Suggested Solution for 2023 NYJC JC1 FM EOY

1	Suggested Solution
(a)	$\mathbf{A}^2 = \mathbf{A}$ $\left\|\mathbf{A}^2\right\| = \left\|\mathbf{A}\right\|$ $\left\|\mathbf{A}^2\right\| - \left\|\mathbf{A}\right\| = 0$ $\left\|\mathbf{A}\right\|\left(\left\|\mathbf{A}\right\| - 1\right) = 0$ $\left\|\mathbf{A}\right\| = 0$ or 1
(b)	$\mathbf{A}^2 = \mathbf{A}$ $\mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}\mathbf{A}$ since the matrix is invertible. $\mathbf{A} = \mathbf{I}$
(c)	$\mathbf{A}$ is non-invertible: $\left\|\mathbf{A}\right\| = wz - xy = 0$ -- (1) $\mathbf{A}^2 = \mathbf{A}$ $\begin{pmatrix} w^2 + xy & wx + xz \\ wy + zy & xy + z^2 \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ $w^2 + xy = w$ $w^2 + wz = w$ (using (1)) $w + z = 1$ Note that solving $wy + zy = y$ or $wx + xz = x$ is also possible. As qn did not mention y or x is non-zero, need to at least mention that if x or y is zero, then A will be the identity matrix whose determinant is not zero. Note: $x \neq 0$ and $y \neq 0$. This is because if $x = 0$ or $y = 0$, det (A) = 0, means $wz = 0$, which is a contradiction.

2	**Suggested Solution**
	Let $\mathrm{P}(n)$ be the proposition that $\dfrac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(\mathrm{f}(x)\right)=(-4)^n\sin(2x)$, for $n\in\mathbb{Z}^+$. When $n=1$, LHS $=\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\sin(2x)\right)$ $=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(2\cos(2x)\right)$ $=2\left(-2\sin(2x)\right)=-4\sin(2x)=\text{RHS}$ $\mathrm{P}(1)$ is true. Assume $\mathrm{P}(k)$ is true for some $k\in\mathbb{Z}^+$. i.e. $\dfrac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}\left(\mathrm{f}(x)\right)=(-4)^k\sin(2x)$. To show $\mathrm{P}(k+1)$ is true. i.e. $\dfrac{\mathrm{d}^{2(k+1)}}{\mathrm{d}x^{2(k+1)}}\left(\mathrm{f}(x)\right)=(-4)^{k+1}\sin(2x)$. LHS $=\dfrac{\mathrm{d}^{2k+2}}{\mathrm{d}x^{2k+2}}\left(\sin(2x)\right)$ $=\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left((-4)^k\sin(2x)\right)$ By inductive hypothesis $=\dfrac{\mathrm{d}}{\mathrm{d}x}\left((-4)^k\,2\cos(2x)\right)$ $=(-4)^k\,2(-2)\sin(2x)$ $=(-4)^{k+1}\sin(2x)=\text{RHS}$ $\therefore \mathrm{P}(k)$ true $\Rightarrow$ $\mathrm{P}(k+1)$ true and since $\mathrm{P}(1)$ is true, by mathematical induction $\dfrac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(\mathrm{f}(x)\right)=(-4)^n\sin(2x)$, for $n\in\mathbb{Z}^+$. Since Maclaurin's expansion of $\mathrm{f}(x)$ is given by $\mathrm{f}(x)=\mathrm{f}(0)+x\mathrm{f}'(0)+\dfrac{x^2}{2!}\mathrm{f}''(0)+\ldots+\dfrac{x^n}{n!}\mathrm{f}^{(n)}(0)+\ldots$ and from the result of induction $\dfrac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(\mathrm{f}(x)\right)=(-4)^n\sin(2x)$, for $n\in\mathbb{Z}^+$ and $\sin(0)=0$ $\mathrm{f}(0)$=0 and all even derivatives are zero $\Rightarrow$ coefficient of even powers of x is equal to 0 $\dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}x^{2n+1}}\left(\mathrm{f}(x)\right)=2(-4)^n\cos(2x)$ When $x=0$, $\dfrac{\mathrm{d}^{2n+1}}{\mathrm{d}x^{2n+1}}\left(\mathrm{f}(x)\right)=2(-4)^n\cos(0)=2(-4)^n\neq 0$ Therefore, all the odd powers of x $(n\geq 3)$ has non-zero coefficient. Clearly the first derivative, . $\mathrm{f}'(0)=2\cos(0)\neq 0$ Thus the Maclaurin expansion only contains odd powers of x.

3	**Suggested Solution**
(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = -a\sin t + a\frac{\sec^2\frac{t}{2}}{2\tan\frac{t}{2}}$ $= -a\sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}} = -a\sin t + \frac{a}{\sin t}$ $\frac{\mathrm{d}y}{\mathrm{d}t} = a\cos t$ Arc length $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{a^2\sin^2 - 2a^2 + \frac{a^2}{\sin^2 t} + a^2\cos^2 t}\ \mathrm{d}t$ $= a\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{-1 + \frac{1}{\sin^2 t}}\ \mathrm{d}t = a\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos t}{\sin t}\ \mathrm{d}t$ $= a\left[\ln\sin t\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = a\left(\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{\sqrt{2}}\right) = \frac{a}{2}\ln\frac{3}{2}$
(b)	$I_n = \int_0^1 \frac{x^n}{\sqrt{4-x^2}}\ \mathrm{d}x$ $= \int_0^1 x^{n-1}\frac{x}{\sqrt{4-x^2}}\ \mathrm{d}x$ $= \left[x^{n-1}\left(-\sqrt{4-x^2}\right)\right]_0^1 - \int_0^1 (n-1)x^{n-2}\left(-\sqrt{4-x^2}\right)\mathrm{d}x$ $= -\sqrt{3} + (n-1)\int_0^1 x^{n-2}\left(\sqrt{4-x^2}\right)\mathrm{d}x$ $= -\sqrt{3} + (n-1)\int_0^1 x^{n-2}\left(\frac{4-x^2}{\sqrt{4-x^2}}\right)\mathrm{d}x$

	$= -\sqrt{3} + 4(n-1)\int_0^1 \frac{x^{n-2}}{\sqrt{4-x^2}}\,dx - (n-1)\int_0^1 \frac{x^n}{\sqrt{4-x^2}}\,dx$ $I_n + (n-1)I_n = -\sqrt{3} + 4(n-1)I_{n-2}$ $nI_n = 4(n-1)I_{n-2} - \sqrt{3}$
(i)	Area $= 1\left(\frac{1}{\sqrt{3}}\right) - \int_0^1 \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{1}{3}\sqrt{3} - I_3$ Using $nI_n = 4(n-1)I_{n-2} - \sqrt{3}$ $3I_3 = 4(2)I_1 - \sqrt{3} \quad \Rightarrow I_3 = \frac{8}{3}I_1 - \frac{1}{3}\sqrt{3}$ Area $= \frac{1}{3}\sqrt{3} - \left(\frac{8}{3}I_1 - \frac{1}{3}\sqrt{3}\right) = \frac{2}{3}\sqrt{3} - \frac{8}{3}I_1$ $I_1 = \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx = -\left[\sqrt{4-x^2}\right]_0^1 = -\left(\sqrt{3}-2\right) = 2-\sqrt{3}$ Area $= \frac{2}{3}\sqrt{3} - \frac{8}{3}\left(2-\sqrt{3}\right) = \frac{10}{3}\sqrt{3} - \frac{16}{3}$
(ii)	Volume $= \pi(1)^2\left(\frac{1}{\sqrt{3}}\right) - 2\pi\int_0^1 x\frac{x^3}{\sqrt{4-x^2}}\,dx$ $= \frac{\pi}{\sqrt{3}} - 2\pi I_4$ $nI_n = 4(n-1)I_{n-2} - \sqrt{3}$ $4I_4 = 4(3)I_2 - \sqrt{3} \quad \Rightarrow I_4 = 3I_2 - \frac{1}{4}\sqrt{3}$ $2I_2 = 4I_0 - \sqrt{3} \quad \Rightarrow I_2 = 2I_0 - \frac{1}{2}\sqrt{3}$ $I_0 = \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 = \frac{\pi}{6}$

	$I_2 = 2\left(\frac{\pi}{6}\right) - \frac{1}{2}\sqrt{3} = \frac{1}{3}\pi - \frac{1}{2}\sqrt{3}$ $I_4 = 3\left(\frac{1}{3}\pi - \frac{1}{2}\sqrt{3}\right) - \frac{1}{4}\sqrt{3} = \pi - \frac{7}{4}\sqrt{3}$ Volume $= \frac{\pi}{3}\sqrt{3} - 2\pi I_4$ $= \frac{\pi}{3}\sqrt{3} - 2\pi\left(\pi - \frac{7}{4}\sqrt{3}\right) = \frac{23}{6}\pi\sqrt{3} - 2\pi^2$

4(a)	Suggested Solution
(i)	As $n \to \infty$, $u_n \to l$ and $u_{n+1} \to l$ $l = \dfrac{kl + a}{l + k}$ $l^2 + kl = kl + a$ $l = \sqrt{a}$ (since the sequence is positive)
(i)	$u_{n+1} - u_n = \dfrac{ku_n + a}{u_n + k} - u_n$ $= \dfrac{ku_n + a - (u_n)^2 - ku_n}{u_n + k}$ $= \dfrac{a - (u_n)^2}{u_n + k}$ $= \dfrac{(\sqrt{a} - u_n)(\sqrt{a} + u_n)}{u_n + k} < 0$ Since $u_n > \sqrt{a}$ and $k > 0$ We have $u_{n+1} - u_n < 0$. Hence $u_{n+1} < u_n$.

5(b)	Suggested Solution
(i)	$x_n = 0.8x_{n-1} + 0.1y_{n-1}$ --- (1) $y_n = 0.2x_{n-1} + 0.9y_{n-1}$ --- (2) From (1), $0.1y_{n-1} = x_n - 0.8x_{n-1}$ $y_{n-1} = 10x_n - 8x_{n-1}$ Substitute into (2), $y_n = 0.2x_{n-1} + 0.9(10x_n - 8x_{n-1})$ $y_n = 9x_n - 7x_{n-1}$ $1 - x_n = 9x_n - 7x_{n-1}$ $10x_n = 7x_{n-1} + 1$ $x_n = 0.7x_{n-1} + 0.1$ Alternatively, From (1): $x_n = 0.8x_{n-1} + 0.1y_{n-1}$ $= 0.8x_{n-1} + 0.1(1 - x_{n-1})$ using given info $= 0.7x_{n-1} + 0.1$
(ii)	General solution: $x_n = A(0.7)^n + B$

	$x_1 = 0.7A + B = 0.9$ $B = 0.7B + 0.1$ Solving, $B = \frac{1}{3}$, $A = \frac{17}{21}$ Thus $x_n = \frac{17}{21}(0.7)^n + \frac{1}{3}$ $y_n = 1 - x_n = \frac{2}{3} - \frac{17}{21}(0.7)^n$

5	**Suggested Solution**
(a)	Since $\dim(\mathbb{R}^4) = 4$ and there are 4 vectors in the set. It is sufficient to prove that if the vectors are linearly independent, the set is a basis for $\mathbb{R}^4$. Using GC, $\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & -1 & 3 & 4 \end{pmatrix} = 6 \neq 0$, the vectors are linearly independent. Hence the set forms a basis for $\mathbb{R}^4$.
(b)	$\mathbf{M}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & -1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 6 & 11 \\ 13 & 5 & 13 & 27 \\ 9 & 4 & 7 & 16 \\ 10 & 6 & 7 & 17 \end{pmatrix}$ $\mathbf{M} = \begin{pmatrix} 4 & 1 & 6 & 11 \\ 13 & 5 & 13 & 27 \\ 9 & 4 & 7 & 16 \\ 10 & 6 & 7 & 17 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & -1 & 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 7 & 10 & 1 \\ 2 & 5 & 7 & 0 \\ 3 & 4 & 7 & 0 \end{pmatrix}$
(c)	$\text{rref}(\mathbf{M}) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Hence basis for range space of $T = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

	Alternatively, $\text{rref}(\mathbf{M}^T)=\begin{pmatrix}1&0&-1&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$ Hence basis for range space of $T=\left\{\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$ Dim(Range space of T) + Dim(Kernel of T) = 4 Dim(Kernel of T) = 4 -3 = 1
(d)	$\lambda_1\begin{pmatrix}1\\3\\2\\3\end{pmatrix}+\lambda_2\begin{pmatrix}2\\7\\5\\4\end{pmatrix}+\lambda_3\begin{pmatrix}1\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\b\\c\\d\end{pmatrix}$ $\left(\begin{array}{ccc\|c}1&2&1&0\\3&7&1&b\\2&5&0&c\\3&4&0&d\end{array}\right)\to\left(\begin{array}{ccc\|c}1&2&1&0\\0&1&-2&b\\0&1&-2&c\\0&-2&-3&d\end{array}\right)$ $\to\left(\begin{array}{ccc\|c}1&2&1&0\\0&1&-2&b\\0&0&0&c-b\\0&0&-7&d+2b\end{array}\right)\to\left(\begin{array}{ccc\|c}1&2&1&0\\0&1&-2&b\\0&0&1&\frac{d+2b}{-7}\\0&0&0&c-b\end{array}\right)$ Alternatively, $\lambda_1\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}+\lambda_2\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+\lambda_3\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\b\\c\\d\end{pmatrix}$ $\left(\begin{array}{ccc\|c}1&0&0&0\\0&1&0&b\\-1&1&0&c\\0&0&1&d\end{array}\right)\to\left(\begin{array}{ccc\|c}1&0&0&0\\0&1&0&b\\0&1&0&c\\0&0&1&d\end{array}\right)\to\left(\begin{array}{ccc\|c}1&0&0&0\\0&1&0&b\\0&0&1&d\\0&0&0&c-b\end{array}\right)$ Hence $c=b$, $d\in\mathbb{R}$ for $\begin{pmatrix}0\\b\\c\\d\end{pmatrix}$ to be in the range space of T.

6	**Suggested Solution**
	$\frac{dt}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)=\frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)=\frac{1}{2}\left(1+t^2\right)$ $\Rightarrow dx=\frac{2}{1+t^2}dt$ $t=\tan\left(\frac{x}{2}\right)$ $\tan x=\frac{2t}{1-t^2}$; $\cos x=\frac{1-t^2}{1+t^2}$; $\sin x=\frac{2t}{1+t^2}$ $\int\frac{1}{3-2\sin x+\cos x}dx$ $=\int\frac{1}{3-2\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt$ $=\int\frac{2}{3+3t^2-4t+1-t^2}dt$ $=\int\frac{1}{t^2-2t+2}dt$ $=\int\frac{1}{(t-1)^2+1}dt$ $=\tan^{-1}(t-1)+c$ $=\tan^{-1}\left(\tan\left(\frac{x}{2}\right)-1\right)+c$
(i)	$\int\cos 3mx\sin 3nx\,dx=\frac{1}{2}\int\sin\left(3m+3n\right)x-\sin\left(3m-3n\right)x\,dx$ $=\frac{1}{2}\left(-\frac{\cos(3m+3n)x}{(3m+3n)}\right)+\frac{1}{2}\left(\frac{\cos(3m-3n)x}{(3m-3n)}\right)+C$
(ii)	$\int_0^{\pi}\cos^2(3mx)+\sin^2(3nx)+2\cos 3mx\sin 3nx\,dx$ $=\int_0^{\pi}\frac{1+\cos 6mx}{2}+\frac{1-\cos 6nx}{2}+\sin(3m+3n)x-\sin(3m-3n)x\,dx$ $=\left[x+\frac{\sin 6mx}{12m}-\frac{\sin 6nx}{12n}\right]_0^{\pi}+\left[-\frac{\cos(3m+3n)x}{3(m+n)}+\frac{\cos(3m-3n)x}{3(m-n)}\right]_0^{\pi}$ $=\pi+\left(-\frac{\cos(3m+3n)\pi}{3(m+n)}\right)+\frac{\cos(3m-3n)\pi}{3(m-n)}-\left(-\frac{1}{3(m+n)}+\frac{1}{3(m-n)}\right)$ Since $\sin k\pi=0$ for all integer k. If m is even and n is odd, then $3\left(m+n\right)$ and $3\left(m-n\right)$ will be

	odd, hence $\cos(3m+3n)\pi=\cos(3m-3n)\pi=-1$ Answer: $=\pi+\dfrac{2}{3(m+n)}-\dfrac{2}{3(m-n)}=\pi+\dfrac{2(m-n-m-n)}{3(m^2-n^2)}=\pi-\dfrac{4n}{3(m^2-n^2)}$

7	Suggested Solution
(a)	$x=\frac{k}{2}\left(e^{\frac{y}{k}}+e^{-\frac{y}{k}}\right)$ $\frac{dx}{dy}=\frac{k}{2}\left(e^{\frac{y}{k}}\left(\frac{1}{k}\right)-e^{-\frac{y}{k}}\left(\frac{1}{k}\right)\right)=\frac{1}{2}\left(e^{\frac{y}{k}}-e^{-\frac{y}{k}}\right)$ $\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+\frac{1}{4}\left(e^{\frac{2y}{k}}-2+e^{-\frac{2y}{k}}\right)}$ $=\frac{1}{2}\sqrt{\left(e^{\frac{2y}{k}}+2+e^{-\frac{2y}{k}}\right)}$ $=\frac{1}{2}\sqrt{\left(e^{\frac{y}{k}}+e^{-\frac{y}{k}}\right)^2}=\frac{1}{2}\left(e^{\frac{y}{k}}+e^{-\frac{y}{k}}\right)$ $S=\int_{-a}^{a}2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ $=\int_{-a}^{a}2\pi\frac{k}{2}\left(e^{\frac{y}{k}}+e^{-\frac{y}{k}}\right)\frac{1}{2}\left(e^{\frac{y}{k}}+e^{-\frac{y}{k}}\right)dy$ $=\frac{\pi k}{2}\int_{-a}^{a}e^{\frac{2y}{k}}+2+e^{-\frac{2y}{k}}\,dy$ $=\frac{\pi k}{2}\left[\frac{k}{2}e^{\frac{2y}{k}}-\frac{k}{2}e^{-\frac{2y}{k}}+2y\right]_{-a}^{a}$ $=\frac{\pi k}{2}\left[\frac{k}{2}e^{\frac{2a}{k}}-\frac{k}{2}e^{-\frac{2a}{k}}-\frac{k}{2}e^{-\frac{2a}{k}}+\frac{k}{2}e^{\frac{2a}{k}}+2a-(-2a)\right]$ $=\frac{\pi k}{2}\left[ke^{\frac{2a}{k}}-ke^{-\frac{2a}{k}}+4a\right]$
(b)	$A=2\pi\frac{k}{2}\left(e^{\frac{a}{k}}+e^{-\frac{a}{k}}\right)(2a)=2\pi ka\left(e^{\frac{a}{k}}+e^{-\frac{a}{k}}\right)$ $S=2A$ $\frac{\pi k}{2}\left[ke^{\frac{2a}{k}}-ke^{-\frac{2a}{k}}+4a\right]=4\pi ka\left(e^{\frac{a}{k}}+e^{-\frac{a}{k}}\right)$ $\left[ke^{\frac{2a}{k}}-ke^{-\frac{2a}{k}}+4a\right]=8a\left(e^{\frac{a}{k}}+e^{-\frac{a}{k}}\right)$

	Dividing by k throughout $$e^{\frac{2a}{k}}-e^{-\frac{2a}{k}}+4\frac{a}{k}=8\frac{a}{k}\left(e^{\frac{a}{k}}+e^{-\frac{a}{k}}\right)$$ Let $x=\frac{a}{k}$ $$e^{2x}-e^{-2x}+4x=8x\left(e^{x}+e^{-x}\right)$$ Using G.C. $x=3.23526\ \simeq 3.235$ (3 d.p.)
(c)	$V=\pi\int_{-a}^{a}x^2\,dy$ $=\pi\int_{-a}^{a}\frac{k^2}{4}\left(e^{\frac{2y}{k}}+2+e^{-\frac{2y}{k}}\right)dy$ $=\frac{k^2}{4}\pi\left[\frac{k}{2}e^{\frac{2y}{k}}-\frac{k}{2}e^{-\frac{2y}{k}}+2y\right]_{-a}^{a}$ $=\frac{\pi k^2}{4}\left[\frac{k}{2}e^{\frac{2a}{k}}-\frac{k}{2}e^{-\frac{2a}{k}}-\frac{k}{2}e^{-\frac{2a}{k}}+\frac{k}{2}e^{\frac{2a}{k}}+4a\right]$ $=\frac{\pi k^2}{4}\left[ke^{\frac{2a}{k}}-ke^{-\frac{2a}{k}}+4a\right]$ $=\frac{1}{2}k\frac{\pi k}{2}\left[ke^{\frac{2a}{k}}-ke^{-\frac{2a}{k}}+4a\right]=\frac{1}{2}kS$

8	**Suggested Solution**
	a $\quad \mathbf{AB}=\mathbf{0}$ has non trivial solution if $\lvert\mathbf{AB}\rvert=0 \Rightarrow \lvert\mathbf{A}\rvert=0$ or $\lvert\mathbf{B}\rvert=0$ i.e. $\left\lvert\begin{pmatrix} k & 0 & 1 \\ 2 & 0 & k \\ 1 & 2 & k \end{pmatrix}\right\rvert=0$ or $\left\lvert\begin{pmatrix} 0 & k & 1 \\ 2 & k & 0 \\ 1 & 2 & k \end{pmatrix}\right\rvert=0$ $-2k^2+4=0$ or $-2k^2-k+4=0$ $k=\pm\sqrt{2}$ or $k=\dfrac{-1\pm\sqrt{33}}{4}$ b(i) $\quad \mathbf{A}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}=\begin{pmatrix} k & 0 & 1 \\ 2 & 0 & k \\ 1 & 2 & k \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow k=2$ b(ii) $\quad$ Characteristic equation $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert=0$ $\det\begin{pmatrix} 2-\lambda & 0 & 1 \\ 2 & -\lambda & 2 \\ 1 & 2 & 2-\lambda \end{pmatrix}=0$ Using GC, $\lambda=-1,\ 1$ and 4 When $\lambda=1$, $\mathbf{Ax}=\mathbf{x}$, we have from **(b)(i)** that $\mathbf{x}=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ When $\lambda=-1, (\mathbf{A}+\mathbf{I})\mathbf{x}=\begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}\mathbf{x}=\mathbf{0}$ and $\mathbf{x}=\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ When $\lambda=4, (\mathbf{A}-4\mathbf{I})\mathbf{x}=\begin{pmatrix} -2 & 0 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -2 \end{pmatrix}\mathbf{x}=\mathbf{0}$ and $\mathbf{x}=\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ $\therefore \mathbf{P}=\begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & 3 \\ -1 & -3 & 4 \end{pmatrix}^{-1}=\begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{10} & \frac{1}{5} & -\frac{1}{10} \\ \frac{2}{15} & \frac{1}{15} & \frac{2}{15} \end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

9	**Suggested Solution**
(a)	$W=\{\mathbf{w}\in\mathbb{R}^3:\mathbf{w}\cdot\mathbf{v}=0 \text{ for every } \mathbf{v}\in V\}$ $\mathbf{0}\in W$ since $\mathbf{0}\cdot\mathbf{v}=0$ for every $\mathbf{v}\in V$ Therefore W is non-empty. Let $\mathbf{w}_1,\mathbf{w}_2\in W$ and $\alpha,\beta\in\mathbb{R}$. For every $\mathbf{v}\in V$, $(\alpha\mathbf{w}_1+\beta\mathbf{w}_2)\cdot\mathbf{v}=\alpha\mathbf{w}_1\cdot\mathbf{v}+\beta\mathbf{w}_2\cdot\mathbf{v}$ $=\alpha 0+\beta 0$ $=0$ Hence $\alpha\mathbf{w}_1+\beta\mathbf{w}_2\in W$ and W is closed under vector addition and scalar multiplication. Hence W is a subspace.
(b) (i)	Clearly, $\begin{pmatrix}1&2&4\\5&10&20\\-2&-4&-8\end{pmatrix}\to\begin{pmatrix}1&2&4\\0&0&0\\0&0&0\end{pmatrix}$ $x+2y+4z=0$ Let $y=\lambda,\ z=\mu$. Then $x=-2\lambda-4\mu$. Hence basis for nullspace is $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix},\begin{pmatrix}-4\\0\\1\end{pmatrix}\right\}$.
(b) (ii)	The blind spots of the detection system lie on a plane passing through the origin (i.e. enemy command centre) and perpendicular to $\begin{pmatrix}1\\2\\4\end{pmatrix}$.
(b) (iii)	The orthogonal complement of the blind spots of the detection system is a line parallel to $\begin{pmatrix}1\\2\\4\end{pmatrix}$ and passing through the origin (i.e. enemy command centre).

10	Suggested Solution
	a) $A_1=5, A_2=8$ and $A_{n+1}=3A_n+2A_{n-1}-8$ for $n\ge 2$. Auxiliary equation: $m^2-3m-2=0$ $\Rightarrow m=\frac{3\pm\sqrt{9+8}}{2}=\frac{3\pm\sqrt{17}}{2}$ Let particular solution be $A_n=\alpha$, then $\alpha=3\alpha+2\alpha-2\Rightarrow\alpha=2$ General solution: $A_n=a\left(\frac{3+\sqrt{17}}{2}\right)^n+b\left(\frac{3-\sqrt{17}}{2}\right)^n+2$ When $n=1$, $5=a\left(\frac{3+\sqrt{17}}{2}\right)+b\left(\frac{3-\sqrt{17}}{2}\right)+2$ $\Rightarrow a\left(\frac{3+\sqrt{17}}{2}\right)+b\left(\frac{3-\sqrt{17}}{2}\right)=3$ ------(1) When $n=2$, $8=a\left(\frac{3+\sqrt{17}}{2}\right)^2+b\left(\frac{3-\sqrt{17}}{2}\right)^2+2$ $6=a\left[3\left(\frac{3+\sqrt{17}}{2}\right)+2\right]+b\left[3\left(\frac{3-\sqrt{17}}{2}\right)+2\right]$ ∵ roots of aux eqn $6=3\left[a\left(\frac{3+\sqrt{17}}{2}\right)+\left(\frac{3-\sqrt{17}}{2}\right)b\right]+2a+2b$ $6=9+2a+2b$ using (1) $b=-\frac{3}{2}-a$ ------(2) Sub into (1): $a\left(\frac{3+\sqrt{17}}{2}\right)+\left(-\frac{3}{2}-a\right)\left(\frac{3-\sqrt{17}}{2}\right)=3$ $\sqrt{17}a-\frac{9}{4}+\frac{3\sqrt{17}}{4}=3$ $a=\frac{21-3\sqrt{17}}{4\sqrt{17}}=\frac{-51+21\sqrt{17}}{68}$ Using (2): $b=-\frac{3}{2}-\frac{-51+21\sqrt{17}}{68}=\frac{-51-21\sqrt{17}}{68}$ $A_n=\frac{-51+21\sqrt{17}}{68}\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)^n+\frac{-51-21\sqrt{17}}{68}\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)^n+2$

b)i) Since $3B_n = 2A_n - 1, n \geq 1$ or alternatively using GC.

$$B_1 = \frac{1}{3}(2A_1 - 1) = 3 \quad B_2 = \frac{1}{3}(2A_2 - 1) = 5 \quad B_3 = \frac{1}{3}(2A_2 - 1) = 17$$

$$B_4 = \frac{1}{3}(2A_3 - 1) = 57 \quad B_5 = \frac{1}{3}(2A_4 - 1) = 201$$

b)ii) Let $B_n = aB_{n-1} + bB_{n-2} + c$, for $n \geq 3$

$B_1 = 3$

$B_2 = 5$

$B_3 = aB_2 + bB_1 + c = 5a + 3b + c = 17$

$B_4 = aB_3 + bB_2 + c = 17a + 5b + c = 57$

$B_4 = aB_4 + bB_3 + c = 57a + 17b + c = 201$

Using GC, $a = 3, b = 2, c = -4$

$\therefore B_n = 3B_{n-1} + 2B_{n-2} - 4$, for $n \geq 3$

Alternatively,

$$A_n = \frac{3B_n + 1}{2}$$

Sub into $A_{n+1} = 3A_n + 2A_{n-1} - 8$

$$\frac{3B_{n+1} + 1}{2} = 3\frac{3B_n + 1}{2} + 2\frac{3B_{n-1} + 1}{2} - 8$$

$$3B_{n+1} + 1 = 9B_n + 3 + 6B_{n-1} + 2 - 16$$

$$3B_{n+1} = 9B_n + 6B_{n-1} - 12$$

$B_{n+1} = 3B_n + 2B_{n-1} - 4$, for $n \geq 2$

c) To find year such that ≥ 2000 rabbit

i.e. find n such that $B_n \geq 20$

From GC,

n	B_n
3	17
4	57

$\therefore$ in 2026 there will be more than 2000 rabbits