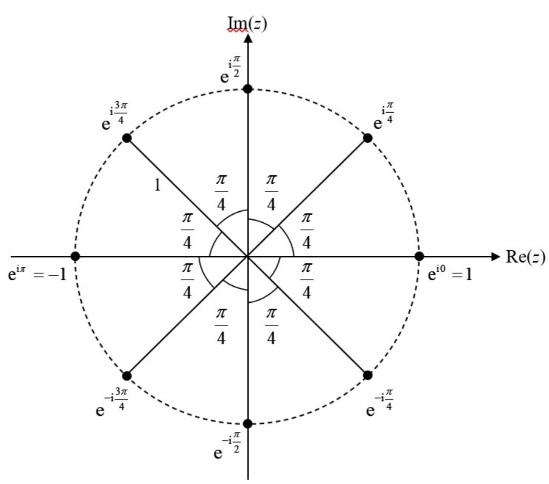
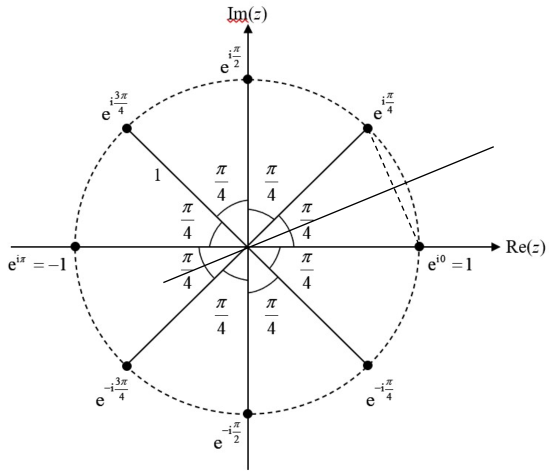
J1 2023 Further Math Year End Exam Solutions and Mark Scheme

1(i)	$p=125\left(\frac{4}{5}\right)^{n-1}$ $\ln p=\ln\left[125\left(\frac{4}{5}\right)^{n-1}\right]$ $=\ln\left[5^3\left(\frac{4}{5}\right)^{n-1}\right]$ $=\ln\left(\frac{2^{2n-2}}{5^{n-4}}\right)$ $=\ln(2^{2n-2})-\ln(5^{n-4})$ $=(2n-2)\ln 2-(n-4)\ln 5$ $A=2,\ B=-2,\ C=1$ and $D=-4$	
1(ii)	Area of banner $=130\times 5=650\ cm^2$ $S_\infty=\frac{125}{1-\frac{4}{5}}=625\ cm^2<650\ cm^2$ Therefore the boy will never finish colouring the banner.	
2(i)		

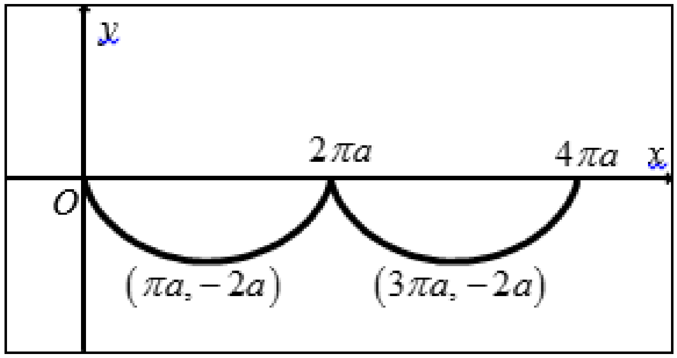
2(ii)	 Substituting $z = 0$ into $\lvert z - z_1 \rvert = \lvert z - z_2 \rvert$, LHS = $\lvert 0 - z_1 \rvert = \lvert z_1 \rvert = 1$ RHS = $\lvert 0 - z_2 \rvert = \lvert z_2 \rvert = 1$, LHS = RHS Since $z = 0$ satisfies the equation $\lvert z - z_1 \rvert = \lvert z - z_2 \rvert$, the locus passes through the origin. Equation of locus is $y = \left[\tan\left(\frac{\frac{\pi}{4}}{2}\right)\right]x$ $y = x\tan\left(\frac{\pi}{8}\right)$	

3(i)	$u_n = u_{n-1} + \frac{1}{2}u_{n-2}$ Characteristic equation: $m^2 = m + \frac{1}{2}$ $2m^2 - 2m - 1 = 0$ $m = \frac{2 \pm \sqrt{4-4(2)(-1)}}{4}$ $= \frac{2 \pm \sqrt{12}}{4}$ $= \frac{2 \pm 2\sqrt{3}}{4}$ $= \frac{1 \pm \sqrt{3}}{2}$ General solution: $u_n = c\left(\frac{1+\sqrt{3}}{2}\right)^n + d\left(\frac{1-\sqrt{3}}{2}\right)^n$, where c and d are constants	
3(ii)	When $n = 0, u_0 = c\left(\frac{1+\sqrt{3}}{2}\right)^0 + d\left(\frac{1-\sqrt{3}}{2}\right)^0 = 2$ $c + d = 2$ $d = 2 - c$ When $n = 1, u_0 = c\left(\frac{1+\sqrt{3}}{2}\right)^1 + d\left(\frac{1-\sqrt{3}}{2}\right)^1 = 1$ Sub $d = 2 - c$, $c\left(\frac{1+\sqrt{3}}{2}\right) + (2-c)\left(\frac{1-\sqrt{3}}{2}\right) = 1$ $c(1+\sqrt{3}) + (2-c)(1-\sqrt{3}) = 2$ $c + \sqrt{3}c + 2 - 2\sqrt{3} - c + \sqrt{3}c = 2$ $2\sqrt{3}c - 2\sqrt{3} = 0$ $c = 1$ $d = 1$ $u_n = \left(\frac{1+\sqrt{3}}{2}\right)^n + \left(\frac{1-\sqrt{3}}{2}\right)^n$	

3(iii)	$u_n = c\left(\frac{1+\sqrt{3}}{2}\right)^n + d\left(\frac{1-\sqrt{3}}{2}\right)^n$, where c and d are constants Since, as $n \to \infty, \left(\frac{1-\sqrt{3}}{2}\right)^n \to 0, \left(\frac{1+\sqrt{3}}{2}\right)^n \to \infty$ and $u_n \to 0 \Rightarrow c = 0$ and $u_0 = 2 \Rightarrow d = 2 - c$ $d = 2$ $\therefore u_n = 2\left(\frac{1-\sqrt{3}}{2}\right)^n$ $u_1 = 2\left(\frac{1-\sqrt{3}}{2}\right)^1 = 1-\sqrt{3}$	
4(i)	Since there are n rectangles between 0 and 1, each rectangle have a width of $\frac{1}{n}$ Area of 1st rectangle = $\frac{1}{n}\left(\frac{1}{n}+a\right)^2 = \frac{1}{n^3}(1+an)^2$ Area of 2nd rectangle = $\frac{1}{n}\left(\frac{2}{n}+a\right)^2 = \frac{1}{n^3}(2+an)^2$ Area of 3rd rectangle = $\frac{1}{n}\left(\frac{3}{n}+a\right)^2 = \frac{1}{n^3}(3+an)^2$ ⋮ Area of last (n^{th}) rectangle = $\frac{1}{n}\left(\frac{n}{n}+a\right)^2 = \frac{1}{n^3}(n+an)^2$ Total area $= \frac{1}{n^3}(1+an)^2 + \frac{1}{n^3}(2+an)^2 + \frac{1}{n^3}(3+an)^2 + \ldots + \frac{1}{n^3}(n+an)^2$ $= \frac{1}{n^3}\left[(1+an)^2 + (2+an)^2 + (3+an)^2 + \ldots + (n+an)^2\right]$ $= \frac{1}{n^3}\sum_{r=1}^{n}(r+an)^2$	

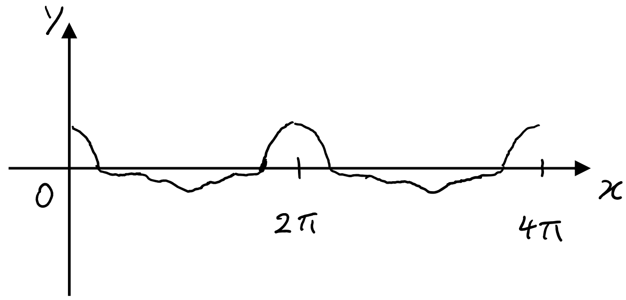
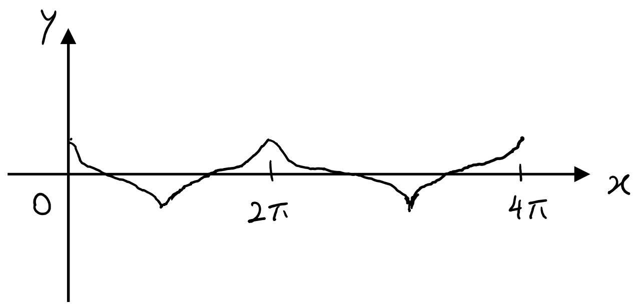
4(ii)	$\frac{1}{n^3}\sum_{r=1}^{n}\left(r+an\right)^2=\frac{1}{n^3}\sum_{r=1}^{n}\left(r^2+2anr+a^2n^2\right)$ $=\frac{1}{n^3}\left[\sum_{r=1}^{n}r^2+2an\sum_{r=1}^{n}r+\sum_{r=1}^{n}a^2n^2\right]$ $=\frac{1}{n^3}\left[\frac{1}{6}n(n+1)(2n+1)+2an\left(\frac{n}{2}(1+n)\right)+a^2n^3\right]$ $=\frac{1}{6n^2}(2n^2+3n+1)+\frac{a+an}{n}+a^2$ $=\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}+\frac{a}{n}+a+a^2$ Area $=\lim_{n\to\infty}\left[\frac{1}{n^3}\sum_{r=1}^{n}\left(r+an\right)^2\right]$ $=\lim_{n\to\infty}\left[\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}+\frac{a}{n}+a+a^2\right]$ $=\frac{1}{3}+0+0+0+a+a^2$ $=a^2+a+\frac{1}{3}$	
5	$z^3(1+2\text{i})+z^2(1-2\text{i})+4z+t=0,$ Given one root is $k+k\text{i}$, then $(k+k\text{i})^3(1+2\text{i})+(k+k\text{i})^2(1-2\text{i})+4(k+k\text{i})+t=0$ $k^3(1+\text{i})^3(1+2\text{i})+k^2(1+\text{i})^2(1-2\text{i})+4(k+k\text{i})+t=0$ $k^3(-2+2\text{i})(1+2\text{i})+k^2(2\text{i})(1-2\text{i})+4(k+k\text{i})+t=0,$ using GC $-6k^3-2k^3\text{i}+4k^2+2k^2\text{i}+4k+4k\text{i}+t=0$ $(-6k^3+4k^2+4k+t)+(-2k^3+2k^2+4k)\text{i}=0$ Comparing real and imaginary parts, $-6k^3+4k^2+4k+t=0$ $-2k^3+2k^2+4k=0 \Rightarrow k^2-k-2=0$ $(k+1)(k-2)=0$ $k=-1$ *or* $k=2$ (rej since k is negative) When $k=-1, t=-6$ then $z^3(1+2\text{i})+z^2(1-2\text{i})+4z-6=0$	

	Using long division, $z^3(1+2\mathrm{i})+z^2(1-2\mathrm{i})+4z-6$ $=[z-(-1-\mathrm{i})][z^2(1+2\mathrm{i})+z(2-5\mathrm{i})+(-3+3\mathrm{i})]$ Using quadratic formula, $z=\dfrac{-(2-5\mathrm{i})\pm\sqrt{(2-5\mathrm{i})^2-4(1+2\mathrm{i})(-3+3\mathrm{i})}}{2(1+2\mathrm{i})}$ $=\dfrac{(-2+5\mathrm{i})\pm(4-i)}{2(1+2\mathrm{i})}$ $=1$ or $\dfrac{3}{5}+\dfrac{9}{5}\mathrm{i}$	
6(a)	To find intersection, solve $\cos 3\theta-3\theta=0$. Using GC, $\theta=0.2463617$. Area $=\dfrac{1}{2}\int_{-\frac{\pi}{6}}^{0.2463617}(\cos 3\theta)^2\,\mathrm{d}\theta-\dfrac{1}{2}\int_{0}^{0.2463617}(3\theta)^2\,\mathrm{d}\theta$ $\approx 0.211549=0.212\text{ units}^2$ (3 s.f.)	
6(b)	Surface area $=\int_0^1 2\pi(2\sqrt{x+1})\sqrt{1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $=4\pi\int_0^1(\sqrt{x+1})\sqrt{1+\dfrac{1}{x+1}}\,\mathrm{d}x$ $=4\pi\int_0^1(x+2)^{\frac{1}{2}}\,\mathrm{d}x$ $=\dfrac{8\pi}{3}\left[(x+2)^{\frac{3}{2}}\right]_0^1$ $=\dfrac{8\pi}{3}\left[(3)^{\frac{3}{2}}-2^{\frac{3}{2}}\right]$ $=\dfrac{8\pi}{3}\left[3\sqrt{3}-2\sqrt{2}\right]$, where $a=3$ and $b=-2$.	

7(i)	$\frac{dy}{dt} = -a\sin t$ and $\frac{dx}{dt} = a(1-\cos t)$ $\Rightarrow \frac{dy}{dx} = \frac{-\sin t}{1-\cos t}$  	
7(ii)	Distance = arc length between $t=\pi$ to $t=3\pi$ $= \int_{\pi}^{3\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\ dt$ $= a\int_{\pi}^{3\pi} \sqrt{(\sin t)^2 + (1-\cos t)^2}\ dt$ $= a\int_{\pi}^{3\pi} \sqrt{2-2\cos t}\ dt$ $= 2a\int_{\pi}^{3\pi} \left\lvert \sin\frac{t}{2} \right\rvert dt$ $= 2a\left(\int_{\pi}^{2\pi} \sin\frac{t}{2}\ dt + \int_{2\pi}^{3\pi} -\sin\frac{t}{2}\ dt \right)$ $= 4a\left(\left[-\cos\frac{t}{2}\right]_{\pi}^{2\pi} + \left[\cos\frac{t}{2}\right]_{2\pi}^{3\pi} \right)$ $= 4a([1-0]+[0-(-1)])$ $= 8a$	

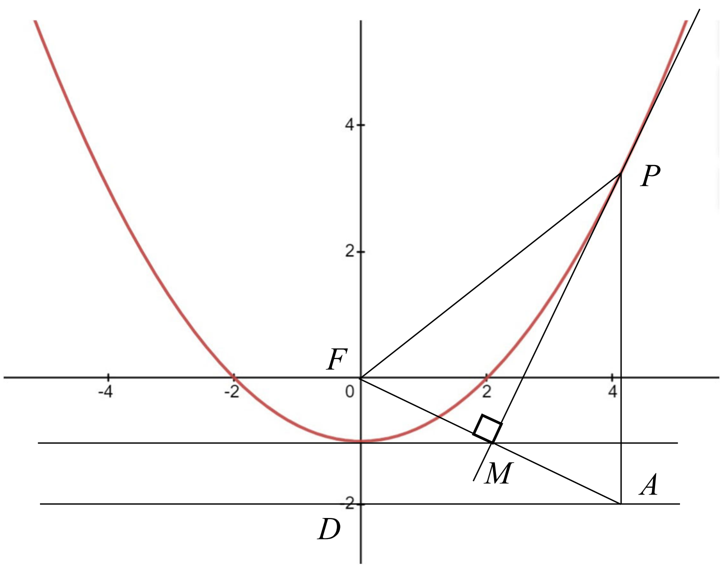
8(i)	$C_{n+1} = 0.6C_n + 10, \quad C_0 = 50, \quad n \geq 0$ $C_n = c(0.6^n) + d \quad$ where $d = \dfrac{b}{1-a}$ $d = \dfrac{10}{1-0.6} = 25$ Sub $n = 0, C_0 = c(0.6^0) + 25$ $50 = c + 25$ $c = 25$ $\therefore C_n = 25(0.6^n) + 25, \quad n \geq 0$	
8(ii)	As $n \to \infty, 0.6^n \to 0, C_n \to 25$ Eventually the number of bacteria will stabilise at 25 million and hence the antibiotic is NOT effective.	
8(iii)	$d' = 15$ $\dfrac{10}{1-a'} = 15$ $1-a' = \dfrac{2}{3}$ $a' = \dfrac{1}{3}$ $\dfrac{1}{3} \times 100\% = 33.333\%$ The antibiotic needs to eliminate $100\% - 33.333\% = 66.667\% \approx 67\%$ of the bacteria every month in order for the number of bacteria to be reduced to 15 million eventually.	
8(iv)	Assumption: The bacteria do not multiply on their own OR No bacteria will live forever unless eliminated by the antibiotic OR any logical/equivalent answer.	

9(i)	$y=\tan^{-1}(e^x)$ $\frac{dy}{dx}=\frac{e^x}{1+e^{2x}}=e^x(1+e^{2x})^{-1}$ $\frac{d^2y}{dx^2}=e^x(1+e^{2x})^{-1}-2e^{3x}(1+e^{2x})^{-2}$ $=e^x(1+e^{2x})^{-1}-2e^x\left[e^x(1+e^{2x})^{-1}\right]^2$ $\frac{d^2y}{dx^2}=\frac{dy}{dx}-2e^x\left(\frac{dy}{dx}\right)^2$ (shown) $\frac{d^3y}{dx^3}=\frac{d^2y}{dx^2}-2\left[e^x\left(\frac{dy}{dx}\right)^2+2e^x\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)\right]$ When $x=0$, $y=\frac{\pi}{4}$ $\frac{dy}{dx}=\frac{1}{2}$ $\frac{d^2y}{dx^2}=0$ $\frac{d^3y}{dx^3}=-\frac{1}{2}$ $y\approx\frac{\pi}{4}+\frac{x}{2}-\frac{x^3}{12}$	
9(ii)	$\int_0^{0.05}\tan^{-1}(e^x)\,dx=\int_0^{0.05}\frac{\pi}{4}+\frac{x}{2}-\frac{x^3}{12}\,dx$ $=\left[\frac{\pi}{4}x+\frac{x^2}{4}-\frac{x^4}{48}\right]_0^{0.05}$ $=0.03989$ (5 dp)	
9(iii)	Using GC, $\int_0^{0.05}\tan^{-1}(e^x)\,dx=0.03989$ (5 dp)	
9(iv)	The 2 answers are the same up to 5 decimal place. The approximation by the Maclaurin series is accurate as the interval [0, 0.05] is close to 0.	

10(a)	$y_1 = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + \frac{1}{4}\cos 4x + \frac{1}{5}\cos 5x$ 	
10(b)	$y_2 = \cos x + \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x + \frac{1}{9}\cos 9x$ 	

10(c)	$y_3 = \cos x + \frac{1}{9}\cos 3x + \frac{1}{81}\cos 5x + \ldots + \frac{1}{3^{2n-2}}\cos(2n-1)x + \ldots$ $= \text{Re}\left[\sum_{r=1}^{\infty}\frac{1}{3^{2r-2}}\text{e}^{\text{i}(2r-1)x}\right]$ $= \text{Re}\left[\text{e}^{\text{i}x} + \frac{1}{9}\text{e}^{\text{i}(3x)} + \frac{1}{81}\text{e}^{\text{i}(5x)} + \ldots\right]$ $= \text{Re}\left(\frac{\text{e}^{\text{i}x}}{1-\frac{1}{9}\text{e}^{\text{i}(2x)}}\right)$ $= \text{Re}\left(\frac{9\text{e}^{\text{i}x}}{9-\text{e}^{\text{i}(2x)}} \times \frac{9-\text{e}^{\text{i}(-2x)}}{9-\text{e}^{\text{i}(-2x)}}\right)$ $= \text{Re}\left(\frac{81\text{e}^{\text{i}x} - 9\text{e}^{\text{i}(-x)}}{81-9[\text{e}^{\text{i}(2x)} - \text{e}^{\text{i}(-2x)}]+1}\right)$ $= \text{Re}\left(\frac{81\text{e}^{\text{i}x} - 9\text{e}^{\text{i}(-x)}}{82-18\cos 2x}\right)$ $= \frac{81\cos x - 9\cos x}{82-18\cos 2x}$ $= \frac{72\cos x}{82-18\cos 2x}$ $= \frac{36\cos x}{41-9\cos 2x}$	

11(i)	$r=\dfrac{2}{3-\sin\theta}$ $3r-r\sin\theta=2$ $3\sqrt{x^2+y^2}=2+y$ $9(x^2+y^2)=(2+y)^2$ $9(x^2+y^2)=4+4y+y^2$ $9x^2+8y^2-4y=4$ $\dfrac{9}{4}x^2+2y^2-y=1$ $\dfrac{9}{4}x^2+2\left(y^2-\dfrac{y}{2}\right)=1$ $\dfrac{9}{4}x^2+2\left(y-\dfrac{1}{4}\right)^2-\dfrac{1}{8}=1$ $\dfrac{9}{4}x^2+2\left(y-\dfrac{1}{4}\right)^2=\dfrac{9}{8}$ $2x^2+\dfrac{16}{9}\left(y-\dfrac{1}{4}\right)^2=1$ $\dfrac{x^2}{\left(\dfrac{1}{\sqrt{2}}\right)^2}+\dfrac{\left(y-\dfrac{1}{4}\right)^2}{\left(\dfrac{3}{4}\right)^2}=1$ Therefore, the curve is an ellipse with centre $\left(0,\dfrac{1}{4}\right)$ and semi-minor $\dfrac{1}{\sqrt{2}}$ and semi-major $\dfrac{3}{4}$.	
11(ii)	$r=\dfrac{2}{3-\sin\theta}=\dfrac{2/3}{1-\frac{1}{3}\sin\theta}$ Eccentricity = $\dfrac{1}{3}$ and $c=ae=\left(\dfrac{3}{4}\right)\left(\dfrac{1}{3}\right)=\dfrac{1}{4}$	

	So the foci are $\frac{1}{4}$ units above and below the centre $\left(0, \frac{1}{4}\right)$ respectively. Hence foci are $\underline{\underline{\left(0, \frac{1}{2}\right) \text{ and } (0, 0)}}$.	
11(iii)	$a = 1$	
11(iv)	 From the definition of parabola, PF = PA. By reflection property of parabola, the tangent at *P* bisects ∠FPA and hence perpendicular to the base FA of isosceles triangle FPA, and intersects at its midpoint *M*. Since the tangent at the vertex is parallel to the directrix and bisects FD, then it also bisects FA at *M*. (shown)	

End