Integration Practice Work Solutions

Differentiation and Integration

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(a)

$$\int \frac{1-e^{2x}}{1-e^{x}}\,dx = \int \frac{(1-e^{x})(1+e^{x})}{1-e^{x}}\,dx$$

$$= \int 1+e^{x}\,dx \quad \text{M1}$$

$$= x+e^{x}+c \quad \text{A1 [Deduct A1 for missing '+c']}$$

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(b)

$$\int_1^3 ax^2\,dx = 7$$

$$\left[\frac{ax^3}{3}\right]_1^3 = 7 \quad \text{M1}$$

$$\frac{27}{3}a - \frac{1}{3}a = 7$$

$$a = \frac{21}{26} \quad \text{M1}$$

$$\int_2^6 ax^2\,dx = \left[\frac{ax^3}{3}\right]_2^6$$

$$= \frac{208}{3}a$$

$$= 56 \quad \text{A1}$$

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(a)

$$\frac{d}{dx}\left(4e^{-x}(\cos x+\sin x)\right) = \underline{-4e^{-x}(\cos x+\sin x)} + \underline{4e^{-x}(-\sin x+\cos x)} \quad \text{M1, M1}$$

$$= 4e^{-x}(-\cos x-\sin x-\sin x+\cos x)$$

$$= -8e^{-x}\sin x \quad \text{A1}$$

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(b)

$$\int_0^{\pi/2} e^{-x}\sin x\,dx = -\frac{1}{8}\int_0^{\pi/2} -8e^{-x}\sin x\,dx$$

$$= -\frac{1}{8}\left[4e^{-x}(\cos x+\sin x)\right]_0^{\pi/2} \quad \text{M1}$$

$$= -\frac{1}{2}\left[e^{-\frac{\pi}{2}}-1\right]$$

$$= \frac{1}{2}\left(1-e^{-\pi/2}\right) \quad \text{A1 [accept } \tfrac{1}{2}-\tfrac{1}{2}e^{-\frac{\pi}{2}} \text{ or } 0.396\text{]}$$

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(a)

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Find $\frac{d}{dx}(x\sin 2x)$.

$= \sin 2x + x(\cos 2x)(2)$

$= \sin 2x + 2x\cos 2x$ — (B1)

(b) Hence, find $\int 2x\cos 2x \, dx$.

$$\int \sin 2x + 2x\cos 2x \, dx = x\sin 2x + C$$

$$\int 2x\cos 2x \, dx = x\sin 2x - \int \sin 2x \, dx + c$$

$$= x\sin 2x - \left(\frac{-\cos 2x}{2}\right) + d$$

$$= x\sin 2x + \frac{1}{2}\cos 2x + d \quad \text{— (A1)}$$

(c)

$$\frac{d}{dx}\left(x^2\cos 2x\right) = 2x\cos 2x + x^2(-\sin 2x)(2) \quad \text{— (M1)}$$

$$= 2x\cos 2x - 2x^2\sin 2x$$

$x^2\sin 2x = 0$ — (M1)

$x = 0$ or $\sin 2x = 0$

$x = 0, \frac{\pi}{2}, \pi, \ldots$

$= 0, \frac{\pi}{2}$.

$$\int_0^{\frac{\pi}{2}} 2x\cos 2x - 2x^2\sin 2x \, dx = \left[x^2\cos 2x\right]_0^{\frac{\pi}{2}}$$ — [M1 for limit, M1 for $\int \frac{d}{dx} x^2\cos 2x \, dx$]

$$-2\int_0^{\frac{\pi}{2}} x^2\sin 2x \, dx = \left[x^2\cos 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x\cos 2x \, dx$$

$$= \left(\frac{\pi}{2}\right)^2(-1) - \left[x\sin 2x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi^2}{4} - \left[\frac{1}{2}(-1) - \frac{1}{2}(1)\right]$$

$$= \underline{\underline{-\frac{\pi^2}{4}}} + \underline{\underline{1}}$$ — (M1 for either)

$$\int_0^{\frac{\pi}{2}} x^2\sin 2x \, dx = -\frac{1}{2}\left(\frac{-\pi^2}{4} + 1\right)$$

$$= \frac{\pi^2}{8} - \frac{1}{2}$$ (also accept: 0.734) — (A1)

4	$y=\dfrac{x}{(3x+2)^2}$ $\dfrac{dy}{dx}=\dfrac{(3x+2)^2-x(2)(3x+2)(3)}{(3x+2)^4}$ $=\dfrac{(3x+2)\left[(3x+2)-6x\right]}{(3x+2)^4}$ $=\dfrac{2-3x}{(3x+2)^3}$ $\int_0^1 \dfrac{2-3x}{(3x+2)^3}\ dx=\left[\dfrac{x}{(3x+2)^2}\right]_0^1$ $\int_0^1 \dfrac{2}{(3x+2)^3}+\dfrac{-3x}{(3x+2)^3}\ dx=\dfrac{1}{25}$ $\int_0^1 \dfrac{-3x}{(3x+2)^3}\ dx=\dfrac{1}{25}-\int_0^1 2(3x+2)^{-3}\ dx$ $=\dfrac{1}{25}-\left[-\dfrac{(3x+2)^{-2}}{3}\right]_0^1$ $=\dfrac{1}{25}+\dfrac{1}{3}\left[\dfrac{1}{25}-\dfrac{1}{4}\right]$ $=-\dfrac{3}{100}$
5	**(i)** $\dfrac{dy}{dx}=e^{2x}(5)+(5x-4)(2e^{2x})$ $=5e^{2x}+10xe^{2x}-8e^{2x}$ $=10xe^{2x}-3e^{2x}$ or $e^{2x}(10x-3)$
	(ii) $\dfrac{d}{dx}\left[e^{2x}(5x-4)\right]=10xe^{2x}-3e^{2x}$ $10xe^{2x}=\dfrac{d}{dx}\left[e^{2x}(5x-4)\right]+3e^{2x}$ $3xe^{2x}=\dfrac{3}{10}\times\left[\dfrac{d}{dx}\left[e^{2x}(5x-4)\right]+3e^{2x}\right]$ $\int 3xe^{2x}dx=\dfrac{3}{10}\int\left[\dfrac{d}{dx}\left[e^{2x}(5x-4)\right]+3e^{2x}\right]dx$ $=\dfrac{3}{10}\left[e^{2x}(5x-4)\right]+\dfrac{3}{10}\left(\dfrac{3e^{2x}}{2}\right)+C$ $=\dfrac{3}{2}xe^{2x}-\dfrac{3}{4}e^{2x}+C$ or $\dfrac{3}{4}e^{2x}(2x-1)+C$

Integration to find equation of a graph

1	$f(x) = \int \frac{6}{3x+5}\, dx$ $= \frac{6\ln(3x+5)}{3} + c$ — (M1) $= 2\ln(3x+5) + c$ Since $f(x)$ passes through the origin, $\underbrace{0 = 2\ln 5 + c}_{\text{(M1)}} \Rightarrow c = -2\ln 5$. $\therefore f(x) = 2\ln(3x+5) - 2\ln 5$. $= 2[\ln(3x+5) - \ln 5]$ $= 2\ln\left(\frac{3x+5}{5}\right)$ (A1, accept either). $= \ln\left(\frac{3x+5}{5}\right)^2$
2	**(i)** $y = \int \cos 2\theta - 3\sin 2\theta \quad d\theta$ $= \frac{1}{2}\sin 2\theta + \frac{3}{2}\cos 2\theta + c$ *When* $\theta = \frac{\pi}{2}, \quad y = -\frac{1}{2}$. $\therefore \quad -\frac{1}{2} = \frac{1}{2}\sin\pi + \frac{3}{2}\cos\pi + c$ $\Rightarrow \quad -\frac{1}{2} = -\frac{3}{2} + c$ $\Rightarrow \quad c = 1$ *Hence,* $\quad y = \frac{1}{2}\sin 2\theta + \frac{3}{2}\cos 2\theta + 1$ **(ii)**

		For turning pts, $f'(\theta)=0.$ $\Rightarrow \quad \cos 2\theta - 3\sin 2\theta = 0$ $\Rightarrow \quad \tan 2\theta = \frac{1}{3}$ $(basic \measuredangle = 0.321751)$ $2\theta = 0.321751, \quad 3.463344$ $\theta = 0.160876, \quad 1.731672$ $\therefore \; y = 2.58, \quad -0.581$ *Coordinates of turning pts are* $(0.161,\ 2.58)$ *and* $(1.73,\ -0.581)$
3	**(i)**	$\frac{d^2y}{dx^2} = 6x - 2$ $\frac{dy}{dx} = \frac{6x^2}{2} - 2x + C = 3x^2 - 2x + C$ Sub $x = 2$, $\frac{dy}{dx} = 3$ $3 = 3(2)^2 - 2(2) + C$ $\Rightarrow C = 3 - 12 + 4 = -5$ $\frac{dy}{dx} = 3x^2 - 2x - 5$

		$y=\int\left(3x^2-2x-5\right)\,\mathrm{d}x=x^3-x^2-5x+D$ Sub $(2,-9)$ $\quad -9=2^3-2^2-5(2)+D$ $\Rightarrow D=-9-8+4+10=-3$ $y=x^3-x^2-5x-3$
	(ii)	$\frac{dy}{dx}=3x^2-2x-5=3\left(x^2-\frac{2}{3}x\right)-5$ $=3\left[\left(x-\frac{1}{3}\right)^2-\frac{1}{9}\right]-5$ $=3\left(x-\frac{1}{3}\right)^2-\frac{1}{3}-5$ $=3\left(x-\frac{1}{3}\right)^2-\frac{16}{3}$ Since $\left(x-\frac{1}{3}\right)^2\geq 0$ $3\left(x-\frac{1}{3}\right)^2-\frac{16}{3}\geq 0-\frac{16}{3}$ $\frac{\mathrm{d}y}{\mathrm{d}x}\geq-\frac{16}{3}$ Hence gradient is never less than $-\frac{16}{3}$ (shown)

Integration: Area under Curve

1	**XMS 2020 AM P1, Q1**

when $x = 2$,

$y = \ln 2^2$

$= \ln 4$ — (M1) (full credit if leave in surf)

(1.3862)

$y = \ln x^2$

$e^y = x^2$

$x = \sqrt{e^y}$ — (M1) : Making x the subject

$= e^{\frac{1}{2}y}$

shaded area $= \int_0^{\ln 4} e^{\frac{1}{2}y}\, dy$

$= \left[2e^{\frac{1}{2}y}\right]_0^{\ln 4}$ — (M1) : correct integration.

$= 2e^{\frac{1}{2}(\ln 4)} - 2e^0$

$= 2e^{\ln 2} - 2$

$= 2(2) - 2$

$= 2 \text{ units}^2$ # — (A1)

2	$\ln(3-x)^2 = 0$ $(3-x)^2 = 1$ $3-x = 1$ or -1 $x = 2$ or 4 (M1) (rej) Also accept $2\ln(3-x) = 0$ $3-x = 1$ $x = 2$ $\therefore P(2, 0)$ at $x=0$, $y = \ln 9$ — (M1) $\therefore Q(0, \ln 9)$ $\frac{dy}{dx} = 2\left(\frac{1}{3-x}\right)(-1)$ $= -\frac{2}{3-x}$ — (M1) at $x=0$, $\frac{dy}{dx} = -\frac{2}{3}$. $\therefore$ M normal : $\frac{3}{2}$ — (M1) Equation of normal: $y = \frac{3}{2}x + \ln 9$ — (M1) at $y=0$, $0 = \frac{3}{2}x + \ln 9$ $\frac{3}{2}x = -\ln 9$ $x = -\frac{2}{3}\ln 9$ — (A1) $\therefore R\left(-\frac{2}{3}\ln 9, 0\right)$

Area of ΔORQ: $\frac{1}{2} \times \ln 9 \times \frac{2}{3} \ln 9$ — (M1)

$= \frac{1}{3}(\ln 9)^2$

$= 1.6092$ } (A1, accept either)

$y = \ln(3-x)^2$

$\frac{y}{2} = \ln(3-x)$

$e^{\frac{y}{2}} = 3-x$

$x = 3 - e^{\frac{y}{2}}$ — (M1)

∴ Area bounded by curve, x & y axes

$\Rightarrow \int_0^{\ln 9} 3 - e^{\frac{y}{2}} \, dy$

$= \left[3y - \frac{e^{\frac{y}{2}}}{\left(\frac{1}{2}\right)}\right]_0^{\ln 9}$ — (M1)

$= \left[3y - 2e^{\frac{y}{2}}\right]_0^{\ln 9}$

$= 3\ln 9 - 2e^{\frac{1}{2}\ln 9} - \left(3(0) - 2e^0\right)$

$= 3\ln 9 - 6 + 2$

$= 3\ln 9 - 4$ (also accept 2.5916) — (A1)

∴ total area: $(3\ln 9 - 4) + \frac{1}{3}(\ln 9)^2$

$= 4.20$ (3 sf) — (A1)

3 Find A:

$\sqrt{2x-1} = 0$

$x = \frac{1}{2}$

$\left(\frac{1}{2}, 0\right)$

Find B:

$$y=(2x-1)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=(2x-1)^{-\frac{1}{2}}$$

$$(2x-1)^{-\frac{1}{2}}=\frac{1}{3}$$

$$2x-1=3$$

$$x=5$$

$$y=3$$

$$(5,3)$$

Gradient of normal $=-3$

Let C be $(a, 0)$

$$\frac{0-3}{a-5}=-3$$

$$a-5=1$$

$$a=6$$

$$(6,0)$$

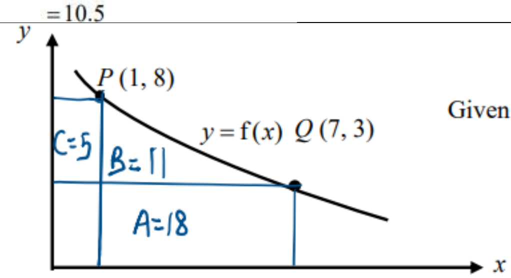
$$\text{Shaded Area}=\int_{\frac{1}{2}}^{5}(2x-1)^{\frac{1}{2}}dx+\frac{1}{2}(1)(3)$$

$$=\frac{1}{3}\left[(2x-1)^{\frac{3}{2}}\right]_{\frac{1}{2}}^{5}+\frac{3}{2}$$

$$=\frac{1}{3}[27-0]+\frac{3}{2}$$

$$=10.5$$

4



Given $\int_1^7 y\,dx=29$

Area A $=6\times3=18$ units^2 Area B $=29-18=11$ units^2 Area C $=1\times5=5$ units^2	M2 (any 2 areas)
$\int_3^8 x\,dy=$ area B + area C = 11 + 5 =16 units^2	A1

Kinematics

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XMS 2020 AM P2, Q8

(i) Show that $k = 10$. [2]

$a = 2t - k$ M1

When $t = 0$, $a = -10$

$-10 = -k$

$k = 10$ (shown) A1

ii) Find the time when the particle reaches ~~maximum~~ minimum velocity. [3]

For min v, $\underline{a = 0}$, M1

$2t - 10 = 0$

$t = 5s$ A1

$\underline{\frac{d^2v}{dt^2} = 2 > 0}$. Hence particle reaches $\underline{\text{min } v}$ when $t = 5$. A1

(iii) Find the distance travelled by the particle in the tenth second. [4]

$s = \int t^2 - 10t + 24\, dt$

$= \frac{t^3}{3} - 5t^2 + 24t + c$ M1

When $s = 0$, $t = 0$, $c = 0$.

$s = \frac{t^3}{3} - 5t^2 + 24t$ M1

When $t = 9$, $s = 54$

$t = 10$, $s = 73\frac{1}{3}$ M1 (both correct)

$\therefore$ distance travelled $= 73\frac{1}{3} - 54$

$= 19\frac{1}{3}m$ A1

OR total dist $= \int_9^{10} t^2 - 10t + 24\, dt$ M1

M1 $= \left[\frac{t^3}{3} - 5t^2 + 24t\right]_9^{10}$

(sub t=10 & t=9) M1 $= 73\frac{1}{3} - 54$

A1 $= 19\frac{1}{3}m$

iv) Explain with workings, whether the particle will return to O for $t > 9$. [2]

① At $t = 5s$, particle reaches minimum velocity.

② At $t = 9s$, $v = 15\ m/s$

③ When $v = 0$, $(t-4)(t-6) = 0$

$t = 4s, 6s$ M1

OR ③ & Indicate s when t=4, 6 & draw out/map out distance travelled — M1

Since $\underline{v > 0 \text{ and } s > 0}$ for $t > 9$, particle will not return to O. A1

OR When $s = 0$,

$t(t^2 - 15t + 72) = 0$

$t = 0$ or $b^2 - 4ac = -63 < 0$ — M1

$\therefore$ no roots — A1

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XMS 2022 AM P1, Q13

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(a) Find the exact time taken for the journey from A to B. [4]

$s = 400 - 400e^{-\frac{t}{10}}$

$v = \frac{ds}{dt}$

$= -400(-\frac{1}{10})e^{-\frac{t}{10}}$ — (M1)

$= 40e^{-\frac{t}{10}}$

at $t=0$, $v = 40$.

at B, $v = 20$.

$\therefore 40e^{-\frac{t}{10}} = 20$ — (M1)

$e^{-\frac{t}{10}} = \frac{1}{2}$

$-\frac{t}{10} = -\ln 2$

$t = 10\ln 2$ — (A1)

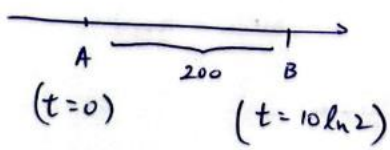
(b) Find the average speed of the motorcycle for the journey from A to B. [2]

at $t = 10\ln 2$, $s = 400(1 - e^{-\frac{10\ln 2}{10}})$

$= 400(1 - e^{-\ln 2})$

$= 400(1 - \frac{1}{2}) = 200$ — (M1)

at $t=0$, $s = 400(1 - e^{0}) = 0$.



$\therefore$ total dist: 200m.

Average speed: $\frac{200\text{m}}{(10\ln 2)\text{s}} = 28.9\text{m/s}$ (3sf) — (A1)

	(c) Find the exact acceleration of the motorcycle at B. [2] $a = \frac{dv}{dt}$ $= 40\left(-\frac{1}{10}\right)e^{-\frac{t}{10}}$ $= -4e^{-\frac{t}{10}}$ — (M1) at B, $a = -4e^{-\frac{1}{10}(10\ln 2)}$ $= -4\left(\frac{1}{2}\right)$ $= -2$ — (A1) Ans: $-2\ m/s^2$
3	**<u>MFSS 2022 AM P1, Q12</u>** **(i)** *When* $v = 0$, $\frac{2}{5}e^{3t} = 6e^{\frac{1}{2}-t}$ $e^{4t-\frac{1}{2}} = 15$ $4t - \frac{1}{2} = \ln 15$ $t = \frac{1}{4}\left(\frac{1}{2} + \ln 15\right)$ $= \frac{1}{8}(1 + 2\ln 15)$ (*shown*) **(ii)** $a = \frac{6}{5}e^{3t} + 6e^{\frac{1}{2}-t}$ *When* $t = \frac{1}{8}(1 + 2\ln 15) \approx 0.802013$ sec, $a = 17.74389 \approx 17.7\ m/s^2$

	iii)	$s=\int \frac{2}{5}e^{3t}-6e^{\frac{1}{2}-t}\ \mathrm{d}t$ $=\frac{2}{15}e^{3t} \quad + \quad 6e^{\frac{1}{2}-t} \quad + \quad c$ *When* $t=0, \quad s=0.$ $\Rightarrow \quad 0=\frac{2}{15}+6e^{\frac{1}{2}}+c$ $\Rightarrow \quad c=-\left(\frac{2}{15}+6e^{\frac{1}{2}}\right)$ *When* $t=\frac{1}{8}(1+2\ln 15)\approx 0.802013$ sec, *Displacement* $OA=-4.111031$ *Distance* $OA=4.11\,m$
	iv)	*When* $t=1, \quad s=-3.71\,m$ *When* $t=2, \quad s=45.1\,m$ During the 2nd second, the displacement changes from negative to positive. Hence, the particle will pass through point *O* again during the 2nd second.
4	**<u>FMSS 2022 AM P1, Q11</u>**	

	(a)	$v = 16 + 6t - at^2$ When $v = 25$, $6t - at^2 = 9$ ----- (1) $\frac{dv}{dt} = 6 - 2at$ When v is max, $\frac{dv}{dt} = 0$ Hence, $6 - 2at = 0$ $t = \frac{3}{a}$ ---- (2) Solving, $6\left(\frac{3}{a}\right) - a\left(\frac{3}{a}\right)^2 = 9$ $a = 1$
	(b)	When $t = 0$, $v = 16$ Hence, $6t - t^2 = 0$. $t = 0$ (NA), 6 secs
	(c)	When $v = 0$, $16 + 6t - t^2 = 0$ $(t-8)(t+2) = 0$ $t = -2$ (NA), 8 secs
	(d)	s = $\int 16 + 6t - t^2 dt$ $= 16t + 3t^2 - \frac{t^3}{3} + c$ When $t = 0$, $s = 3$, hence $c = 3$ When $t = 8$, $s = 152\frac{1}{3}$ m When $t = 9$, $s = 147$m Hence, distance travelled = $154\frac{2}{3}$ m
		OR $c = -3$ When $t = 0$, $s = 3$, hence $c = -3$ When $t = 8$, $s = 146\frac{1}{3}$ m When $t = 9$, $s = 141$m Hence, distance travelled = $154\frac{2}{3}$ m