CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 3
JC2 Preliminary Examination

MATHEMATICS 9820/01

Paper 1

17 September 2021

3 hours

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet and a graph paper booklet will be provided with this question paper. You should follow the instruction on the front cover of both booklets. If you need additional answer paper or graph paper**,** ask the invigilator for a continuation booklet or graph paper booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise. Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

1. Let $\mathrm{f}(n) = 13^{n+1}(6n+7) - 1$. By considering $\mathrm{f}(k+1) - \mathrm{f}(k)$, where $k \in \mathbb{Z}^+$, prove using Mathematical Induction that $\mathrm{f}(n)$ is divisible by 18 for every positive integer *n*. [6]

2. **(a)** For any real number *x*, the largest integer less than or equal to *x* is denoted by $\lfloor x \rfloor$. For example, $\lfloor 3.7 \rfloor = 3$ and $\lfloor 4 \rfloor = 4$.

(i) Use a sketch graph of $y = \lfloor x \rfloor$ for $0 \le x < 5$ to evaluate $\int_0^5 \lfloor x \rfloor \, \mathrm{d}x$. [2]

(ii) Use a sketch graph of $y = \lfloor \mathrm{e}^x \rfloor$ for $0 \le x < \ln n$, where *n* is an integer, to show that $\int_0^{\ln n} \lfloor \mathrm{e}^x \rfloor \, \mathrm{d}x = n \ln n - \ln(n!)$. [3]

(iii) Hence, show that $n! \ge n^n \mathrm{e}^{1-n}$. [3]

(b) Find the exact value of the integral $\int_{-2}^{2} \sqrt{9 - x^2} \left(x^3 \cos \frac{x}{2} + \frac{1}{2} \right) \mathrm{d}x$. [4]

3. An *n*-digit number uses only digits 1, 2 and 3. It does not contain any occurrence of '12' or '21'. Let there be T_n such numbers, with X_n of these having first digit 1 and Y_n having first digit 3.

(i) Prove that, for $n \ge 2$,

(a) $X_n = X_{n-1} + Y_{n-1}$, [1]

(b) $Y_n = 2X_{n-1} + Y_{n-1}$. [2]

(ii) Hence, find a recurrence relation for Y_{n+1} in terms of Y_n and Y_{n-1} for $n \ge 2$. [2]

(iii) Prove that $Y_n = \frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{1}{2}\left(1-\sqrt{2}\right)^n$ for $n \in \mathbb{Z}^+$. [5]

(iv) Find and simplify an expression for T_n for $n \in \mathbb{Z}^+$. [2]

4. The Bernoulli polynomials, $\mathrm{B}_n(x)$, where n = 0, 1, 2, …, are defined by $\mathrm{B}_0(x)=1$ and, for $n \geq 1$, $\dfrac{\mathrm{dB}_n}{\mathrm{d}x}=n\mathrm{B}_{n-1}(x)$ and $\int_0^1 \mathrm{B}_n(x)\mathrm{d}x=0$.

(i) Show that $\mathrm{B}_4(x)=x^2(x-1)^2+A$, where A is a constant (that need not be evaluated). [4]

(ii) Show that, for $n \geq 2$, $\mathrm{B}_n(1)-\mathrm{B}_n(0)=0$. [2]

(iii) Show that $\mathrm{B}_n(x+1)-\mathrm{B}_n(x)=nx^{n-1}$ for all positive integers n. [4]

(iv) Hence, for any positive integer N, show that $\sum_{m=1}^{N} m^{n-1}=\dfrac{1}{n}\left[\mathrm{B}_n(N+1)-\mathrm{B}_n(1)\right]$, and deduce that $\sum_{m=1}^{N} m^3=\left(\dfrac{N(N+1)}{2}\right)^2$. [3]

5. (a) New Chang Lee sells 5 types of puffs: curry, sardine, black-pepper chicken, tuna and yam.

Mr. Ong wants to order a total of 26 puffs such that each type is included and there is an even number of puffs of each type. Find the number of ways he can make the order. [4]

(b) 4 married couples are randomly seated at a round table with 8 chairs.
By using the principle of inclusion and exclusion, find the probability that no wife sits next to her husband. [4]

(c) Let $a_1,\ a_2,\ ...,\ a_{10}$ be a sequence of 10 natural numbers.
By considering the sums $a_1,\ a_1+a_2,\ \ldots,\ a_1+a_2+\ldots+a_{10}$, and using the pigeonhole principle, prove that there is a sequence of n consecutive term(s) whose sum is divisible by 10, for some $1 \leq n \leq 10$. [5]

6. (a) Let p be a prime number and $r, s \in \mathbb{Z}$ such that $0 < r, s < p$.

(i) For any $a \in \mathbb{Z}$, show that $ra \equiv sa \pmod{p}$ if and only if $r=s$. [3]

(ii) By considering the product $a \times 2a \times 3a \times ... \times (p-1)a$, prove that for any $a \in \mathbb{Z}$ but not divisible by p, $a^{p-1} \equiv 1 \pmod{p}$ (Fermat's Little Theorem). [2]

(iii) Hence, show that for any integers a, b, p divides $ab^p - a^p b$. [2]

(b) Let $m, n, k \in \mathbb{Z}^+$ and $\gcd(m,n)=1$.
Prove that $n+m \mid n^2+km^2$ if and only if $n+m \mid k+1$. [7]

[Turn over

7. **(a)** **(i)** Show that $x+\dfrac{1}{y}\geq 2\sqrt{\dfrac{x}{y}}$ for $x,\ y>0$. [1]

(ii) Hence, show that

$$\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_3}\right)\ldots\left(x_{49}+\frac{1}{x_{50}}\right)\left(x_{50}+\frac{1}{x_1}\right)\geq 2^{50}\text{, for } x_1,\ x_2,\ x_3,\ \ldots,\ x_{50}>0.$$ [2]

(iii) Deduce the positive solutions of the system of equations

$$x_1+\frac{1}{x_2}=8$$
$$x_2+\frac{1}{x_3}=\frac{1}{2}$$
$$x_3+\frac{1}{x_4}=8$$
$$\vdots$$
$$x_{49}+\frac{1}{x_{50}}=8$$
$$x_{50}+\frac{1}{x_1}=\frac{1}{2}.$$ [6]

(b) **(i)** Let x,y,z be positive real numbers satisfying $xyz=1$.
Determine, with proof, the minimum value of

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}.$$ [4]

(ii) Hence, prove that if a,b,c are positive real numbers satisfying $abc=1$, then

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$ [2]

8. **(a)** For each positive integer r, let

$$a_r = \frac{1}{r+1} + \frac{1}{(r+1)(r+2)} + \frac{1}{(r+1)(r+2)(r+3)} + \dots ,$$

$$b_r = \frac{1}{r+1} + \frac{1}{(r+1)^2} + \frac{1}{(r+1)^3} + \dots .$$

(i) Find b_r in terms of r. [2]

(ii) Deduce that $0 < a_r < \frac{1}{r}$. [2]

(iii) Show that $a_r = r!\mathrm{e} - \lfloor r!\mathrm{e} \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. [3]

(iv) Hence show that e is irrational. [3]

(b) Given that a sequence $y_1, y_2, y_3, \dots$ is defined by

$$y_1 = 2,\ y_{n+1} = \frac{y_n}{2} + \frac{1}{y_n} \text{ , for all } n \in \mathbb{N} .$$

Show that the sequence converges.
Find the exact limit of the sequence. [5]