus either $\lvert\mathbf{c}-\mathbf{a}\rvert=0$ or $\sin\theta=0$ (Note $\mathbf{b}\neq\mathbf{0}$) |
| **(b)**<br>**[2]** | From (a), since $\mathbf{c}=\mathbf{a}$ or $(\mathbf{c}-\mathbf{a})$ and $\mathbf{b}$ are parallel.<br>$(\mathbf{c}-\mathbf{a})=k\mathbf{b},\ k\in\mathbb{R}$<br>$\mathbf{c}=\mathbf{a}+k\mathbf{b},\ k\in\mathbb{R}$<br><br>Point $C$ lies on the line passing through the point $A$ and the line is parallel to the vector $\mathbf{b}$.<br><br>Note that $\mathbf{c}=\mathbf{a}$ when $k=0$. | |

**2** Complete the square for $2x^2-9x+5$. [1]

The function f is defined by

$$\mathrm{f}: x \mapsto \left|2x^2-9x+5\right|, \quad \text{for } x \in \mathbb{R}.$$

**(a)** Sketch the graph of $y=\mathrm{f}(x)$, stating the coordinates of any turning points and points of intersection with the axes. [2]

**(b)** If the domain of f is further restricted to $1 \le x \le k$, state with a reason the largest value of $k$ for which the function $\mathrm{f}^{-1}$ exists. [2]

In the rest of the question, the domain of f is $x \in \mathbb{R}, 1 \le x \le 2$.

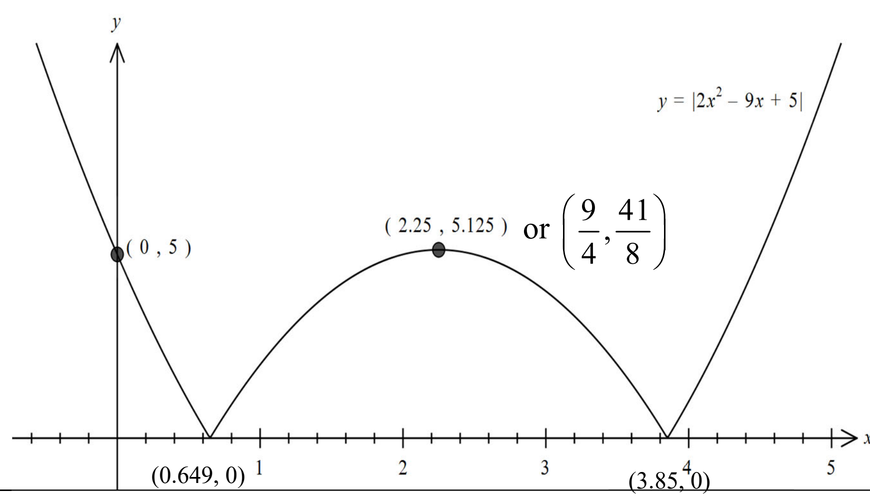
**(c)** Find the solution set of $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [1]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{1}{x}, \quad \text{for } x \in \mathbb{R},\ 0 < x \le 5.$$

**(d)** State whether the composite function $\mathrm{f}^{-1}\mathrm{g}$ exists, justifying your answer. [2]

**(e)** If the domain of g is further restricted to $p \le x \le q$ and the composite function gg exists, find an equation involving $p$ and $q$. [2]

| **Solutions** | | **Comments** |
|---|---|---|
| **[1]** | $2x^2-9x+5=2\left(x-\frac{9}{4}\right)^2-\frac{41}{8}$ | |
| **(a)** **[2]** |  | Take note of details when drawing a graph. 5.125 > 5, so the turning point should be slightly higher than the $y$-intercept (0,5)<br><br>Graph should also be symmetrical about $x=2.25$ |

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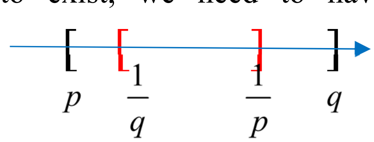
| **(b)**<br>**[2]** | Largest value of $k$ is $\frac{9}{4}$.<br><br>Since the line $y=m,\ 2\le m\le\frac{41}{8}$, cuts the graph of $y=\mathrm{f}(x)$ at exactly one point, it is a 1-1 function and so $\mathrm{f}^{-1}$ exists. | |
| **(c)**<br>**[1]** | Note : since $1\le x\le 2$, $\mathrm{f}^{-1}$ exists.<br><br>**Method 1**<br>The solution set of $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ is the same as that of $\mathrm{f}(x)=x$.<br>$-(2x^2-9x+5)=x$<br>$-2x^2+8x-5=0$<br>$x=0.775$ or $3.22$ (3 s.f.)<br>No solution for $1\le x\le 2$.<br>The solution set is { } or $\varnothing$.<br><br>**Method 2**<br><br>The solution set of $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ is the same as that of $\mathrm{f}(x)=x$.<br>Since the graph of f do not intersect the line $y=x$ for $1\le x\le 2$, the solution set is { } or $\varnothing$. | While we can solve the quadratic equation to get $x=0.775, 3.22$, we need to check them against the domain.<br>Note that set of no solution is either { } or $\varnothing$, known as the empty set. |
| **(d)**<br>**[2]** | For $\mathrm{f}^{-1}\mathrm{g}$ to exist, we need $\mathrm{R_g}\subseteq \mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R_f}$.<br><br>Since $\mathrm{R_g}=\left[\frac{1}{5},\infty\right)$ and $\mathrm{R_f}=[2,5]$, $\mathrm{R_g}\not\subset \mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R_f}$, and so $\mathrm{f}^{-1}\mathrm{g}$ does not exist. | Take note that the domain of f is $x\in\mathbb{R},\ 1\le x\le 2$. |
| **(e)**<br>**[2]** | Let $\mathrm{D_g}=[p,q]$. Then $\mathrm{R_g}=\left[\frac{1}{q},\frac{1}{p}\right]$.<br>For the composite function gg to exist, we need to have $\mathrm{R_g}\subseteq\mathrm{D_g}\Rightarrow\left[\frac{1}{q},\frac{1}{p}\right]\subseteq[p,q]$.<br><br>Therefore $\frac{1}{q}\ge p$ and $\frac{1}{p}\le q\Rightarrow pq\le 1$ and $pq\ge 1$<br>so $pq=1$, where $0<p<q\le 5$. | Note that g is a decreasing function |

**3** Cadence, the pedal speed in cycling, is measured in pedal stroke revolutions per minute (RPM). For example, a cadence of 60 RPM means that one pedal makes a complete revolution 60 times in one minute. Every complete revolution of the pedal stroke of a stationary bicycle is equivalent to cycling an approximate distance of 0.006 km.

Lisa signed up for a beginner spinning class. The workout routine requires her to start cycling at 30 RPM consistently for the first minute and progressively increase the cadence by 2.5 RPM for every subsequent minute.

**(a)** At which minute will Lisa cycle at 120 RPM? [2]

**(b)** Find Lisa's average speed of cycling, in km/h, for the first 15 minutes. [3]

Celina signed up for an intermediate spinning class. The workout routine requires her to start cycling at 25 RPM consistently for the first minute and increase the cadence by 15% for every subsequent minute.

**(c)** At which minute does Celina's cadence first exceed 120 RPM? [2]

**(d)** If Lisa and Celina start their workout routine at the same time, at which minute does Celina's total distance cycled first exceed Lisa's total distance cycled? [3]

| **Solutions** | | **Comments** |
|---|---|---|
| **(a)** **[2]** | AP: $a = 30,\ d = 2.5$ <br> $120 = 30 + (n-1)(2.5)$ <br> $n = 37$ <br><br> Lisa will cycle at 120 RPM at the 37th minute. | |
| **(b)** **[3]** | $S_{15} = \frac{15}{2}\left[2(30) + (14)(2.5)\right]$ <br> $= 712.5$ revolutions <br> Average speed $= 712.5 \times 0.006 \div \left(\frac{15}{60}\right)$ <br> $= 17.1$ km/h | |

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| | <u>Alternative Method using mean PRM</u><br>Mean PRM for first 15 minutes $=\frac{1}{2}\left[2(30)+(14)(2.5)\right]=47.5$<br>Average speed $=47.5\times 0.006\times 60$<br>$=17.1$ km/h | |
| **(c)**<br>**[2]** | GP: $a=25,\ r=1.15$<br>$25\left(1.15\right)^{n-1}>120$<br>$n-1>11.223$<br>$n>12.223$<br><br>Celina's cadence will first exceed 120 RPM at the 13th minute. | |
| **(d)**<br>**[3]** | Let the time taken by Celina's total distance to first exceed Lisa's to be *n*.<br>$\frac{25(1.15^n-1)}{1.15-1}\times 0.006>\frac{n}{2}\left(2(30)+2.5\left(n-1\right)\right)\times 0.006$<br>$\frac{25}{0.15}(1.15^n-1)-\frac{n}{2}(57.5+2.5n)>0$<br>Let $Y=\frac{25}{0.15}(1.15^n-1)-\frac{n}{2}(57.5+2.5n)$, by GC, $n>5.8627$<br><br>**<u>Or</u>** use the table of values:<br><br><table><tr><td>$n$</td><td>$Y$</td></tr><tr><td>5</td><td>$-6.44<0$</td></tr><tr><td>6</td><td>$1.34>0$</td></tr></table><br>Celina's total distance cycled first exceed Lisa's total distance cycled at the 6th minute. | Reminder: when using GC, you still need to provide working, such as the inequalities.<br><br>Table of values can be used if the unknown to be solved is an integer and you need to show the change in values, in this case < 0 and > 0. |

**4** A curve $C$ has parametric equations

$$x=\left(a^2-t^2\right)^{\frac{1}{2}}, \qquad y=\ln t, \qquad \text{for } 0<t\le a \text{ and } a>1.$$

**(a)** $C$ crosses the $x$-axis at the point $P$. Find the $x$-coordinate of $P$ in terms of $a$. [1]

**(b)** Show that $\frac{\mathrm{d}y}{\mathrm{d}x}\le 0$ for $0<t\le a$. [3]

**(c)** Describe the behaviour of the tangent to $C$ as $t\to 0$. [1]

**(d)** Sketch the graph of $C$, stating the equations of any asymptotes and the coordinates of the points where the curve meets the axes. [2]

**(e)** The tangent at $P$ cuts the $y$-axis at the point $Q$ and the normal at $P$ cuts the $y$-axis at the point $R$. Find the area of triangle $PQR$. [4]

| **Solutions** | | **Comments** |
|---|---|---|
| **(a)** **[1]** | When $y=0$, $t=1$, $x=\sqrt{a^2-1}$ | |
| **(b)** **[3]** | $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{2}\left(a^2-t^2\right)^{-\frac{1}{2}}(-2t)=-\frac{t}{\sqrt{a^2-t^2}}$<br>$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{t}$<br>$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\sqrt{a^2-t^2}}{t^2}$<br>Since $\sqrt{a^2-t^2}\ge 0$ and $t^2>0$ for $0<t\le a$,<br>$\frac{\mathrm{d}y}{\mathrm{d}x}\le 0$ for $0<t\le a$. (shown) | |
| **(c)** **[1]** | When $t\to 0$, $x\to a$ and $\frac{\mathrm{d}y}{\mathrm{d}x}\to -\infty$.<br>The tangent to $C$ as $t\to 0$ approaches the vertical line $x=a$. | Note that tangent is a line. We need to describe the behaviour of the line. |

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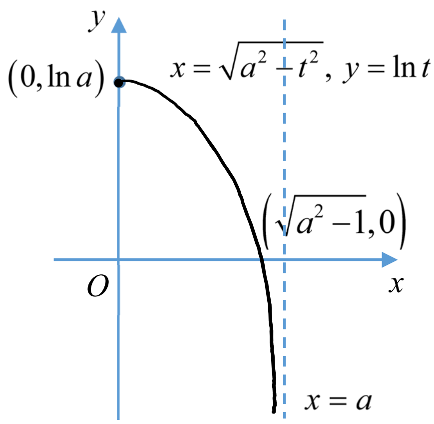
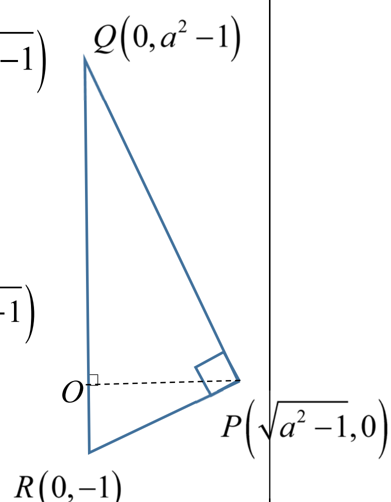
| **(d)** **[2]** | When $t=a,\ x=0,\ y=\ln a,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=0$  | |
| **(e)** **[4]** | At $P$, $t=1,\ x=\sqrt{a^2-1},\ y=0,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{a^2-1}$.<br>Equation of tangent at $P$ is $y=-\sqrt{a^2-1}\left(x-\sqrt{a^2-1}\right)$<br>When $x=0,\ y=a^2-1$<br>i.e coordinates of $Q$ are $\left(0,\ a^2-1\right)$.<br>Equation of normal at $P$ is $y=\dfrac{1}{\sqrt{a^2-1}}\left(x-\sqrt{a^2-1}\right)$<br>When $x=0,\ y=-1$<br>i.e coordinates of $R$ are $(0,-1)$.<br>$\therefore$ Area of triangle $PQR=\dfrac{1}{2}(QR)(OP)=\dfrac{1}{2}a^2\sqrt{a^2-1}$  | |

**5** It is given that

$$\frac{\mathrm{d}w}{\mathrm{d}x}=(w-1)(2-w)\,, \quad \text{for } 1<w<2\,. \qquad \textbf{(I)}$$

**(a)** By solving **(I)**, show that $w=1+\dfrac{A\mathrm{e}^x}{1+A\mathrm{e}^x}$, where $A$ is a positive constant. [5]

**(b)** Sketch the graph of $w$ against $x$, stating the equation of any asymptotes and the coordinates of the point(s) where the curve meets the axes. [2]

It is also given that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-(3+2x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+x^2+3x+1=0\,, \quad \text{for } 1<\frac{\mathrm{d}y}{\mathrm{d}x}-x<2\,. \qquad \textbf{(II)}$$

**(c)** Using the substitution $\dfrac{\mathrm{d}y}{\mathrm{d}x}=w+x$, show that **(II)** can be transformed to **(I)**. Hence find $y$ in terms of $x$. [5]

| **Solutions** | | **Comments** |
|---|---|---|
| **(a)** **[5]** | $\dfrac{\mathrm{d}w}{\mathrm{d}x}=(w-1)(2-w) \quad \Rightarrow \quad \dfrac{1}{(w-1)(2-w)}\dfrac{\mathrm{d}w}{\mathrm{d}x}=1$<br>$\Rightarrow \quad \displaystyle\int \frac{1}{w-1}+\frac{1}{2-w}\,\mathrm{d}w=\int 1\,\mathrm{d}x$<br>$\Rightarrow \quad \ln\lvert w-1\rvert-\ln\lvert 2-w\rvert=x+c$<br>$\Rightarrow \quad \ln\left\lvert\dfrac{w-1}{2-w}\right\rvert=x+c$<br>$\Rightarrow \quad \ln\left(\dfrac{w-1}{2-w}\right)=x+c$<br>$\left(\text{Since } 1<w<2,\ \dfrac{w-1}{2-w}>0 \ \Rightarrow\ \left\lvert\dfrac{w-1}{2-w}\right\rvert=\dfrac{w-1}{2-w}\right)$<br>$\Rightarrow \quad \dfrac{w-1}{2-w}=A\mathrm{e}^x \quad \text{where } A=\mathrm{e}^c>0$<br>$\Rightarrow \quad w-1=2A\mathrm{e}^x-wA\mathrm{e}^x$<br>$\Rightarrow \quad w=\dfrac{2A\mathrm{e}^x+1}{1+A\mathrm{e}^x}=1+\dfrac{A\mathrm{e}^x}{1+A\mathrm{e}^x}$ (shown) | $1<w<2\Rightarrow$ $(w-1)(2-w)\neq 0$<br><br>This “show” question specifically states $A$ is a positive constant. We need to explain in the solution why $A$ is positive. |

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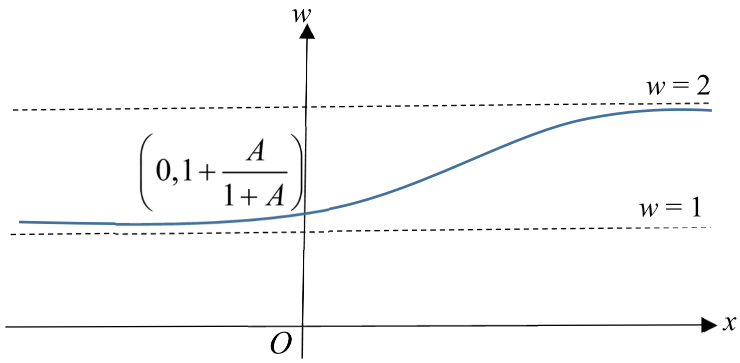
| **(b)**<br>**[2]** |  | |
| **(c)**<br>**[5]** | $\dfrac{dy}{dx} = w + x \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{dw}{dx} + 1$, and so<br>$\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 - (3+2x)\left(\dfrac{dy}{dx}\right) + x^2 + 3x + 1 = 0$<br>$\dfrac{dw}{dx} + 1 + (w+x)^2 - (3+2x)(w+x) + x^2 + 3x + 1 = 0$<br>$\dfrac{dw}{dx} + 1 + w^2 + 2wx + x^2 - 3w - 3x - 2wx - 2x^2 + x^2 + 3x + 1 = 0$<br>$\dfrac{dw}{dx} = -w^2 + 3w - 2 = (w-1)(2-w)$ (shown)<br>From **(a)**, $w = 1 + \dfrac{Ae^x}{1+Ae^x}$.<br>$\therefore \dfrac{dy}{dx} = w + x = 1 + \dfrac{Ae^x}{1+Ae^x} + x$<br>Integrating with respect to $x$,<br>$\int 1 \, dy = \int 1 + \dfrac{Ae^x}{1+Ae^x} + x \, dx$<br>$y = x + \dfrac{x^2}{2} + \ln\lvert 1 + Ae^x \rvert + B$<br>From **(b)**, $A > 0$. Hence $y = x + \dfrac{x^2}{2} + \ln(1 + Ae^x) + B$. | Note that $w$ and $x$ are variables. Thus<br>$\dfrac{d}{dx}(w+x)$<br>$= \dfrac{d}{dx}(w) + \dfrac{d}{dx}(x)$<br>$= \dfrac{dw}{dx} + 1$ |

**6** The plane $\pi_1$ contains the point $(2,-2,1)$ and the line $\mathbf{r}=\begin{pmatrix}0\\1\\8\end{pmatrix}+\lambda\begin{pmatrix}0\\2\\3\end{pmatrix}$,

where $\lambda$ is a parameter.

**(a)** Find an equation of $\pi_1$ in scalar product form. [2]

The plane $\pi_2$ has vector equation $\mathbf{r}=a\begin{pmatrix}0\\-1\\4\end{pmatrix}+b\begin{pmatrix}1\\2\\-1\end{pmatrix}$, where $a$ and $b$ are parameters.

**(b)** Find the acute angle between $\pi_1$ and $\pi_2$. [3]

The line $l$ passes through the point $A$ with position vector $4\mathbf{i}+5\mathbf{j}+6\mathbf{k}$ and is parallel to both $\pi_1$ and $\pi_2$.

**(c)** Find a vector equation for $l$. [2]

The plane $\pi_3$ with the cartesian equation $3x+2y+mz=0$ is also parallel to $l$.

**(d)** Show that $m=-6$. [1]

**(e)** Find $\overrightarrow{OA}\cdot(3\mathbf{i}+2\mathbf{j}-6\mathbf{k})$. [1]

**(f)** Find the distance between $l$ and $\pi_3$. [2]

**(g)** Hence or otherwise, find an equation of the line which is a reflection of $l$ in $\pi_3$. [2]

<table>
<tr><th colspan="2">Solutions</th></tr>
<tr><td>(a)<br>[2]</td><td>

A normal vector to $\pi_1 = \left(\begin{pmatrix}0\\1\\8\end{pmatrix}-\begin{pmatrix}2\\-2\\1\end{pmatrix}\right)\times\begin{pmatrix}0\\2\\3\end{pmatrix}=\begin{pmatrix}-2\\3\\7\end{pmatrix}\times\begin{pmatrix}0\\2\\3\end{pmatrix}=\begin{pmatrix}-5\\6\\-4\end{pmatrix}$

$\pi_1 : \mathbf{r}\cdot\begin{pmatrix}-5\\6\\-4\end{pmatrix}=\begin{pmatrix}2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}-5\\6\\-4\end{pmatrix}=-10-12-4=-26$

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<tr><td>(b)<br>[3]</td><td>

A normal vector to $\pi_2 = \begin{pmatrix}0\\-1\\4\end{pmatrix}\times\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}-7\\4\\1\end{pmatrix}$

The acute angle between $\pi_1$ and $\pi_2 = \cos^{-1}\dfrac{\left|\begin{pmatrix}-5\\6\\-4\end{pmatrix}.\begin{pmatrix}-7\\4\\1\end{pmatrix}\right|}{\sqrt{(-5)^2+6^2+(-4)^2}\sqrt{(-7)^2+4^2+1^2}}$

$=\cos^{-1}\dfrac{55}{\sqrt{77}\sqrt{66}}=39.5°$ (to 1 d.p.)

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<tr><td>(c)<br>[2]</td><td>

A vector parallel to both $\pi_1$ and $\pi_2$ is perpendicular to both normal vectors, one such vector will be:

$\begin{pmatrix}-5\\6\\-4\end{pmatrix}\times\begin{pmatrix}-7\\4\\1\end{pmatrix}=\begin{pmatrix}6+16\\28+5\\-20+42\end{pmatrix}=\begin{pmatrix}22\\33\\22\end{pmatrix}//\begin{pmatrix}2\\3\\2\end{pmatrix}$

Therefore $l$: $\mathbf{r}=\begin{pmatrix}4\\5\\6\end{pmatrix}+\beta\begin{pmatrix}2\\3\\2\end{pmatrix}$, where $\beta\in\mathbb{R}$.

<u>Alternative Method</u>

$\begin{aligned}\pi_1 &: -5x+6y-4z=-26\\ \pi_2 &: -7x+4y+z=0\end{aligned} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-\frac{52}{11}+z\\ -\frac{91}{11}+\frac{3}{2}z\\ z\end{pmatrix}$ by GC

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| | Therefore $l$: $\mathbf{r}=\begin{pmatrix}4\\5\\6\end{pmatrix}+\beta\begin{pmatrix}1\\\frac{3}{2}\\1\end{pmatrix}$, where $\beta\in\mathbb{R}$. |
| **(d)** **[1]** | $l$ is parallel to $\pi_3$, therefore the direction vector of $l$ is perpendicular to the normal of $\pi_3$ <br> $\begin{pmatrix}2\\3\\2\end{pmatrix}.\begin{pmatrix}3\\2\\m\end{pmatrix}=0\Rightarrow 6+6+2m=0\Rightarrow m=-6$ (shown) |
| **(e)** **[1]** | $\overrightarrow{OA}\cdot\begin{pmatrix}3\\2\\-6\end{pmatrix}=\begin{pmatrix}4\\5\\6\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-6\end{pmatrix}=-14$ |
| **(f)** **[2]** | Since $l$ is parallel to $\pi_3$, the distance between $l$ and $\pi_3$ is the same as the perpendicular distance from any point on $l$ to $\pi_3$. <br> Observe that point $A$ is on $l$ and the origin is on $\pi_3$. <br> The distance between $l$ and $\pi_3$ <br> =length of projection of $\overrightarrow{OA}$ on $\begin{pmatrix}3\\2\\-6\end{pmatrix}$ <br> $=\dfrac{\left\lvert\begin{pmatrix}4\\5\\6\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-6\end{pmatrix}\right\rvert}{\sqrt{3^2+2^2+(-6)^2}}=\dfrac{14}{7}=2$ units |

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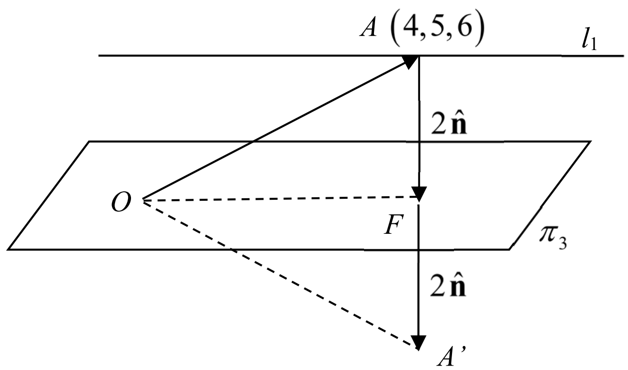
| **(g)** <br> **[2]** | **Hence Method 1:** <br><br> From part (e), note that the angle between $\overrightarrow{OA}$ and $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$ is an obtuse angle because the dot product is $-14$ which is a negative number. <br><br> Therefore $\overrightarrow{OA'} = \overrightarrow{OA} + 4\hat{\mathbf{n}}$ as illustrated in the diagram below. <br><br>   <br><br> where point $F$ is the foot of perpendicular from the point $A(4,5,6)$ to $\pi_3$ and point $A'$ is the point of reflection of $A$ in $\pi_3$. <br><br> Hence, $\overrightarrow{OA'} = \begin{pmatrix}4\\5\\6\end{pmatrix} + 2\times 2\times \frac{1}{7}\begin{pmatrix}3\\2\\-6\end{pmatrix} = \frac{1}{7}\begin{pmatrix}40\\43\\18\end{pmatrix}$ <br><br> The reflection of $l$ in $\pi_3$ is $\mathbf{r} = \frac{1}{7}\begin{pmatrix}40\\43\\18\end{pmatrix} + \gamma\begin{pmatrix}2\\3\\2\end{pmatrix}$, where $\gamma \in \mathbb{R}$. |
| | **Hence Method 2:** <br><br> Check that the point with position vector $\begin{pmatrix}4\\5\\6\end{pmatrix} + 2\times \frac{1}{7}\begin{pmatrix}3\\2\\-6\end{pmatrix} = \frac{1}{7}\begin{pmatrix}34\\39\\30\end{pmatrix}$ lies on the plane $\pi_3$, so $\begin{pmatrix}3\\2\\-6\end{pmatrix}$ is in the same direction as $\overrightarrow{AF}$. <br><br> Therefore $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AF} = \begin{pmatrix}4\\5\\6\end{pmatrix} + 2\times 2\times \frac{1}{7}\begin{pmatrix}3\\2\\-6\end{pmatrix} = \frac{1}{7}\begin{pmatrix}40\\43\\18\end{pmatrix}$ |

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|---|---|
| | The reflection of $l$ in $\pi_3$ is $\mathbf{r}=\frac{1}{7}\begin{pmatrix}40\\43\\18\end{pmatrix}+\gamma\begin{pmatrix}2\\3\\2\end{pmatrix}$, where $\gamma\in\mathbb{R}$. |
| | **Otherwise Method (not the most efficient way in this context):**<br>Let $F$ be the foot of perpendicular from the point $A(4,5,6)$ to $\pi_3$ and let $A'$ be the reflection point of point $A$ in $\pi_3$.<br>Then $\overrightarrow{OF}=\begin{pmatrix}4\\5\\6\end{pmatrix}+p\begin{pmatrix}3\\2\\-6\end{pmatrix}$ for some $p\in\mathbb{R}$, sub into $\pi_3$:<br>$\left(\begin{pmatrix}4\\5\\6\end{pmatrix}+p\begin{pmatrix}3\\2\\-6\end{pmatrix}\right)\cdot\begin{pmatrix}3\\2\\-6\end{pmatrix}=0\Rightarrow -14+49p=0\Rightarrow p=\frac{2}{7}$<br>$\overrightarrow{OF}=\begin{pmatrix}4\\5\\6\end{pmatrix}+\frac{2}{7}\begin{pmatrix}3\\2\\-6\end{pmatrix}=\frac{1}{7}\begin{pmatrix}34\\39\\30\end{pmatrix}$<br>By ratio theorem,<br>$\overrightarrow{OF}=\frac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}\Rightarrow \overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}=\frac{2}{7}\begin{pmatrix}34\\39\\30\end{pmatrix}-\begin{pmatrix}4\\5\\6\end{pmatrix}=\frac{1}{7}\begin{pmatrix}40\\43\\18\end{pmatrix}$<br>The reflection of $l$ in $\pi_3$ is $\mathbf{r}=\frac{1}{7}\begin{pmatrix}40\\43\\18\end{pmatrix}+\gamma\begin{pmatrix}2\\3\\2\end{pmatrix}$, where $\gamma\in\mathbb{R}$. |

**7** A firm claims that, on average, their workers spend no more than 8 hours a day at the construction site. The amount of time clocked, $x$ hours, by each worker from a random sample of 150 workers, is summarised as follows.

$$\sum x = 1305\,,\ \ \sum x^2 = 12751\,.$$

Test, at the 1% significance level, whether the firm's claim is valid. [5]

| **Solutions** | |
|---|---|
| **[5]** | Unbiased estimate of population mean, $\bar{x} = \dfrac{1305}{150} = 8.7$ hours<br>Unbiased estimate of population variance $= s^2$<br>$= \dfrac{1}{149}\left[\sum x^2 - \dfrac{\left(\sum x\right)^2}{150}\right]$<br>$= \dfrac{1}{149}\left[12751 - \dfrac{(1305)^2}{150}\right]$<br>$= \dfrac{2795}{298}$<br>$= 9.3792 \text{ hours}^2 (5\text{s.f})$<br>Let $X$ hours be the time spent by each worker at the site daily and let $\mu$ hours be the population mean time spent by each worker at the site daily.<br>Null Hypothesis $\mathrm{H}_0 : \mu = 8$<br>Alternative Hypothesis $\mathrm{H}_1 : \mu > 8$<br>Perform a 1-tail test at 1% significance level.<br>Under $\mathrm{H}_0$, since sample size 150 is large,<br>$\overline{X} \sim \mathrm{N}\left(8, \dfrac{2795}{298(150)}\right)$ approximately by Central Limit Theorem.<br>From the sample, $\bar{x} = 8.7$<br>Using a $z$-test, $p$-value$= \mathrm{P}\left(\overline{X} \geq 8.7\right) = 0.00256 < 0.01$<br>Since $p$-value = 0.00256 < 0.01, we reject $\mathrm{H}_0$. There is sufficient evidence, at the 1% significance level, to conclude that the firm's claim is not valid. |

**8** Colour Vision Deficiency is the decreased ability to see differences in colours. Someone with normal colour vision will be able to see 6 colours (Red, Orange, Yellow, Green, Blue, Violet) as 6 bands of colours.

Deutan, who has colour vision deficiency, is not able to differentiate the following two pairs of colours:

(I) Red and Green,

(II) Blue and Violet.

For 6 colours (Red, Orange, Yellow, Green, Blue, Violet) of a particular shade, when the above pairs of colours are placed next to each other, it will look like a single band of colour to Deutan. Some examples of his vision of these 6 colours are as follows.

| **Arrangement of these 6 colours in a line** | **Number of bands it will appear to Deutan** |
|---|---|
| Orange, Yellow, Red, Green, Blue, Violet | 4 |
| Red, Orange, Yellow, Green, Violet, Blue | 5 |
| Violet, Red, Orange, Yellow, Green, Blue | 6 |

Find the number of ways these 6 colours can be arranged such that Deutan will see exactly

**(a)** 4 bands of colour, [2]

**(b)** 5 bands of colour, [3]

**(c)** 6 bands of colour. [2]

| **Solutions** | |
|---|---|
| **(a)** **[2]** | To have only 4 colour bands, red/green must be together and blue/violet must be together.<br>Red and Green   Blue and Violet   Orange   Yellow<br>So we have 4 objects to arrange, 2 of which are groups of 2. So the total number of ways $= 4!\times 2!\times 2! = 96$. |

| | |
|---|---|
| **(b)**<br>**[3]** | To have 5 colour bands, one of the pairs (red/green or blue/violet) must be separated, and the other pair together.<br>Consider red/green together.<br>**<u>Method 1 (Grouping followed by Slotting)</u>**<br>$\downarrow$ Red and Green $\downarrow$ Orange $\downarrow$ Yellow $\downarrow$<br>Step 1 : arrange Orange, Yellow, Red/Green<br>Number of ways =3!2!<br>Step 2 : insert blue and violet in between Orange, Yellow, Red/Green<br>Number of ways = $\binom{4}{2}\times 2!$ or ${}^4P_2$<br>Total number of ways (after considering Blue/Violet as the grouped item in Step 1) = $\left[3!2!\binom{4}{2}\times 2!\right]2 = 288$<br>**<u>Method 2</u>**<br>Step 1 : arrange Orange, Yellow, Blue, Violet, Red/Green<br>Red and Green \| Blue \| Violet \| Orange \| Yellow<br>Number of ways = $2!5!$<br>Step 2 : Consider those cases with Blue and Violet together also<br>Red and Green \| Blue and Violet \| Orange \| Yellow<br>Number of ways = 96 (from part (i))<br>Total number of ways (after considering Blue/Violet as the grouped item in Step 1) = $2[2!5!-96]=288$ |
| **(c)**<br>**[2]** | Since Duetan will see either 4, 5 or 6 colour bands, required number of ways = 6! – (i)'s answer – (ii)'s answer = 720 – 96 – 288 = 336 |

| | |
|---|---|
| | **<u>Alternatively [Consider Cases]</u>**<br>Case 1: arrange O/Y, then slot in R/G, followed by B/V<br>Number of ways $=2!\times\binom{3}{2}2!\times\binom{5}{2}2!=240$<br>Note that the cases with RBG, RVG, GBR, GVR are not included in this case.<br>Case 2 : either one of RBG, RVG, GBR, GVR is a group<br>For each of the 4 cases, there will be a lone V, B, V and B, respectively, and we need to arrange the group, the lone V or B, and the other two colours O and Y, so therefore 4!.<br>Number of ways = $4!\times 4=96$<br>Total number of ways = 240 + 96 = 336 |

**9** Two families plan to go on a hiking trip together. There are 10 people from the Lee family, consisting of 3 married couples, 3 single men and 1 single woman. There are 14 people from the Tan family, consisting of 4 married couples, 1 single man and 5 single women. Two people are randomly chosen to organise the hiking trip.

**(a)** Show that the probability that they are both from the Lee family is $\frac{15}{92}$. [1]

**(b)** Find the probability that they are a man and woman from the same family. [3]

**(c)** Find the probability that they are married to each other given that they are a man and a woman from the same family. [3]

| **Solutions** | | **Comments** |
|---|---|---|
| **(a)**<br>**[1]** | P(both are from the Lee family)<br>$=\frac{^{10}C_2}{^{24}C_2}=\frac{45}{276}=\frac{15}{92}$ (shown)<br>OR : $\frac{10}{24}\times\frac{9}{23}$ | |
| **(b)**<br>**[3]** | P(man and woman from the same family)<br>=P(man and woman from the Lee family)<br>+ P(man and woman from the Tan family) | |

| | | |
|---|---|---|
| | $=\dfrac{{}^{6}C_1\,{}^{4}C_1}{{}^{24}C_2}+\dfrac{{}^{5}C_1\,{}^{9}C_1}{{}^{24}C_2}$<br>$=\dfrac{1}{4}$<br>OR : $\left(\dfrac{6}{24}\times\dfrac{4}{23}\right)2!+\left(\dfrac{5}{24}\times\dfrac{9}{23}\right)2!$ | |
| **(c)** **[3]** | $\mathrm{P}(\text{married to each other } \vert \text{man and a woman from the same family})$<br>$=\dfrac{\text{no. of ways to choose a couple}}{\text{no. of ways to choose a man and woman from the same family}}$<br>$=\dfrac{{}^{3}C_1+{}^{4}C_1}{{}^{6}C_1\,{}^{4}C_1+{}^{5}C_1\,{}^{9}C_1}$<br>$=\dfrac{7}{69}$<br><br>**OR**<br>$A$ : the two chosen people are a married couple<br>$B$ : man and woman from the same family<br>$\mathrm{P}(A\vert B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$<br>$=\dfrac{\mathrm{P}(\text{married couple from the Lee family})+\mathrm{P}(\text{married couple from the Tan family})}{\mathrm{P}(B)}$<br>$=\dfrac{\left(\frac{6}{24}\times\frac{1}{23}\right)+\left(\frac{8}{24}\times\frac{1}{23}\right)}{\frac{1}{4}}$<br>$=\dfrac{7}{69}$ | Applying reduced sample space<br><br>Applying conditional probability formula |

**10** In this question you should state clearly the parameters of any normal distributions you use.

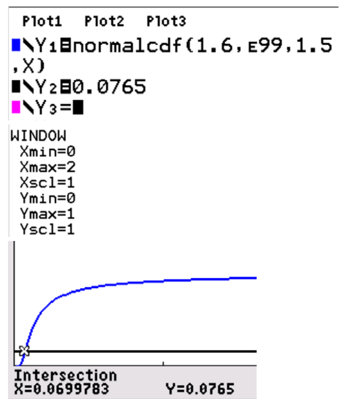
The masses in kilograms of chickens have the distribution $\mathrm{N}(1.5, \sigma^2)$ such that 7.65% of them have a mass greater than 1.6 kg.

**(a)** Find the value of $\sigma$. [2]

The masses in kilograms of ducks have the distribution $\mathrm{N}(2.0, 0.08^2)$.

**(b)** Find the probability that the mass of a randomly chosen chicken is less than 1.6 kg and the mass of a randomly chosen duck is more than 1.9 kg. [2]

**(c)** Chickens and ducks are sold at \$11 per kg and \$18 per kg respectively. Find the probability that the total selling price of 4 randomly chosen chickens differs from twice the selling price of a randomly chosen duck by at least \$5. [4]

| **Solutions** | | **Comments** |
|---|---|---|
| **(a)**<br>**[2]** | Let $X$ be the mass in kg of a randomly chosen chicken.<br>Then $X \sim \mathrm{N}\left(1.5, \sigma^2\right)$<br>$\mathrm{P}\left(X > 1.6\right) = 0.0765$<br>$\mathrm{P}\left(Z > \dfrac{1.6 - 1.5}{\sigma}\right) = 0.0765$<br>From GC, $\dfrac{0.1}{\sigma} = 1.4290$<br>$\sigma = 0.069979$ (5 s.f.)<br>$\sigma = 0.0700$ (3 s.f.) | **Alternative : Use GC**<br><br> |
| **(b)**<br>**[2]** | Let $Y$ be the mass in kg of a randomly chosen duck.<br>Then $Y \sim \mathrm{N}\left(2.0, 0.08^2\right)$<br>Required probability<br>$= \mathrm{P}\left(X < 1.6\right)\mathrm{P}\left(Y > 1.9\right)$<br>$= \left(1 - 0.0765\right)\left(0.89435\right)$<br>$= 0.826$ (3 s.f) | |
| **(c)**<br>**[4]** | Let $S = 11\left(X_1 + X_2 + X_3 + X_4\right) - 18(2Y)$<br>$\mathrm{E}\left(S\right) = 11\left(4\right)1.5 - 18(2)2 = -6$<br>$\mathrm{Var}\left(S\right) = 11^2\left(4\right)0.069979^2 + 18^2\left(2\right)^2 0.08^2 = 10.665$<br>Then $S \sim \mathrm{N}\left(-6, 10.665\right)$<br>$\mathrm{P}\left(\lvert S \rvert \geq 5\right) = 1 - \mathrm{P}\left(-5 \leq S \leq 5\right) = 0.621$ (3 s.f.) | Since (a) was not a 'Show' question, we should use 5 s.f. or higher for $\sigma$ in the calculation of $\mathrm{Var}\left(S\right)$. |

**11** By the end of 2022, the percentages of a population who has received $V$ dose(s) of Covid-19 vaccine are shown in the table below, with no one receiving 5 or more doses of vaccine.

| $v$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Percentage of population | 7 | $a$ | $b$ | 69 | 9 |

**(a)** Given that $\mathrm{E}(V) = 2.72$, find $\mathrm{Var}(V)$. [4]

Mr Lee is doing a survey on people's perspectives on the Covid-19 vaccine.

**(b)** Find the probability that the total number of doses of vaccine received by 3 randomly selected surveyees is more than 10. [2]

Moderna's and Pfizer-BioNTech's bivalent vaccine are available at a particular vaccination centre.

On average 60% of walk-ins opt for Moderna's bivalent vaccine at this centre. Let $M$ denote the number of people, out of $n$ walk-ins surveyed by Mr Lee on a particular day, that choose to receive the Moderna's bivalent vaccine.

**(c)** Explain, in the context of this question, the assumptions needed to model $M$ by a binomial distribution. [2]

Assume now that these assumptions do in fact hold.

**(d)** Find $\mathrm{P}(4 \le M < 8)$ when $n = 10$. [2]

**(e)** Find the least number of people Mr Lee needs to survey to have at least a 90% chance of at least 15 surveyees choosing to receive Moderna's bivalent vaccine. [3]

| | **Solutions** | **Comments** |
|---|---|---|
| **(a)** **[4]** | $7+a+b+69+9=100 \Rightarrow a+b=15$<br>$\mathrm{E}(V)=0.01a+0.02b+3\times0.69+4\times0.09=2.72$<br>$\Rightarrow 0.01a+0.02b=0.29$<br><br>By GC, $\begin{array}{r} a+\quad b=15 \\ 0.01a+0.02b=0.29 \end{array} \Rightarrow \begin{array}{l} a=1 \\ b=14 \end{array}$<br><br>$\mathrm{E}(V^2)=1^2(0.01)+2^2(0.14)+3^2(0.69)+4^2(0.09)=8.22$<br>$\mathrm{Var}(V)=\mathrm{E}(V^2)-[\mathrm{E}(V)]^2=8.22-2.72^2=0.8216$ | Note that "percentage" instead of "probability" was used in the table. |
| **(b)** **[2]** | Let $S$ denote the total number of doses of vaccine received by 3 randomly selected surveyees.<br>$\mathrm{P}(S>10)=\mathrm{P}(S=11)+\mathrm{P}(S=12)$<br>$=\frac{3!}{2!}\mathrm{P}(V=3)[\mathrm{P}(V=4)]^2+[\mathrm{P}(V=4)]^3$<br>$=3\times0.69\times0.09^2+0.09^3$<br>$=0.017496$ (exact) | |
| **(c)** **[2]** | In order to model $M$ by a binomial distribution, the assumptions are:<br>1. one person's decision to choose to receive the Moderna's bivalent vaccine is independent of another person's decision to choose to receive the Moderna's bivalent vaccine;<br>2. the probability of each person choosing to receive the Moderna's bivalent vaccine is a constant 0.6. | |
| **(d)** **[2]** | $M \sim \mathrm{B}(10, 0.6)$ when $n=10$<br>$\mathrm{P}(4\le M<8)=\mathrm{P}(M\le7)-\mathrm{P}(M\le3)$<br>$=0.83271-0.054762$<br>$=0.778$ (3s.f.) | |
| | Alternatively, $\mathrm{P}(4\le M<8)=\sum_{m=4}^{7}\mathrm{P}(M=m)=0.778$ (3s.f.) | |
| **(e)** **[3]** | $M \sim \mathrm{B}(n, 0.6)$<br>$\mathrm{P}(M\ge15)\ge0.9$<br>$\mathrm{P}(M\le14)\le0.1$ | |

| $n$ | $\mathrm{P}(M \le 14)$ |
|---|---|
| 29 | $0.1362 > 0.1$ |
| 30 | $0.0971 < 0.1$ |

From GC, least $n = 30$