Qn	Solution
1	**Differential Equations**
(a)	$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1-\frac{P}{k}\right)-h$ $= -\frac{r}{k}\left(P-\frac{k}{2}\right)^2+\frac{rk}{4}-h$ Since there are two distinct and positive equilibrium population values, $0 < h < \frac{rk}{4}$.
(b)	$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{r}{k}\left(P-\frac{k}{2}\right)^2+\frac{rk}{4}-h = 0$ $P = \frac{k}{2} \pm \frac{k}{2}\sqrt{1-\frac{4h}{kr}}$ Let $\alpha = \frac{k}{2}-\frac{k}{2}\sqrt{1-\frac{4h}{kr}}, \beta = \frac{k}{2}+\frac{k}{2}\sqrt{1-\frac{4h}{kr}}$

Qn	Solution
2	**Recurrence Relations**
2(a)	$R_k = pR_{k+1} + \left(1-p\right)R_{k-1}$ $pR_{k+1} - R_k + \left(1-p\right)R_{k-1} = 0$ $A.E.: pm^2 - m + \left(1-p\right) = 0$ $m = \frac{1 \pm \sqrt{1-4p\left(1-p\right)}}{2p}$ $= \frac{1 \pm \sqrt{\left(1-2p\right)^2}}{2p}$ When $p = \frac{1}{2}, m = 1$ $G.S.: R_k = \left(Ak + B\right)$ $R_0 = 0 \Rightarrow B = 0$ $R_N = 1 \Rightarrow AN = 1 \Rightarrow A = \frac{1}{N}$ $\therefore R_k = \frac{k}{N}$ When $p \neq \frac{1}{2}, m = \frac{1-p}{p}$ or 1 G.S.: $R_k = C\left(\frac{1-p}{p}\right)^k + D$ $R_0 = 0 \ \Rightarrow \ C + D = 0 \qquad ---(1)$ $R_N = 1 \ \Rightarrow \ C\left(\frac{1-p}{p}\right)^N + D = 1 \quad ---(2)$ Solving (1) and (2): $C = \frac{1}{\left(\frac{1-p}{p}\right)^N - 1}, D = -\frac{1}{\left(\frac{1-p}{p}\right)^N - 1}$ $\therefore R_k = \left(\frac{1}{\left(\frac{1-p}{p}\right)^N - 1}\right)\left(\frac{1-p}{p}\right)^k - \left(\frac{1}{\left(\frac{1-p}{p}\right)^N - 1}\right) = \frac{\left(\frac{1-p}{p}\right)^k - 1}{\left(\frac{1-p}{p}\right)^N - 1}$
(b)	In a fair game i.e. $p = \frac{1}{2},\ R_k = \frac{k}{N} \to 0$ since *k* is finite but *N* is infinite. Adam is likely to lose.

	If $p < \frac{1}{2}$, $R_k = \frac{\left(\frac{1-p}{p}\right)^k - 1}{\left(\frac{1-p}{p}\right)^N - 1} \to 0$ since $\frac{1-p}{p} > 1 \Rightarrow \left(\frac{1-p}{p}\right)^N \to \infty$ Adam is likely to lose. If $p > \frac{1}{2}, 0 < \frac{1-p}{p} < 1 \Rightarrow \left(\frac{1-p}{p}\right)^N \to 0$ $\therefore R_k = \frac{\left(\frac{1-p}{p}\right)^k - 1}{\left(\frac{1-p}{p}\right)^N - 1} \to 1 - \left(\frac{1-p}{p}\right)^k$ which increases as *k* increases. Adam has a chance to win if *k* is large enough.

Qn	Solution
3	**Complex Numbers**
(a)	$\lvert w-1\rvert = 2\lvert w-3\rvert$ $\lvert x+\mathrm{i}y-1\rvert = 2\lvert x+\mathrm{i}y-3\rvert$ $(x-1)^2+y^2=4\left((x-3)^2+y^2\right)$ $3x^2-22x+35+3y^2=0$ $3\left(x-\frac{11}{3}\right)^2+3y^2=\frac{16}{3}$ $\left(x-\frac{11}{3}\right)^2+y^2=\left(\frac{4}{3}\right)^2$ $\lvert w-1\rvert = 2\lvert w-3\rvert$ describes a circle centred at $\left(\frac{11}{3},0\right)$ with radius $\frac{4}{3}$ units.
(b)(i)	Point on line: $\frac{2+3\mathrm{i}+(-2+5\mathrm{i})}{2}=4\mathrm{i}$ $2+3\mathrm{i}-(-2+5\mathrm{i})=4-2\mathrm{i}$ Direction of line: $\mathrm{i}(4-2\mathrm{i})=2+4\mathrm{i}$ To find the closest point of L_1 to L_2, $5+\mathrm{i}+\lambda(4-2\mathrm{i})=4\mathrm{i}+\mu(2+4\mathrm{i}),\quad \lambda,\mu\in\mathbb{R}$ Comparing real and imaginary components, $5+4\lambda=2\mu \qquad -----(1)$ $1-2\lambda=4+4\mu \quad -----(2)$ Solving, $\lambda=-\frac{13}{10},\ \mu=-\frac{1}{10}$. $z_1=-\frac{1}{5}+\frac{18}{5}\mathrm{i}$
(b)(ii)	Distance between point and centre of circle $=\sqrt{\left(5-(-2)\right)^2+(1-5)^2}$ $=\sqrt{65}$ $\theta=-\tan^{-1}\left(\frac{4}{7}\right)\pm\sin^{-1}\left(\frac{3}{\sqrt{65}}\right)$ $\theta=-0.138\text{ rad}\ \text{ or }\ -0.900\text{ rad}\ \ (3\text{ s.f.})$

Qn	Solution
4	**Definite Integral**
(a)(i)	$I_n = \int_0^{\frac{\pi}{6}} \cos x \cos^{n-1} x \ \mathrm{d}x$ $= \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin x \left(n-1\right)\left(-\sin x\right)\cos^{n-2} x \ \mathrm{d}x$ $= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^{n-1} + \left(n-1\right)\int_0^{\frac{\pi}{6}} \left(1-\cos^2 x\right)\cos^{n-2} x \ \mathrm{d}x$ $nI_n = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^{n-1} + \left(n-1\right) I_{n-2}$ $I_n = \frac{1}{2n}\left(\frac{\sqrt{3}}{2}\right)^{n-1} + \frac{n-1}{n} I_{n-2}$
(ii)	$r = 4\cos^3\left(\frac{\theta}{3}\right)$, for $0 \le \theta < 3\pi$ Required area $= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \ \mathrm{d}\theta$ $= 8\int_0^{\frac{\pi}{2}} \cos^6\left(\frac{\theta}{3}\right) \mathrm{d}\theta$ $= 24\int_0^{\frac{\pi}{6}} \cos^6 x \ \mathrm{d}x \qquad \left(\text{substituting } x = \frac{\theta}{3}\right)$ $= 24\left(\frac{1}{12}\left(\frac{\sqrt{3}}{2}\right)^5 + \frac{5}{6}\int_0^{\frac{\pi}{6}} \cos^4 x \ \mathrm{d}x\right)$ $= \frac{9\sqrt{3}}{16} + 20\left(\frac{1}{8}\left(\frac{\sqrt{3}}{2}\right)^3 + \frac{3}{4}\int_0^{\frac{\pi}{6}} \cos^2 x \ \mathrm{d}x\right)$ $= \frac{9\sqrt{3}}{16} + \frac{15\sqrt{3}}{16} + 15\left(\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 \ \mathrm{d}x\right)$ $= \frac{27\sqrt{3}}{8} + \frac{5\pi}{4}$

(b)	Consider $x^2+y^2=a^2$ $2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y}$ Let the thickness of each slice of cake be d. $S_x=\pi\int_{x_1}^{x_1+d} y\sqrt{1+\left(-\frac{x}{y}\right)^2}\,\mathrm{d}x$ $=\pi\int_{x_1}^{x_1+d}\sqrt{y^2+x^2}\,\mathrm{d}x$ $=\pi\int_{x_1}^{x_1+d} a\ \mathrm{d}x$ $=\pi\left[ax\right]_{x_1}^{x_1+d}$ $=a\pi d\ \ \left(\text{independent of } x_1 \text{ - shown}\right)$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>5</td><td>Differential Equations</td></tr>
<tr><td>(a)</td><td>

$$\left(1+x^4\right)\frac{\mathrm{d}y}{\mathrm{d}x}+8x^3y=x\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x-8x^3y}{1+x^4}$$

$$y_{n+1}=y_n+\left(0.1\right)\left[\frac{x_n-8{x_n}^3y_n}{1+{x_n}^4}\right]$$

Given that $x_0=1,\ y_0=\frac{1}{3}$ and $h=0.1,$

$$y_1=\frac{1}{3}+\left(0.1\right)\left[\frac{1-8(1)\left(\frac{1}{3}\right)}{1+\left(1\right)^4}\right]=0.25$$

When $x_1=1.1,$

When $x_2=1.2,$

$$y_2=0.25+\left(0.1\right)\left[\frac{1.1-8(1.1)^3\left(0.25\right)}{1+\left(1.1\right)^4}\right]$$

$$=0.18661\quad\left(\text{to 5 s.f}\right)$$

$$=0.187\quad\left(\text{to 3 s.f}\right)$$

</td></tr>
<tr><td>(b)</td><td>

n	x_n	$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\rvert_{x=x_n}$
0	1	–0.83333
1	1.1	–0.63390
2	1.2	–0.44889

Since the estimated value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ **is increasing** over [1, 1.2], we expect that the value is an **under-estimate** of the actual value of y when $x=1.2$.

</td></tr>
<tr><td>(c)</td><td>

$$\left(1+x^4\right)\frac{\mathrm{d}y}{\mathrm{d}x}+8x^3y=x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{8x^3}{1+x^4}\right)y=\frac{x}{1+x^4}$$

Integrating factor: $u=\mathrm{e}^{\int\frac{8x^3}{1+x^4}\,\mathrm{d}x}=\mathrm{e}^{2\int\frac{4x^3}{1+x^4}\,\mathrm{d}x}=\mathrm{e}^{2\ln\left(1+x^4\right)}=\left(1+x^4\right)^2$

</td></tr>
</table>

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{8x^3}{1+x^4}\right)y=\frac{x}{1+x^4}$$

$$\left(1+x^4\right)^2\frac{\mathrm{d}y}{\mathrm{d}x}+8x^3\left(1+x^4\right)y=x\left(1+x^4\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\left(1+x^4\right)^2\right)=x+x^5$$

$$y\left(1+x^4\right)^2=\int x+x^5\ \mathrm{d}x$$

$$y\left(1+x^4\right)^2=\frac{x^2}{2}+\frac{x^6}{6}+C$$

When $x=1,\quad y=\frac{1}{3}$

$$\left(\frac{1}{3}\right)(1+1)^2=\frac{1}{2}+\frac{1}{6}+C$$

$$C=\frac{2}{3}$$

Particular solution:

$$y\left(1+x^4\right)^2=\frac{x^2}{2}+\frac{x^6}{6}+\frac{2}{3}$$

$$y=\frac{3x^2+x^6+4}{6\left(1+x^4\right)^2}$$

When $x=1.2$,

$$y=\frac{3(1.2)^2+(1.2)^6+4}{6\left(1+(1.2)^4\right)^2}=0.19946$$

(to 5 s.f)

Percentage error

$$=\left|\frac{0.19946-0.18661}{0.19946}\right|\times 100\%=6.44\%\ (3\text{ s.f})$$

Since the percentage error is only 6.44%, the estimate is close to the actual value.

Qn	Solution
6	**Confidence Interval**
(a)	Let X be the height of a randomly chosen student (in cm). Since $n=80$ is sufficiently large, by Central Limit Theorem, $\overline{X}$ follows an approximate normal distribution. 95% Confidence Interval: $\left(166-1.9600\sqrt{\frac{5.2064}{80}},166+1.9600\sqrt{\frac{5.2064}{80}}\right)$ (165.5,166.5) (1 d.p.) Therefore, a 95% confidence for the population mean height of students in the college is (165.5,166.5). In the long run, if the process of obtaining samples is repeated an infinite number of times, the confidence intervals constructed will contain the actual population mean height of the students for 95% of the samples taken.
(b)	$\hat{p}=\frac{0.598+0.801}{2}=0.6995$ $0.801=0.6995+z\sqrt{\frac{0.6995\left(1-0.6995\right)}{80}}$ $z=1.9801$ $\mathrm{P}\left(-1.9801<Z<1.9801\right)=0.952$ $k=95.2 \quad \left(3\,\text{s.f.}\right)$ **<u>Alternative method</u>** $P_s \sim \mathrm{N}\left(\hat{p},\frac{\hat{p}\left(1-\hat{p}\right)}{n}\right)$ approximately $\mathrm{P}\left(0.598<P_s<0.801\right)=0.95231$ $k=95.2 \quad \left(3\,\text{s.f.}\right)$

Qn	**Solution**
7	**Geometric / Exponential Distribution + Continuous RV**
(a)(i)	Let X be the number of customer who checkouts in one hour. $X \sim \text{Po}(10)$ $\text{P}(X=15)=\text{e}^{-10}\dfrac{10^{15}}{15!}=0.0347$
(ii)	A customer spends 0.1 hour or 6 minutes at checkout counter. Let T be the time, in hours, customer spends at checkout counter. $T \sim \text{Exp}(10)$ Customer spends 9 minutes or 0.15 hour at checkout counter. $\text{P}(T<0.15)$ $=\int_0^{0.15} 10\text{e}^{-10t}\,\text{d}t$ $=0.77687$ $=0.777$ (to 3 sig fig)
(iii)	$\text{P}(T>0.15 \mid T>0.1)$ by memoryless property $=\text{P}(T>0.05)$ $=1-\int_0^{0.05} 10\,\text{e}^{-10t}\,\text{d}t$ $=0.60653$ $=0.607$ (to 3 sig fig)
(b)	$\text{P}(Y=n) \;=\text{P}(\lfloor X \rfloor+1=n)$ $=\text{P}(n-1\leq X<n)$ $=\int_{n-1}^{n}\dfrac{1}{\lambda}\text{e}^{-\frac{1}{\lambda}t}\,\text{d}t$ $=\left[-\text{e}^{-\frac{1}{\lambda}t}\right]_{n-1}^{n}$ $=-\text{e}^{-\frac{1}{\lambda}n}+\text{e}^{-\frac{1}{\lambda}(n-1)}$ $=\text{e}^{-\left(\frac{n-1}{\lambda}\right)}(1-\text{e}^{-\frac{1}{\lambda}})=\left(\text{e}^{-\frac{1}{\lambda}}\right)^{n-1}(1-\text{e}^{-\frac{1}{\lambda}})$ This is the pdf of a geometric random variable with parameter $1-\text{e}^{-\frac{1}{\lambda}}$. (shown)

Qn	Solution
8	**Hypothesis Testing (Chi-Squared)**

(a)

Let x be the number of survey respondents who are aged above 50 years old and are not financially literate.

H_0 : Age and financial literacy are independent.

H_1: Age and financial literacy are not independent.

The contingency table is as follows.

	<25	25 to 50	>50	Total
FL	70	210	150	430
Not FL	130	$440-x$	x	570
Total	200	$650-x$	$150+x$	1000

Using the formula $E_{ij}=\dfrac{O_i\times O_j}{n}$, the expected frequency table is shown below.

	<25	25 to 50	>50	Total
FL	86	$\dfrac{43(650-x)}{100}$	$\dfrac{43(150+x)}{100}$	430
Not FL	114	$\dfrac{57(650-x)}{100}$	$\dfrac{57(150+x)}{100}$	570
Total	200	$650-x$	$150+x$	1000

Degrees of freedom = (3−1)(3−1) = 2

Test Statistic: $\sum_i\sum_j \dfrac{\left(O_{ij}-E_{ij}\right)^2}{E_{ij}} \sim \chi_2^2$

Contributions in each cell

	<25	25 to 50	>50
FL	$\dfrac{128}{43}$	$\dfrac{(43x-6950)^2}{4300(650-x)}$	$\dfrac{(8550-43x)^2}{4300(150+x)}$
Not FL	$\dfrac{128}{57}$	$\dfrac{(6950-43x)^2}{5700(650-x)}$	$\dfrac{(43x-8550)^2}{5700(150+x)}$

Using MF26, at the 2.5% significance level , reject H_0 if

χ^2 cal ≥ 7.378

Since H_0 is not rejected,

$$\frac{12800}{2451}+\frac{\left(43x-6950\right)^2}{2451\left(650-x\right)}+\frac{\left(8550-43x\right)^2}{2451\left(150+x\right)}<7.378$$

Using GC,

$169.28 < x < 197.55$

Therefore, the minimum number of survey respondents who are aged above 50 years old and are not financially literate is 170.

(b) Contributions in each cell

	<25	25 to 50	>50
FL	$\frac{128}{43}\approx 2.98$	$\frac{27}{430}\approx 0.0628$	$\frac{961}{860}\approx 1.12$
Not FL	$\frac{128}{57}\approx 2.25$	$\frac{27}{570}\approx 0.0474$	$\frac{961}{1140}\approx 0.843$

The cells belonging to those under 25 years old are the highest contributors to the test statistics at 2.98 and 2.25. There are fewer financially literate respondents than expected, which might mean that there are more Singaporeans under 25 years old than expected who are not financial literate and may not have the knowledge to manage their finances well. Hence, this may be a potential concern.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>9</td><td>Non Parametric Test</td></tr>
<tr><td>(a)</td><td>Since $n=10$,
Sum of the ranks $1+2+3+...+10=\frac{10\times 11}{2}=55$.</td></tr>
<tr><td>(b)</td><td>Let <i>m</i> be the population median of the post-test scores minus pre-test scores
$\text{H}_0: m=0$
$\text{H}_1: m>0$
<table>
<tr><td>Student</td><td><i>A</i></td><td><i>B</i></td><td><i>C</i></td><td><i>D</i></td><td><i>E</i></td><td><i>F</i></td><td><i>G</i></td><td><i>H</i></td><td><i>I</i></td><td><i>J</i></td></tr>
<tr><td>Sign</td><td>+</td><td>+</td><td>+</td><td>+</td><td>–</td><td>–</td><td>+</td><td>+</td><td>–</td><td>+</td></tr>
<tr><td>Abs Diff</td><td>2</td><td>10</td><td>9</td><td>4</td><td>7</td><td>8</td><td>6</td><td>5</td><td>1</td><td>12</td></tr>
<tr><td>Rank</td><td>2</td><td>9</td><td>8</td><td>3</td><td>6</td><td>7</td><td>5</td><td>4</td><td>1</td><td>10</td></tr>
</table>
$P=2+9+8+3+5+4+10=41$
$Q=6+7+1=14$
$T=\min\left(P,Q\right)=14$

From MF26, for $n=10$, 1-tailed test at 5% significance level, $c=10$
To reject H_0, $T\le 10$.
Since $T=14>10$, we do not reject H_0 and conclude that there is insufficient evidence at 5% significance level that the innovative teaching strategy can improve the Mathematics test scores of students.</td></tr>
<tr><td>(c)</td><td>The underlying population of differences is symmetrically distributed.</td></tr>
<tr><td>(d)</td><td>Using a sign test,
Let $S_-=$ number of $'-'$.
Under H_0, $S_-\sim \text{B}\left(10,0.5\right)$
From the data, $S_-=3$
Using GC, <i>p</i>-value = $\text{P}\left(S\le 3\right)=0.172>0.05$ (3 s.f)
We do not reject H_0 and conclude that there is insufficient evidence at 5% significance level that the innovative teaching strategy can improve the Mathematics test scores of students.
This is the same conclusion as part (b) and hence the researcher’s claim is incorrect.</td></tr>
<tr><td>(e)</td><td>Paired-sample <i>t</i>-test can be used. However, this will only be valid if the difference in the test scores follows a normal distribution.</td></tr>
</table>

Qn	Solution
10	**Poisson Distribution**
(a)	$\sum_{r=1}^{\infty}\mathrm{P}(Y=r)=1$ $\sum_{r=1}^{\infty}k\mathrm{P}(X=r)=1$ $k\left(\sum_{r=0}^{\infty}\mathrm{P}(X=r)-\mathrm{P}(X=0)\right)=1$ $k\left(1-\frac{\lambda^{0}\mathrm{e}^{-\lambda}}{0!}\right)=1 \qquad \left(\text{since } X\sim\mathrm{Po}(\lambda)\right)$ $k=\frac{1}{1-\mathrm{e}^{-\lambda}}$ $X\sim\mathrm{Po}(6.5)$ $\mathrm{P}(Y>7)=\frac{1}{1-\mathrm{e}^{-\lambda}}\mathrm{P}(X>7)$ $=\frac{1}{1-\mathrm{e}^{-6.5}}\left(1-\mathrm{P}(X\le 7)\right)$ $=0.328$
(b)	$\mathrm{E}(Y)=k\,\mathrm{E}(X)=\frac{\lambda}{1-\mathrm{e}^{-\lambda}}$ $\frac{\lambda}{1-\mathrm{e}^{-\lambda}}=4.5$ $\lambda=4.45$
(c)	$\mathrm{E}(Y)=\frac{\lambda}{1-\mathrm{e}^{-\lambda}}\to\lambda$ for large values of λ. The claim is valid for large values of λ
(d)	$\mathrm{E}\left[Y(Y-1)\right]=\sum_{y=1}^{\infty}\left(y(y-1)\right)\mathrm{P}(Y=y)$ $=\sum_{y=2}^{\infty}\left(y(y-1)\right)\mathrm{P}(Y=y)$ $=\sum_{y=2}^{\infty}\left(y(y-1)\right)\left(\frac{1}{1-\mathrm{e}^{-\lambda}}\frac{\lambda^{y}\mathrm{e}^{-\lambda}}{y!}\right)$ $=\frac{\lambda^{2}}{1-\mathrm{e}^{-\lambda}}\sum_{y=2}^{\infty}\frac{\lambda^{y-2}\mathrm{e}^{-\lambda}}{(y-2)!}$ $=\frac{\lambda^{2}}{1-\mathrm{e}^{-\lambda}}\sum_{r=0}^{\infty}\frac{\lambda^{r}\mathrm{e}^{-\lambda}}{r!}$ $=\frac{\lambda^{2}}{1-\mathrm{e}^{-\lambda}}$ $\mathrm{E}\left(Y^{2}-Y\right)=\mathrm{E}\left(Y^{2}\right)-\mathrm{E}(Y)=\frac{\lambda^{2}}{1-\mathrm{e}^{-\lambda}}$

	$\mathrm{E}\left(Y^2\right)=\dfrac{\lambda^2+\lambda}{1-\mathrm{e}^{-\lambda}}$ $\mathrm{Var}\left(Y\right)=\mathrm{E}\left(Y^2\right)-\left[\mathrm{E}\left(Y\right)\right]^2$ $\qquad\qquad=\dfrac{\lambda^2+\lambda}{1-\mathrm{e}^{-\lambda}}-\dfrac{\lambda^2}{\left(1-\mathrm{e}^{-\lambda}\right)^2}$