H2 Further Mathematics

2023 JPJC JC2 Prelim Examination Paper 2 Solutions

1	
	$\sqrt{X_{n+2}} = \frac{X_{n+1}}{\left(X_n\right)^2}$ $\frac{1}{2}\ln\left(X_{n+2}\right) = \ln\left(X_{n+1}\right) - 2\ln\left(X_n\right)$ $\ln\left(X_{n+2}\right) = 2\ln\left(X_{n+1}\right) - 4\ln\left(X_n\right)$ Let $Y_n = \ln\left(X_n\right)$, we have $Y_{n+2} = 2Y_{n+1} - 4Y_n$ $Y_{n+2} - 2Y_{n+1} + 4Y_n = 0$ Auxiliary equation: $m^2 - 2m + 4 = 0$ $m = \frac{2 \pm \sqrt{2^2 - 4(4)}}{2}$ $= 1 \pm \sqrt{3}\mathrm{i}$ $= 2\mathrm{e}^{\pm\frac{\pi}{3}\mathrm{i}}$ Hence, general solution, $Y_n = 2^n\left[A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right]$ $\ln\left(X_n\right) = 2^n\left[A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right]$ $X_n = \mathrm{e}^{2^n\left[A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right]}$

2

(a)

$$I_n = \int_{-1}^{1} \left(1-x^2\right)^n \mathrm{d}x$$

$$= \left[x\left(1-x^2\right)^n \right]_{-1}^{1} - \int_{-1}^{1} -2nx^2 \left(1-x^2\right)^{n-1} \mathrm{d}x$$

$u = \left(1-x^2\right)^n$ $\quad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -2xn\left(1-x^2\right)^{n-1}$ $\quad v = x$

$$= -2n\int_{-1}^{1} -x^2 \left(1-x^2\right)^{n-1} \mathrm{d}x$$

$$= -2n\int_{-1}^{1} \left(-1+1-x^2\right)\left(1-x^2\right)^{n-1} \mathrm{d}x$$

$$= -2n\int_{-1}^{1} (-1)\left(1-x^2\right)^{n-1} \mathrm{d}x - 2n\int_{-1}^{1} \left(1-x^2\right)\left(1-x^2\right)^{n-1} \mathrm{d}x$$

$$= 2n\int_{-1}^{1} \left(1-x^2\right)^{n-1} \mathrm{d}x - 2n\int_{-1}^{1} \left(1-x^2\right)^{n} \mathrm{d}x$$

$$I_n = 2n\ I_{n-1} - 2n\ I_n$$

$$\left(1+2n\right) I_n = 2n\ I_{n-1} \quad \text{(Shown)}$$

(b)

From above, $\left(2n+1\right) I_n = 2n\ I_{n-1}$

$$I_n = \frac{2n}{2n+1}\ I_{n-1}$$

$$I_{\frac{3}{2}} = \frac{3}{4} I_{\frac{1}{2}}$$

$$= \frac{3}{4}\left(\frac{1}{2} I_{-\frac{1}{2}}\right)$$

$$= \frac{3}{8}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \mathrm{d}x$$

$$= \frac{3}{8}\left[\sin^{-1} x\right]_{-1}^{1}$$

$$= \frac{3}{8}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{3\pi}{8}$$

(c)

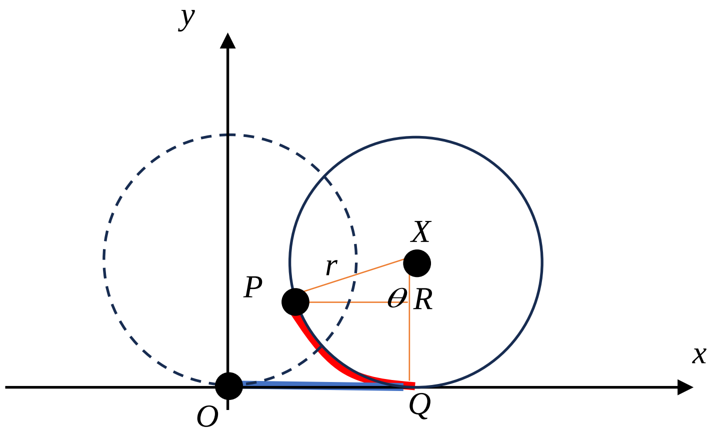
$$I_{\frac{3}{2}} = \int_{-1}^{1} \left(1-x^2\right)^{\frac{3}{2}} \mathrm{d}x$$

Using Simpson's rule with 5 ordinates, let $\mathrm{f}\left(x\right) = \left(1-x^2\right)^{\frac{3}{2}}$

x	-1	-0.5	0	0.5	1
$\mathrm{f}\left(x\right)$	0	$\frac{3\sqrt{3}}{8}$	1	$\frac{3\sqrt{3}}{8}$	0

	$I_{\frac{3}{2}}=\int_{-1}^{1}\left(1-x^2\right)^{\frac{3}{2}}\mathrm{d}x$ $\approx\frac{1}{3}(0.5)\left[0+4\left(\frac{3\sqrt{3}}{8}\right)+2(1)+4\left(\frac{3\sqrt{3}}{8}\right)+0\right]$ $=\frac{1}{6}\left(2+3\sqrt{3}\right)$
(d)	Using (b) and (c), $\frac{3\pi}{8}\approx\frac{1}{6}\left(2+3\sqrt{3}\right)$ $\pi\approx\frac{4}{9}\left(2+3\sqrt{3}\right)$

3

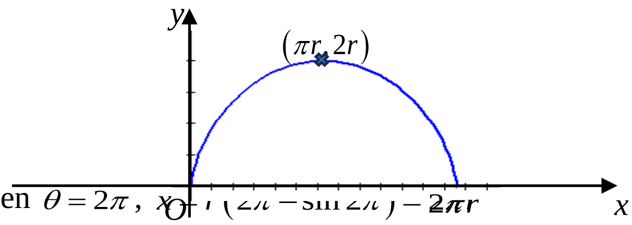


Let X, Q and R be the points as shown in the diagram.
Note that at any time, $OQ = PQ = r\theta$

x-coordinate of P = $OQ - PR = r\theta - r\sin\theta = r\left(\theta - \sin\theta\right)$

y-coordinate of P = $XQ - XR = r - r\cos\theta = r\left(1 - \cos\theta\right)$

(a)



When $\theta = 2\pi$, $x = r\left(2\pi - \sin 2\pi\right) = 2\pi r$

$y = r\left(1 - \cos\theta\right)$

When y is maximum,

$\cos\theta = -1$

$\theta = \pi$

$x = r\left(\pi - \sin\pi\right) = \pi r$ and $y = r\left(1 - \cos\pi\right) = 2r$

Hence, the coordinate of the maximum point is $\left(\pi r, 2r\right)$.

(b)

$$x = r(\theta - \sin\theta) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = r(1 - \cos\theta)$$

$$y = r(1 - \cos\theta) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} = r\sin\theta$$

Surface area of revolution $= 2\pi\int_0^{2\pi} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$

$$= 2\pi\int_0^{2\pi} r(1-\cos\theta)\sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\ \mathrm{d}\theta$$

$$= 2\pi r^2\int_0^{2\pi} (1-\cos\theta)\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\ \mathrm{d}\theta$$

$$= 2\pi r^2\int_0^{2\pi} (1-\cos\theta)\sqrt{2-2\cos\theta}\ \mathrm{d}\theta$$

$$= 2\sqrt{2}\pi r^2\int_0^{2\pi} (1-\cos\theta)^{\frac{3}{2}}\ \mathrm{d}\theta$$

$$= 2\sqrt{2}\pi r^2\int_0^{2\pi} \left(2\sin^2\frac{\theta}{2}\right)^{\frac{3}{2}}\ \mathrm{d}\theta$$

$$= 8\pi r^2\int_0^{2\pi} \sin^3\frac{\theta}{2}\ \mathrm{d}\theta \qquad \text{(Shown)}$$

$$\text{Exact Area} = 8\pi r^2\int_0^{2\pi} \sin^3\frac{\theta}{2}\ \mathrm{d}\theta$$

$$= 8\pi r^2\int_0^{2\pi} \sin\frac{\theta}{2}\left(1-\cos^2\frac{\theta}{2}\right)\mathrm{d}\theta$$

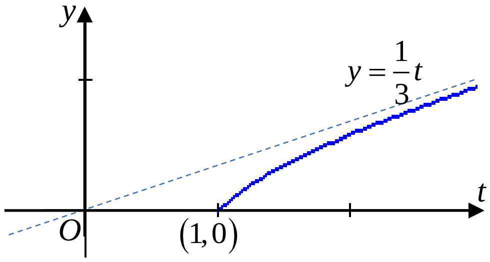
$$= 8\pi r^2\int_0^{2\pi} \left(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)\mathrm{d}\theta$$

$$= 8\pi r^2\left[-2\cos\frac{\theta}{2}+2\cdot\frac{\cos^3\frac{\theta}{2}}{3}\right]_0^{2\pi}$$

$$= 8\pi r^2\left[\left(2-\frac{2}{3}\right)-\left(-2+\frac{2}{3}\right)\right]$$

$$= \frac{64}{3}\pi r^2\ \text{units}^2$$

4	
(a)	$t\frac{\mathrm{d}y}{\mathrm{d}t}+ky=t$ $\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{k}{t}y=1$ Integrating Factor: $\mathrm{e}^{\int\frac{k}{t}\mathrm{d}t}=\mathrm{e}^{k\ln(t)}$ $=\mathrm{e}^{\ln(t)^{k}}$ $=t^{k}$ Multiplying the DE by the IF, $t^{k}\frac{\mathrm{d}y}{\mathrm{d}t}+t^{k}\frac{k}{t}y=t^{k}$ $\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{k}y\right]=t^{k}$ $t^{k}y=\int t^{k}\,\mathrm{d}t \qquad (\text{Since } k\neq -1)$ $t^{k}y=\frac{t^{k+1}}{k+1}+C$ $y=\frac{t}{k+1}+Ct^{-k}$ Given that $y=0$ when $t=1$, $0=\frac{1}{k+1}+C(1)^{-k}$ $C=-\frac{1}{k+1}$ Hence, $y=\frac{t}{k+1}-\frac{1}{k+1}t^{-k}=\frac{1}{k+1}\left(t-t^{-k}\right)$
(b)	When $k=2$, $t\frac{\mathrm{d}y}{\mathrm{d}t}+2y=t$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{t-2y}{t}=1-\frac{2y}{t}=\mathrm{f}(t,y)$ Step size, $h=0.1$ $t_{n+1}=t_{n}+0.1,\ \mathrm{f}(t_{n},y_{n})=1-\frac{2y_{n}}{t_{n}}$ $y_{n+1}=y_{n}+0.1\mathrm{f}(t_{n},y_{n})=y_{n}+0.1\left(1-\frac{2y_{n}}{t_{n}}\right)$ $t_{0}=1,\ y_{0}=0$ $t_{1}=1.1,\ y_{1}=y_{0}+0.1\left(1-\frac{2y_{0}}{t_{0}}\right)=0+0.1\left(1-\frac{2(0)}{1}\right)=0.1$

	$t_2 = 1.2$, $y_2 = y_1 + 0.1\left(1-\frac{2y_1}{t_1}\right) = 0.1+0.1\left(1-\frac{2\left(0.1\right)}{1.1}\right) = 0.181818$ Hence, $y \approx 0.182$ (3sf)
(c)	When $k = 2$, $y = \frac{1}{3}\left(t - t^{-2}\right)$  For $t \geq 1$, the solution curve is **increasing and concave downwards**, the line segments used in Euler's method lies above the curve, this results in an **overestimate** of the exact value of y when $t = 1.2$.

5	
(a)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\sqrt{3}\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$ Auxiliary equation is $m^2 - 2\sqrt{3}\,m + 4 = 0$ $m = \frac{2\sqrt{3} \pm \sqrt{\left(2\sqrt{3}\right)^2 - 4(1)(4)}}{2}$ $= \frac{2\sqrt{3} \pm \sqrt{-4}}{2}$ $= \frac{2\sqrt{3} \pm 2\mathrm{i}}{2}$ $= \sqrt{3} \pm \mathrm{i}$ General solution is $y = \mathrm{f}(x)$ $= \mathrm{e}^{x\sqrt{3}}\left(A\cos x + B\sin x\right)$ $\mathrm{f}(0) = 0 \Rightarrow \quad 0 = \mathrm{e}^0\left(A\cos 0 + 0\right)$ $0 = A$ $\therefore\ \mathrm{f}(x) = B\mathrm{e}^{x\sqrt{3}}\sin x$ $\mathrm{f}'(x) = B\mathrm{e}^{x\sqrt{3}}(\cos x) + B\sqrt{3}\,\mathrm{e}^{x\sqrt{3}}\sin x$ $\mathrm{f}'(0) = 1 \Rightarrow \quad 1 = B\mathrm{e}^0(\cos 0) + 0$ $1 = B$ $\therefore\ \mathrm{f}(x) = \underline{\underline{\mathrm{e}^{x\sqrt{3}}\sin x}}$
(b)	Let P_n be the statement "$\mathrm{f}^{(n)}(x) = 2^n\mathrm{e}^{x\sqrt{3}}\sin\left(x + \frac{1}{6}n\pi\right)$" for all $n \in \mathbb{Z}^+$ LHS of $P_1 = \mathrm{f}'(x)$ $= \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x\sqrt{3}}\sin x$ $= \mathrm{e}^{x\sqrt{3}}\cos x + \sqrt{3}\mathrm{e}^{x\sqrt{3}}\sin x$ $= \mathrm{e}^{x\sqrt{3}}\left(\sqrt{3}\sin x + \cos x\right)$ $= \mathrm{e}^{x\sqrt{3}}\left[2\sin\left(x + \tan^{-1}\frac{1}{\sqrt{3}}\right)\right]$ (*R*-formula) $= 2\mathrm{e}^{x\sqrt{3}}\sin\left(x + \frac{1}{6}\pi\right)$ $=$ RHS of P_1 $\therefore P_1$ is true. Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e., $\mathrm{f}^{(k)}(x) = 2^k\mathrm{e}^{x\sqrt{3}}\sin\left(x + \frac{1}{6}k\pi\right)$.

We want to prove that P_{k+1} is true, i.e., $\mathrm{f}^{(k+1)}(x)=2^{k+1}\mathrm{e}^{x\sqrt{3}}\sin\left[x+\frac{1}{6}(k+1)\pi\right]$.

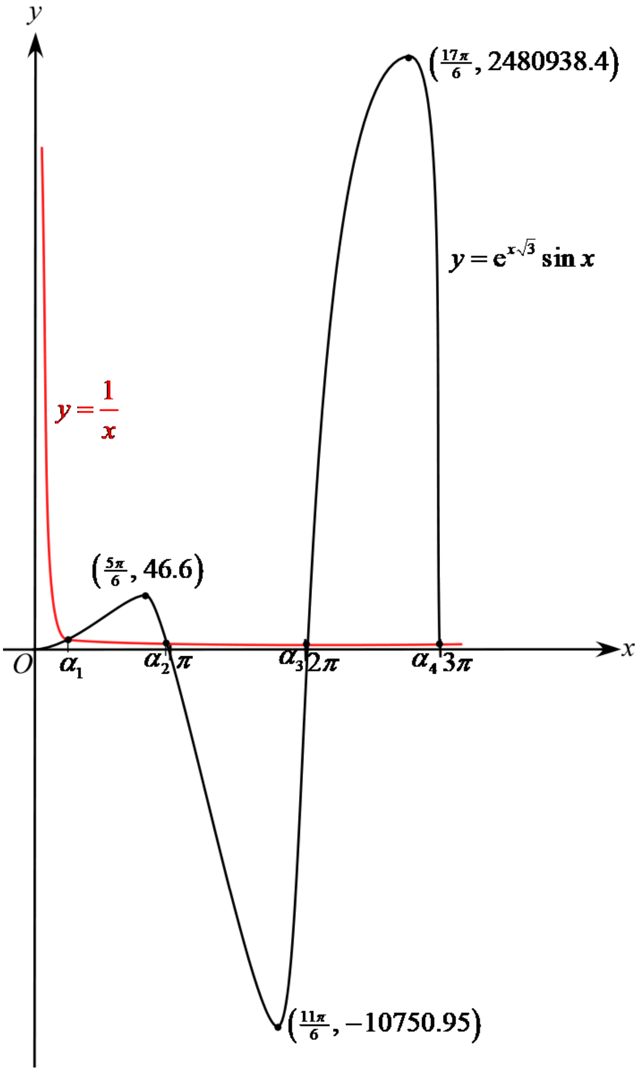
$$\begin{aligned}
\text{LHS of } P_{k+1} &= \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}^{(k)}(x)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[2^{k}\mathrm{e}^{x\sqrt{3}}\sin\left(x+\frac{1}{6}k\pi\right)\right]\\
&= 2^{k}\left[\sqrt{3}\mathrm{e}^{x\sqrt{3}}\sin\left(x+\frac{1}{6}k\pi\right)+\mathrm{e}^{x\sqrt{3}}\cos\left(x+\frac{1}{6}k\pi\right)\right]\\
&= 2^{k}\mathrm{e}^{x\sqrt{3}}\left[\sqrt{3}\sin\left(x+\frac{1}{6}k\pi\right)+\cos\left(x+\frac{1}{6}k\pi\right)\right]\\
&= 2^{k}\mathrm{e}^{x\sqrt{3}}\left[2\sin\left(x+\frac{1}{6}k\pi+\tan^{-1}\frac{1}{\sqrt{3}}\right)\right] \qquad (R\text{-formula})\\
&= 2^{k}\mathrm{e}^{x\sqrt{3}}\left[2\sin\left(x+\frac{1}{6}k\pi+\frac{1}{6}\pi\right)\right]\\
&= 2^{k+1}\mathrm{e}^{x\sqrt{3}}\sin\left[x+\frac{1}{6}(k+1)\pi\right]\\
&= \text{RHS of } P_{k+1}
\end{aligned}$$

$\therefore$ P_k is true $\Rightarrow P_{k+1}$ is true

Since (1) P_1 is true, and (2) P_k is true $\Rightarrow P_{k+1}$ is true,

$\therefore$ by mathematical induction, P_n is true for all $n\in\mathbb{Z}^{+}$.

(c) $y = \mathrm{e}^{x\sqrt{3}} \sin x,\ 0 \le x \le 3\pi$.



(d) **(i)** Given that the positive roots of the equation $\mathrm{f}(x) = \frac{1}{x}$ are denoted by $\alpha_1, \alpha_2, ..., \alpha_n, ...,$ in increasing order.

[Refer to the above graphs of $y = \mathrm{f}(x) = \mathrm{e}^{x\sqrt{3}} \sin x$ and $y = \frac{1}{x}$, and look for the intersection points at $x = \alpha_1, \alpha_2, \alpha_3, \alpha_4$.]

$$\mathrm{f}(x) = \frac{1}{x}$$

$$\mathrm{e}^{x\sqrt{3}} \sin x = \frac{1}{x}$$

$$\mathrm{e}^{x\sqrt{3}} \sin x - \frac{1}{x} = 0$$

Let $$\mathrm{g}(x) = \mathrm{e}^{x\sqrt{3}} \sin x - \frac{1}{x}$$

$$\mathrm{g}'(x) = \mathrm{f}'(x) + \frac{1}{x^2}$$

$$= 2e^{x\sqrt{3}}\sin\left(x+\frac{1}{6}\pi\right)+\frac{1}{x^2}$$

Let $\alpha_0 = 0.6$

$$\alpha_1 = \alpha_0 - \frac{g(x_0)}{g'(x_0)}$$

$$= 0.6 - \frac{g(0.6)}{g'(0.6)}$$

$$= 0.60894 \text{ (5 sf)}$$

$$= \underline{\underline{0.609}} \text{ (3 sf)}$$

(ii) $$e^{x\sqrt{3}}\sin x = \frac{1}{x}$$

When n is large, α_n is large ($\because$ $\alpha_1, \alpha_2, ..., \alpha_n, ...,$ are in increasing order), then $\frac{1}{\alpha_n} \approx 0$.

$$e^{\alpha_n\sqrt{3}}\sin\alpha_n = \frac{1}{\alpha_n} \approx 0$$

$$\sin\alpha_n \approx 0 \qquad \left(\because e^{\alpha_n\sqrt{3}} \text{ will be large}\right)$$

$$\alpha_n \approx k\pi \text{, where } k \in \mathbb{Z}^+ \ (\because \ \alpha_n \text{ is positive})$$

Hence a first approximation to α_n when n is large will be $\underline{\underline{(n-1)\pi}}$.

Section B: Probability and Statistics [50 marks]

6	
(a)	The bacteria occurs independently of each other. Or The average number of bacteria per ml of drugs remains constant.
(b)	Let A_k = the number of bacteria in k ml of drug A $A_1 \sim \text{Po}(0.5)$ $A_k \sim \text{Po}(0.5k)$ $\text{P}(A \le 3) < 0.02$ From GC, When $k = 18$, $\text{P}(A \le 3) = 0.0212 > 0.02$ When $k = 19$, $\text{P}(A \le 3) = 0.0149 < 0.02$ When $k = 20$, $\text{P}(A \le 3) = 0.0103 < 0.02$ Hence, the least integer $k = 19$.
(c)	Let B_n = the number of bacteria in n ml of drug B and $B_1 \sim \text{Po}(0.8)$ Let C = the number of bacteria in 2.5 ml of the drug mixture $C = A_{1.4} + B_{1.1} \sim \text{Po}(0.5 \times 1.4 + 0.8 \times 1.1) = \text{Po}(1.58)$ $\text{P}(C \ge 5) = 1 - \text{P}(C \le 4)$ $= 0.022596$ ≈ 0.0226

7

$\bar{x} = \frac{10.56}{8} = 1.32$ $\qquad$ $\bar{y} = \frac{12.39}{10} = 1.239$

${s_X}^2 = \frac{1}{7}\left(14.1775 - \frac{10.56^2}{8}\right) = 0.034043$ (5 sf) $\qquad$ ${s_Y}^2 = \frac{1}{9}\left(15.894 - \frac{12.39^2}{10}\right) = 0.06031$

$$\text{Pooled estimate of } \sigma^2, s^2 = \frac{7(0.034043) + 9(0.06031)}{8 + 10 - 2}$$

$$= 0.22095^2 \text{ (5 sf)}$$

Let μ_X kg and μ_Y kg be the population mean mass of the ducks on lake A and lake B respectively.

$H_0 : \mu_X - \mu_Y = 0$

$H_1 : \mu_X - \mu_Y > 0$

Assumption

The population variances are equal.

Under H_0, test statistic $T = \frac{\bar{X} - \bar{Y} - 0}{S\sqrt{\frac{1}{8} + \frac{1}{10}}} \sim t(16)$

$\alpha = 0.1$

$t = \frac{1.32 - 1.239}{0.22095\sqrt{\frac{1}{8} + \frac{1}{10}}}$ = 0.77286 (5 sf) (from GC)

p-value = $\text{P}(T \geq 0.77286)$ = 0.225 (3 sf) (from GC)

Since *p*-value = 0.225 > α = 0.1, we do not reject H_0 at the 10% level of significance and conclude there is insufficient evidence that the scientist’s claim is justified (or: that the ducks on lake A are heavier on average than the ducks on lake B).

8

(a) The expected frequency (E_{ij}) is calculated as follow:

Supplier	Poor	Fair	Good	Total
P	$\frac{180\times 30}{240}=22.5$	45	112.5	180
Q	7.5	15	37.5	60
Total	30	60	150	240

(b) Let the number of items rated Poor from Supplier *P* be *m*. Hence, we have

Supplier	Poor	Fair	Good	Total
P	*m*	60 – *m*	120	180
Q	30 – *m*	*m*	30	60
Total	30	60	150	240

$$\chi^2_{CALC}=\sum_i\sum_j\frac{\left(O_{ij}-E_{ij}\right)^2}{E_{ij}}$$

$$=\frac{(m-22.5)^2}{22.5}+\frac{(60-m-45)^2}{45}+\frac{(120-112.5)^2}{112.5}$$

$$+\frac{(30-m-7.5)^2}{7.5}+\frac{(m-15)^2}{15}+\frac{(30-37.5)^2}{37.5}$$

$$=\frac{(m-22.5)^2}{22.5}+\frac{(15-m)^2}{45}+\frac{(7.5)^2}{112.5}$$

$$+\frac{(22.5-m)^2}{7.5}+\frac{(m-15)^2}{15}+\frac{(7.5)^2}{37.5}$$

$$=\frac{4(m-22.5)^2}{22.5}+\frac{4(15-m)^2}{45}+\frac{4(7.5)^2}{112.5}$$

$$6.4=\frac{8(m-22.5)^2+4(15-m)^2}{45}+2$$

$$4.4(45)=8(m-22.5)^2+4(15-m)^2$$

$$12m^2-480m+4752=0$$

$$m^2-40m+396=0$$

$$m=18\ \text{ or }\ m=22$$

Since there are less than 20 items that are rate poor for both suppliers, *m* = 18.

Hence,

Supplier	Poor	Fair
P	18	42
Q	12	18

(c)

H_0 : Quality rating is independent of the supplier.

H_1 : Quality rating is dependent of the supplier.

Under H_0 ,

O_{ij}	Poor	Fair	Good	Total
P	18	42	120	180
Q	12	18	30	60
Total	30	60	150	240

E_{ij}	Poor	Fair	Good	Total
P	22.5	45	112.5	180
Q	7.5	15	37.5	60
Total	30	60	150	240

Degree of freedom = $(2-1)(3-1)=2$

α = 0.05

$\chi^2_{CALC} = 6.4$

p-value = 0.0408

Since *p*-value = 0.0408 < α , we reject H_0. There is **sufficient evidence** at the 5% level of significance to conclude that the quality rating is dependent of the supplier.

9	
(a)	$\mathrm{f}(x)=\begin{cases}200(x-0.1), & 0.1\le x\le 0.2,\\ 0, & \text{otherwise.}\end{cases}$ $\mathrm{P}(X\le x)=\int_{0.1}^{x}200(t-0.1)\,\mathrm{d}t$ $=200\left[\dfrac{(t-0.1)^2}{2}\right]_{0.1}^{x}$ $=100\left[(x-0.1)^2-(0.1-0.1)^2\right]$ $=100(x-0.1)^2$ Hence, $\mathrm{F}(x)=\begin{cases}0, & x<0.1,\\ 100(x-0.1)^2, & 0.1\le x\le 0.2,\\ 1, & x>0.2\end{cases}$ **Alternative:** $\mathrm{P}(X\le x)=\int_{0.1}^{x}200(t-0.1)\,\mathrm{d}t$ $=200\left[\dfrac{t^2}{2}-0.1t\right]_{0.1}^{x}$ $=200\left[\dfrac{x^2}{2}-0.1x-\dfrac{0.1^2}{2}+0.1^2\right]$ $=100x^2-20x+1$
(b)	$\mathrm{G}(y)=\mathrm{P}(Y\le y)$ $=\left[\mathrm{P}(X\le y)\right]\left[\mathrm{P}(X\le y)\right]$ $=\left[\mathrm{F}(y)\right]^2$ (Shown)
(c)	$\mathrm{G}(y)=\left[\mathrm{F}(y)\right]^2$ $=\left[100(y-0.1)^2\right]^2$ $=10000(y-0.1)^4$ $\mathrm{g}(y)=\dfrac{\mathrm{d}}{\mathrm{d}y}\left[10000(y-0.1)^4\right]$ $=10000(4)(y-0.1)^3$ $=40000(y-0.1)^3$ $\mathrm{E}(Y)=\int_{0.1}^{0.2}y\mathrm{g}(y)\,\mathrm{d}y$ $=\int_{0.1}^{0.2}40000y(y-0.1)^3\,\mathrm{d}y$ $=0.18$
(d)	For $0.14\le x\le 0.16$, $E_i=200\left[\mathrm{F}(0.16)-\mathrm{F}(0.14)\right]$ $=200\left[100(0.16-0.1)^2-100(0.14-0.1)^2\right]$ $=40$ For $0.18\le x\le 0.2$, $E_i=200\left[1-\mathrm{F}(0.18)\right]$ $=200\left[1-100(0.18-0.1)^2\right]$ $=72$

x	0.1 – 0.12	0.12 – 0.14	0.14 – 0.16	0.16 – 0.18	0.18 – 0.2
E_i	8	24	40	56	72

(e) H_0 : X has a probability density function as stated above.

H_1 : X does not have a probability density function as stated above.

Under H_0 ,

x	0.1 – 0.12	0.12 – 0.14	0.14 – 0.16	0.16 – 0.18	0.18 – 0.2
O_i	14	18	45	58	65
E_i	8	24	40	56	72

Degree of freedom = 5 – 1 = 4

α = 0.05

Using GC, $\chi^2_{CALC} = 7.38$

p-value = 0.117

Since p-value = 0.117 > α , we do not reject H_0 . There is insufficient evidence at the 5% level of significance to conclude that the greengrocer’s claim is not valid.

i.e. The greengrocer’s claim may be valid.

10

(a) Let X kg and Y kg be the pre-treatment grip and post-treatment grip respectively.
Let $D = Y_i - X_i$.

d	4	5.1	2.3	3.1	0	1.1	0.9	1.2

Using GC ((one-sample) t-test),

$$\bar{d} = 2.2125,\ s_D{}^2 = 1.7406^2$$

Let μ_X kg and μ_Y kg be the mean pre-treatment grip and mean post-treatment grip respectively.
Let $\mu_D = \mu_Y - \mu_X$.

Assumption

The population of differences, $D = Y_i - X_i$, has a **normal** distribution.

$\mathrm{H}_0:\ \mu_D = 0$

$\mathrm{H}_1:\ \mu_D > 0$

Under H_0, test statistic $T = \dfrac{\bar{D} - 0}{\frac{S_D}{\sqrt{8}}} \sim t(7)$

$\alpha = 0.01$

$t = \dfrac{2.2125}{\frac{1.7406}{\sqrt{8}}} = 3.5952$ (5 sf) (from GC)

p-value = $\mathrm{P}(T \geq 3.5952) = \underline{0.00440}$ (3 sf) (from GC)

Since p-value = 0.00440 $\leq$ α = 0.01, we reject $\underline{\underline{\mathrm{H}_0}}$ at the 1% level of significance and conclude there is sufficient evidence that the mean grip of people with the syndrome has increased after undergoing the treatment.

(b) $\mathrm{H}_0:\ \mu_D = w$

$\mathrm{H}_1:\ \mu_D > w$

Under H_0, test statistic $T = \dfrac{\bar{D} - w}{\frac{S_D}{\sqrt{8}}} \sim t(7)$

$\alpha = 0.1$

Evidence showing an increase in the mean grip by more than w means that H_0 is rejected.
At the 10% level of significance,

reject H_0 if $t \geq 1.4149$ (GC: invT: area = 0.9, df: 7)

$$\frac{2.2125 - w}{\frac{1.7406}{\sqrt{8}}} \geq 1.4149$$

$$2.2125 - w \geq 1.4149\left(\frac{1.7406}{\sqrt{8}}\right)$$

$$w \leq 2.2125 - 1.4149\left(\frac{1.7406}{\sqrt{8}}\right)$$

$$\therefore\ \underline{\underline{w \leq 1.34}} \quad \text{(3 sf)}$$

(c)

To perform sign test,
Let *m* be the median of the increase in grip after the treatment

d	4	5.1	2.3	3.1	0	1.1	0.9	1.2
$d-1$	+	+	+	+	–	+	–	+

Let K_+ be the number of '+'

$\mathrm{H_0}: m = 1$

$\mathrm{H_1}: m > 1$

Under $\mathrm{H_0}$, test statistic $K_+ \sim \mathrm{B}(8, 0.5)$

$\alpha = 0.01$

From the data, $k_+ = 6$

p-value = $\mathrm{P}(K_+ \geq 6) = 1 - \mathrm{P}(K_+ \leq 5) = 0.144$

Since *p*-value = 0.144 > α, we do not reject $\mathrm{H_0}$. There is insufficient evidence, at 1% level of significance to conclude that the average grip increases by more than 1 kg after the treatment.

(d)

$\mathrm{H_0}: m = 1$

$\mathrm{H_1}: m > 1$

Under $\mathrm{H_0}$, test statistic $K_+ \sim \mathrm{B}(20, 0.5)$

$\alpha = 0.01$

For $\mathrm{H_0}$ to be rejected,

p-value = $\mathrm{P}(K_+ \geq k_+) \leq 0.01$

$$1 - \mathrm{P}(K_+ \leq k_+ - 1) \leq 0.01$$

$$\mathrm{P}(K_+ \leq k_+ - 1) \geq 0.99$$

From GC,

$$\mathrm{P}(K_+ \leq 14) = 0.9793 < 0.99$$

$$\mathrm{P}(K_+ \leq 15) = 0.9941 > 0.99$$

$$\mathrm{P}(K_+ \leq 16) = 0.9987 > 0.99$$

Hence, $k_+ - 1 \geq 15 \Rightarrow k_+ \geq 16$

There should be at least **16** patients with their grip increases by more than 1 kg after the treatment