Anderson Junior College
Preliminary Examination 2011
H2 Mathematics Paper 2

Section A: Pure Mathematics [40 marks]

<u>Answer ALL questions</u>

1. At a water treatment plant, sewage water is being pumped into a processing tank, with an unknown capacity, at a constant rate of 100 litres per hour. The tank processes the water and discharges it at a rate proportional to the volume of sewage water currently in it. Due to a crack in the tank, sewage water is leaking out of the tank at a constant rate of 3 litres per hour. At time t hours, the volume of sewage water in the tank is u litres.

(i) By setting up and solving a differential equation, show that the general solution is

$u = \frac{1}{k}\left(97 - A\mathrm{e}^{-kt}\right)$ where A and k are constants, and $k > 0$. [3]

(ii) The processing container is initially empty. After an hour, the volume of sewage water in the container is 70 litres. Given that the sewage water that leaked out of the tank is not processed, find the volume of water processed by the container when $t = 2$. [4]

(iii) Sketch, on a single diagram, the graph of the solution found in part (ii) and find the minimum tank capacity for the model to be valid. Give your answer to the nearest litre. [2]

2. (a) The function f is defined by

$$\mathrm{f}(x) = (x+1)\sin^{-1} x + 1,\ 0 \le x \le \frac{1}{2}$$

(i) By using differentiation or otherwise, show that the function f^{-1} exist. Hence find the value of $\mathrm{f}^{-1}(1)$. [3]

(ii) Find the equation of the tangent to the curve $y = \mathrm{f}(x)$ at the point where $x = 0$. [2]

(iii) Write down the equation of the tangent to the curve $y = \mathrm{f}^{-1}(x)$ at the point where $x = 1$. [1]

(b) The functions g and h are defined by

$$\mathrm{g}(x)=\begin{cases}3x-11 & , \quad \text{for } x\in\mathbb{R},\ 0\le x\le 4\\ 5-(x-6)^2 & , \quad \text{for } x\in\mathbb{R},\ 4<x\le 7\end{cases}$$

$$\mathrm{h}(x)=1+\frac{1}{x-a} \qquad \text{for } x\in\mathbb{R},\ x\neq a$$

(i) Find the range of g. [1]

(ii) Given that the composite function hg exists and a is an integer, find the greatest value of a such that the range of hg is a subset of $(1,\infty)$. [2]

3. With reference to the origin *O*, the position vectors of the points *A* and *B* are $2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}$ and $\lambda\boldsymbol{i}+2\boldsymbol{k}$ where λ is a negative integer.

The cosine of the angle between the position vectors of *A* and *B* is $-\frac{3}{\sqrt{70}}$. The point *M* lies on the line segment *AB* such that $AM=3MB$, and the point *C* lies on *OM* produced such that $4OM=3OC$.

(i) Show that $\lambda=-4$. [2]

(ii) Find the position vector of *C*. [2]

(iii) Geometrically, what is the significance of p, where $p=\frac{\left|\overrightarrow{AC}\times\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|}$?

By finding the value of p, deduce the distance between *O* and the line passing through *A* and *B*. [4]

4. The complex number *z* is such that $\left|iz+4+3i\right|=\left|\frac{9}{z+3-4i}\right|$.

(i) Sketch, on an Argand diagram, the locus of the points representing *z*. [2]

(ii) Find the range of values of $\arg(z+i)$. [2]

Another complex number *w* is such that $\left|w^*\right|=\left|w-4\right|$.

(iii) Sketch the locus of the points representing *w* on the same Argand diagram and hence, find the least value of $\left|z+iw\right|$. [3]

5. l is the line of intersection between two perpendicular planes π_1 and π_2. The equations of l and π_2 are given as follows:

$$l : \mathbf{r} = \lambda \begin{pmatrix} 1 \\ a \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\pi_2 : \mathbf{r} \cdot \begin{pmatrix} b \\ 5 \\ -1 \end{pmatrix} = c$$

where a, b and c are constants.

(i) State the value of c. [1]

(ii) Show that $5a + b = -1$. [1]

(iii) Given that the vector $\mathbf{i} + 2\mathbf{j}$ is parallel to π_1, find the value of a and b. [3]

(iv) The plane π_3 is parallel to π_2 and is at a distance of 4 units from π_2. Find a vector equation of π_3 in scalar product form. [2]

Section B: Statistics [60 marks]

Answer ALL questions

6. Using the letters of the word “CORRELATION”, find the number of

(i) arrangements such that the first letter is a vowel and the last letter is a consonant. [2]

(ii) 6-letter code words that can be formed. [3]

7. (i) The Yummy Confectionery manufactures candies from different fruits and randomly packs them in packets of 20. It is known that 56% of the candies are made from mangoes, 20% from strawberries and the rest from lemons. A supermarket buys 60 packets of candies from Yummy Confectionery. Find the probability that the average number of lemon candies in each packet is not less than 5. [3]

(ii) The confectionery wishes to test the popularity of a new flavour target among people of all ages by surveying 300 people.
Comment on the suitability of the sample if the survey is administered to 300 randomly selected students from various secondary schools. [1]
Give a reason why it would be difficult to use a stratified sample and suggest a suitable sampling method. [2]

8. A researcher wants to find out how the ground temperature affects the rate of chirps made by a ground cricket. The data gathered is as follows:

Ground temperature, t, (degree Celsius, ^{o}C)	20.9	22.0	24.0	27.0	29.1	31.6	34.1
Rate of chirps, x, (number of chirps / min)	16.2	16.5	16.9	17.7	20.2	25.3	31.2

(i) Draw a scatter diagram to illustrate the data. [1]

The researcher wants to fit a model of the form $x = Ae^{Bt}$ where A and B are constants.

(ii) Calculate the product moment correlation coefficient, r, between ln x and t. [1]

(iii) Use your answers in part (i) & (ii) to explain why the model is suitable and find the values of the least square estimates of A and B. [3]

(iv) Estimate the percentage increase in the rate of chirps when the ground temperature is increased by 5 ^{o}C.

On a particular day, there is a change in the ground temperature from 10 ^{o}C to 15 ^{o}C. Can the researcher use the answer found above to estimate the percentage change in the rate of chirps? Justify your answer. [2]

(v) The researcher wants to convert the units for the temperature, *t*, from Celsius to Fahrenheit, by using the formula $c = \frac{5}{9}(f - 32)$, where *c* is the temperature in Celsius and *f* is the temperature in Fahrenheit. Explain how this affects the value of *r*. [1]

9. The lifespan of a Liquid Crystal Display (LCD) television, *A*, is normally distributed with a mean of 45000 hours and a standard deviation of 2000 hours. The lifespan of a Plasma television, *B*, is also normally distributed with a standard deviation of 1850 hours. It is known that there is a 50% chance that the lifespan of a Plasma television is less than 30000 hours.

(i) A LCD television and two Plasma televisions are randomly selected. Show that the probability that twice the lifespan of the LCD television exceeds the total lifespan of the two Plasma televisions by at least 25000 hours is 0.852. State an assumption needed for your calculations [3]

(ii) A batch of 50 Plasma televisions is produced. Find the probability that more than 14 but fewer than 22 of the Plasma televisions have a lifespan of more than 30000 hours. [2]

(iii) Find the least value of *n* such that the probability that the average lifespan of *n* LCD televisions is at most 46500 is at least 0.99. [3]

(iv) Explain which of the following two probabilities is greater.

(I) $P(A+B > 50000)$

(II) $P(A > 25000 \text{ and } B > 25000)$ [1]

10. The number of errors on a page of a Mathematics textbook, X, follows a Poisson distribution such that $P(X=2)=10P(X=3)$. Each textbook consists of 50 pages of printed text and the number of errors on each page of the textbook occurs independently.

(i) Show that $E(X)=0.3$. [2]

(ii) Find the probability that a randomly selected textbook has 10 errors. [1]

(iii) A randomly selected textbook is found to have 10 errors. Find the probability that more than eight errors are found between page 11 and page 40, inclusive, of the textbook. [3]

(iv) Each textbook cost $12 to print and the publisher is offering a once-off exchange to its customers if the textbook contains more than 18 errors. Assuming that every textbook printed is sold, find the minimum selling price of a textbook such that the company can expect to make a profit of at least 15%, giving your answer to the nearest cent. [2]

(v) Use a suitable approximation to find the probability that, in 50 randomly selected textbooks, fewer than 6 books each contain more than 18 errors. [3]

11. Mary tosses a six-sided die and a coin. The die is biased in such a way that there is an equal chance of obtaining a 1,3 or 5 and an equal chance of obtaining a 2, 4 or 6 from one throw of the die. In addition, the probability of getting an even number from one throw of the die is twice the probability of getting an odd number. The coin is biased in such a way that the probability of obtaining a 'head' from one throw is twice the probability of obtaining a 'tail'.

(i) Show that the probability of obtaining a '1' from one throw of the die is $\frac{1}{9}$. [1]

(ii) Find the probability that the coin shows a 'tail' and the die shows a number that is at most 4. [3]

(iii) Find the probability that the coin shows a 'tail' or the die shows a number that is at most 4 or both. [2]

A score of 1 is given when the coin shows a 'tail' and 2 when the coin shows a 'head'. The combined score in a toss of the die and the coin is computed as the product of the numbers shown on the die and the coin. (e.g. a ‘head’ from the coin and ‘3’ from the die will give a combined score of $2\times 3=6$)

(iv) Find the probability that the coin shows a ‘tail’ if the score is at most 4. [2]

Mary’s friend, Jane, suggests that they play a game by throwing the die and coin together 60 times. If the score on each throw is at most 4, Mary gives Jane \$1, otherwise, Mary receives \$0.50 from Jane.

(v) Will Mary stand to gain if she plays the game? Justify your answer. [2]

12. The manager of Takayama Shopping Complex claims that customers spend an average of 4 hours shopping in the complex. His superior suspects that this claim is incorrect. A survey was conducted, and the times, x hours, spent by 12 randomly chosen customers were recorded and summarized as follows:

$$\sum x = 44.8 \ , \quad \sum\left(x-\bar{x}\right)^2 = 1.9467$$

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) Carry out a test at 6% significance level to confirm the superior’s suspicion. State any assumption(s) made. [4]

(iii) If the superior has suspected that customers spends less than 4 hours on average at the complex, state with a reason, whether the conclusion of the test in part (ii) would remain the same. [1]

A publicity campaign is held and the manager claims that due to the publicity campaign, the average time that customers spent shopping in Takayama Shopping Complex is now increased to μ_0 hours.

Another survey on 8 customers was conducted after the publicity campaign, and the times, in hours, spent shopping in the complex were recorded as follows:

4.3 3.8 4.5 4.4 3.9 3.5 4.6 4.9

It is known that the times spent by customers in Takayama Shopping Complex are normally distributed with a standard deviation of 0.466.

(iv) Using the data from the given sample of 8 customers, find the largest value of μ_0, to the nearest minute, so that the manager can justify at 6% level of significance that he has not overstated the average time that the customers spent shopping at Takayama Shopping Complex. [4]

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