F03: Further Differential Equations

1 **(i)** Given that y is a function of x and that $x = \mathrm{e}^t$, show that

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \text{ and } x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}.$$ [3]

(ii) Using the substitution $x = \mathrm{e}^t$, find the general solution of

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = \frac{1}{x^\alpha},$$

where α is a real constant, $\alpha \neq 2$. [7]

(iii) It is given that for all $k > 0$, $\lim_{x\to\infty}\left(\frac{\ln x}{x^k}\right) = 0$. Determine, with justification, whether $x^\alpha y$ will remain finite no matter what the initial conditions are as x becomes large and positive when

(a) $\alpha < 2$, and [2]

(b) $\alpha > 2$. [1]

(2016 DHS / JC1 / Promo / Q11)

2 **(i)** Solve the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = \sin 2x$. [6]

(ii) Show that when $x = n\pi$, where n is a sufficiently large positive integer, $y \approx -\frac{1}{4}n\pi$, whatever the initial conditions. [2]

(2016 SRJC / JC1 / Promo / Q6)

3 3D printing is a term that is commonly used to describe any process in which a 3D object is created from a computer model. To be able to print in 3 dimensions, a computer-aided manufacturing (CAM) software is used to control the motors and the movement of a 3D printer nozzle which is the part that does the actual printing.

A certain CAM software is designed to move the nozzle along the path C described by the parametric equations $x = \sin^3\theta,\ y = \cos^3\theta - 2$ for $0 \leq \theta \leq \frac{\pi}{2}$.

(i) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of θ. [4]

(ii) Find the exact length of the path covered by the nozzle, from $\theta = 0$ to $\theta = \frac{\pi}{2}$. [3]

After completion of the printing for C, the printer continues to print on the surface generated by rotating C through 2π radians about the x-axis. Find the exact area of the surface generated. [4]

(2016 HCI / JC1 / Promo / Q10)

4 **(i)** By means of the substitution $y = vx$, show that the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2,\ x \neq 0,$$

can be reduced to $\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{v^2+1}{2vx}$. [3]

(ii) Hence find the general solution. [4]

(iii) Sketch the solution curve passing through (2, 0). [2]

(2016 SRJC / JC1 / Promo / Q9)

5 **(a)** **(i)** Given that $x > 0$, find the general solution of the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

expressing y as a function of x. [3]

(ii) Sketch the graph of the particular solution for which $y \to 2$ as $x \to \infty$. [3]

(b) Water starts pouring into an empty open tank, and t seconds later the volume, V litres, of water in the tank is given by $\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{1}{20}V = \frac{1}{4}t + 1$.

(i) Find V in terms of t, and hence show that, for large values of t, $V \approx 5t - 80$. [5]

(ii) What can we deduce from **(b)(i)**, in the context of this question? [1]

(2016 RI / JC1 / Promo / Q7)

6 Find the general solution of the differential equation

$$20\frac{\mathrm{d}y}{\mathrm{d}x} + y = 5x + 20.$$ [4]

(2016 RVHS / JC1 / Promo / Q1)

7 **(a)** Using the substitution $x = \cos 2t$ where $0 < t < \frac{\pi}{2}$, show that the differential equation

$$4\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \left(4x + 2\sqrt{1-x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2x$$

can be reduced to

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = 2\cos 2t.$$ [3]

(b) The current in a particular electrical circuit is described by the equation

$$\frac{\mathrm{d}^2I}{\mathrm{d}t^2} + \frac{\mathrm{d}I}{\mathrm{d}t} + I = 2\cos 2t,$$

where I is the current in amperes and t is the time in minutes after the power source is turned on.

(i) Find the general solution of the differential equation. [5]

(ii) Show that after the circuit is turned on for a long time, the current fluctuates between $-\frac{2}{\sqrt{13}}$ amperes and $\frac{2}{\sqrt{13}}$ amperes inclusive. [2]

(2016 RVHS / JC1 / Promo / Q9)

8 The spread of information on a social media is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP(50 - P),$$

where P is the number of people in thousands who received the information after t hours, and a is a positive constant.

(i) Explain why this is a reasonable model. [2]

(ii) Explain the significance of a and that of the number '50' in the equation. [3]

(iii) Sketch the graph of P against t and comment on the number of people who received the information in the long-term. [3]

(iv) Given that $a = 0.01$ and 10 thousand people received the information after 1 hour, use the Euler method with step size 0.5 to estimate the number of people who received the information after 2 hours. Explain whether you would expect this value to be an under-estimate or an over-estimate of the true value. [4]

(2016 RVHS / JC1 / Promo / Q11)

9 **(i)** Find the general solution of the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0.$$ [3]

(ii) Hence solve the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = \mathrm{e}^{3x},$$

given that $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8}{7}$ when $x = 0$. [7]

(2017 ACJC / JC2 / BT / Q2)

10 A population P of birds in a bird sanctuary is assumed to be governed by the mathematical model $\frac{\mathrm{d}P}{\mathrm{d}t} = \alpha P - \frac{\beta P^2}{100}$ where t is time in weeks, α and β are real non-zero constants. During a bird flu outbreak, infected birds were killed to prevent the spread of the disease. Let C denotes the number of birds that were killed in each week, where $C > 0$.

(a) Write down the carrying capacity of the sanctuary before the bird flu outbreak, in terms of α and β. [1]

(b) **(i)** At the start of the bird flu outbreak, $\frac{50\alpha}{\beta}$ infected birds were killed. By sketching an appropriate labelled graph of $\frac{\mathrm{d}P}{\mathrm{d}t}$ against P, find in terms of α and β, the maximum possible value of C in order to ensure continual survival of the bird population in the long term. [3]

(ii) Given that $C = \frac{21\alpha^2}{\beta}$, describe the long term behaviour of the bird population. [2]

(2017 TJC / JC2 / BT / Q2)

11 Find the solution of the differential equation

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{2y^2}{x}=-x^2\mathrm{e}^x,$$

using the substitution $z=y^2$. [5]

(2017 ACJC / JC2 / BT / Q3)

12 **(i)** Using the substitution $y=\frac{z}{x^2}$, show that the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(2+16x^2\right)y=0 \text{ ----- [I]}$$

can be reduced to $\frac{\mathrm{d}^2z}{\mathrm{d}x^2}+16z=0$. Find the general solution of the differential equation [I]. [4]

(ii) The current in an electric circuit is modelled by the equation

$$\frac{\mathrm{d}^2I}{\mathrm{d}t^2}+16I=\sin^2 2t$$

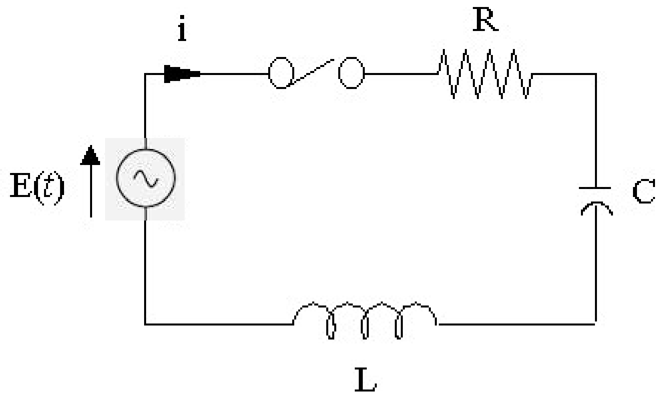
where I is the current in amperes and t is the time in minutes.

Given that $I=0$ and $\frac{\mathrm{d}I}{\mathrm{d}t}=0$ at the instant when $t=\frac{\pi}{4}$, find I in terms of t.

Describe how I changes as t increases. [9]

(2017 TJC / JC2 / BT / Q9)

13 A series RLC circuit is an electrical circuit consisting of a resistor of R ohms, an inductor of L henries, and a capacitor of C farads, connected in series as shown in the diagram below.



When a driving electromotive force $E(t)$ volts is applied to the circuit, the current I amps for this circuit satisfies the equation

$$L\frac{\mathrm{d}^2I}{\mathrm{d}t^2}+R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{1}{C}I=E(t)$$

It is given that $R = 10$, $L = 1$ and $C = 0.02$.

(i) If $E(t) = 0$, find the general solution of the differential equation. [2]

(ii) If $E(t) = 25\sin 5t$, find the particular solution of the differential equation, given that at time $t = 0$, there is no current in the circuit and the rate of change of the current is 94 amps/s. [7]

(2017 VJC / JC2 / BT / Q1)

14 Drugs called cytostatics are used in chemotherapy to inhibit the growth of cancerous tumour. The growth of a tumour, of size P mm^3, in the presence of c mg of cytostatics, can be modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 12P - \frac{P^2}{30} - c\,.$$

(i) What is the least amount of cytostatics that should be administered to prevent a tumour of any size from growing bigger? [2]

(ii) Suppose the amount found in part **(i)** is administered. Without solving the differential equation, discuss the long term behaviour of a tumour for different initial sizes of tumour. [4]

(2017 VJC / JC2 / BT / Q6)

15 **(a)** A tank initially contains 40 litres of pure water. A student carrying out an experiment pours a salt solution containing 3 grams of salt per litre into the tank at the constant rate of 2 litres per minute. At the same time, the mixture is stirred thoroughly and flows out of the tank at the constant rate of 3 litres per minute. This is carried out until the tank is empty.

(i) If x is the amount of salt in the tank after t minutes, explain why x satisfies the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6 - \frac{3x}{40-t}.$$ [2]

(ii) Find the amount of salt in the tank when there are 20 litres of mixture in it. [5]

(iii) Find the maximum amount of salt in the tank during this experiment. (You do not need to show that it is a maximum.) [3]

(b) Another student carries out the same experiment as the first student, but does not stir the tank properly, with the result that the amount of salt in the tank follows the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6 - \frac{3\sqrt{x}}{40-t}.$$

Use the Improved Euler Method with step size $h = 1$ to determine the amount of salt in the tank after 2 minutes. [4]

(2017 ACJC / JC2 / MYE / Q4)

16 By using the substitution $x = e^t$, find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{xy}{x+1} = \frac{1}{x^2-1},\ x > 1.$$ [6]

(2017 AJC / JC2 / MYE / Q5)

17 Find the general solution for

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 8\sin 2t.$$ [7]

(2017 CJC / JC2 / MYE / Q3)

18 An electrical circuit contains a resistor, an inductor, a capacitor and an electromotive force supplied by a generator arranged in series. Kirchhoff's voltage law says that the change in current I (amps) with time t (seconds) in an electrical circuit are related by the following differential equation

$$L\frac{\mathrm{d}I}{\mathrm{d}t}+RI+\frac{Q}{C}=\mathrm{E}(t).$$

where L is the inductance (henries), R is the resistance (ohms), Q is the charge (coulombs), C is the capacitance (farads) and $\mathrm{E}(t)$ is the electromotive force function. The current I is the rate of change of charge Q with respect to time t.

Show that $L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}+R\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{Q}{C}=\mathrm{E}(t)$.

For a circuit with $R=12$ ohms, $L=1$ henry, $C=0.01$ farad and electromotive force function $\mathrm{E}(t)=408\cos 2t$, find an expression for the current I at time t, given that the initial charge and current in the electrical circuit are both zero.

The electrical circuit has been left on for a long period of time, determine the peak value of the current in the circuit, giving your answer in exact form. [12]

(2017 AJC / JC2 / MYE / Q6)

19 The population of fish in a fishery is modelled by

$$\frac{\mathrm{d}P}{\mathrm{d}t}=-\frac{1}{3}P\left(1-\frac{P}{2}\right)-h,$$

where P (in millions) is the population of fish and h (in millions) is the number of fish harvested per year.

Find the stable population P_S in terms of h. [2]

Describe the trend in population when

(a) $0<P_0<P_S$,

(b) $P_0>P_S$,

where P_0 is the initial population of fish. [2]

Given that P_0 is 5 million, find the maximum number of fish the fishery can harvest per year such that the fishery is sustainable. (It is not sustainable if the population of fish becomes zero at some time.) [2]

(2017 CJC / JC2 / MYE / Q2)

20 By Kirchoff's Second Law, the current in an electric RLC circuit is described by the equation

$$\frac{\mathrm{d}^2I}{\mathrm{d}t^2}+2\frac{\mathrm{d}I}{\mathrm{d}t}+4I=6\cos 3t,$$

where $I(t)$ is the current in amperes and t is the time in seconds after the power source is switched on. Find the general solution for the differential equation and hence determine the steady-state current as $t\to\infty$. Find the exact maximum current at steady state. [8]

(2017 DHS / JC2 / MYE P1 / Q4)

21 **(i)** Find the general solution for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x-y}{x},$$

for $x>0$. [2]

(ii) It is given that the equation has a solution curve, C. Two asymptotes of C pass through the origin, and a third line passes through the origin also passes through the minimum point of C. Find the equations of the three lines. Justify your answers briefly. [3]

(iii) Sketch for $x > 0$, two members of the family of solution curves which are non-linear. [2]

(iv) A solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x-y}{x}$$

has $y=\frac{5}{3}$ when $x=1$.

(a) Use the Euler's method with step size 0.5 to estimate y at $x=2$. Explain whether you expect this value of y to be an under-estimate or over-estimate of the true value. [3]

(b) Copy and complete the table showing the use of the improved Euler method with step size 0.5 to estimate y at $x = 2$. [4]

x	y	$\frac{x-y}{x}$	$\tilde{y}$	$\frac{\Delta y}{\Delta x}$
1	$\frac{5}{3}$	$-\frac{2}{3}$		$\frac{-\frac{2}{3}+\frac{1}{9}}{2}$
1.5	1.5278	–0.0185		
2				

(2017 DHS / JC2 / MYE P1 / Q10)

22 **(a)** Given that $(1+4x^2)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2y}{\tan^{-1}2x}=1$ where $x>0$, find the general solution for y in terms of x. [4]

(b) Show, by means of the substitution $y=x^{-3}z$, that the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6(x^2+x)\frac{\mathrm{d}y}{\mathrm{d}x}+(15x^2+18x+6)y=\frac{265\cos 2x}{x}$$

can be reduced to the form

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}+6\frac{\mathrm{d}z}{\mathrm{d}x}+15z=f(x)$$

where $f(x)$ is to be determined. [3]

Find the general solution. [5]

(2017 HCI / JC2 / MYE P1 / Q7)

23 **(a)** Apply the improved Euler's method to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{y}{t}+y^2, \quad y(1)=1$$

to obtain the approximation value of $y\,(1.5)$ using a step size of $h=0.25$. [4]

(b) A fish farmer has a fishing pond. The fish population $P(t)$ can be modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t}=4P-0.02P^2, \quad P(0)=P_0.$$

where t is the time in weeks.

(i) Obtain an expression for P in terms of P_0 and t. Hence find the equilibrium population in the long run. [5]

The owner decides to harvest at a rate of h fishes per week.

(ii) What is the maximum value of h so that there is a chance for fish population to survive in the long run? [3]

(iii) Using the value of h obtained in part (ii), discuss the impact on the fish population if

(a) $P_0=80$,

(b) $P_0=120$. [2]

(2017 HCI / JC2 / MYE P1 / Q10)