5 Differentiation & Applications

Basic Skills

1. Prove these trigonometric identities.

(a) $\frac{1-\cos x}{\sin x}+\frac{\sin x}{1-\cos x}\equiv\frac{2}{\sin x}$ (b) $\sin x+\cos x\equiv\frac{\tan x-\cot x}{\sec x-\operatorname{cosec} x}$

(c) $\frac{1}{2}\left(1+\sec 2x\right)\equiv\frac{\cot^2 x}{\cot^2 x-1}$ (d) $\sec 2x\equiv\frac{\tan x+\cot x}{\cot x-\tan x}$

(e) $\frac{\sin x-\sin 2x+\sin 3x}{\cos x-\cos 2x+\cos 3x}\equiv\tan 2x$

(a) $\frac{1-\cos x}{\sin x}+\frac{\sin x}{1-\cos x}$ $=\frac{\left(1-\cos x\right)^2+\sin^2 x}{\sin x\left(1-\cos x\right)}$ $=\frac{1-2\cos x+\cos^2 x+\sin^2 x}{\sin x\left(1-\cos x\right)}$ $=\frac{2\left(1-\cos x\right)}{\sin x\left(1-\cos x\right)}$ $=\frac{2}{\sin x}$	(b) $\frac{\tan x-\cot x}{\sec x-\operatorname{cosec} x}$ $=\frac{\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}}{\frac{1}{\cos x}-\frac{1}{\sin x}}$ $=\frac{\frac{\sin^2 x-\cos^2 x}{\sin x\cos x}}{\frac{\sin x-\cos x}{\sin x\cos x}}$ $=\frac{\left(\sin x+\cos x\right)\left(\sin x-\cos x\right)}{\sin x-\cos x}$ $=\sin x+\cos x$
(c) $\frac{\cot^2 x}{\cot^2 x-1}$ $=\frac{\frac{\cos^2 x}{\sin^2 x}}{\frac{\cos^2 x}{\sin^2 x}-1}$ $=\frac{\cos^2 x}{\cos^2 x-\sin^2 x}$ $=\frac{\frac{\cos 2x+1}{2}}{\cos 2x}$ $=\frac{1}{2}\left(\frac{\cos 2x+1}{\cos 2x}\right)$ $=\frac{1}{2}\left(1+\sec 2x\right)$	(d) $\frac{\tan x+\cot x}{\cot x-\tan x}$ $=\frac{\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}}{\frac{\cos^2 x-\sin^2 x}{\sin x\cos x}}$ $=\frac{1}{\cos 2x}$ $=\sec 2x$

(e) $\frac{\sin x - \sin 2x + \sin 3x}{\cos x - \cos 2x + \cos 3x}$ $= \frac{2\sin 2x \cos x - \sin 2x}{2\cos 2x \cos x - \cos 2x}$ $= \frac{\sin 2x(2\cos x - 1)}{\cos 2x(2\cos x - 1)}$ $= \tan 2x$	

2. Differentiate the following expressions with respect to x:

(a) $\sqrt[3]{4x - x^2}$ (b) $\frac{(3x+1)^4}{x-3}$, $x \neq 3$ (c) $\cos(\sin x)$

(d) $\sin^{-1}(1-x)$ (e) $\ln\sqrt{\frac{x+3}{x-3}}$ (f) $\frac{e^x}{e^x + e^{-x}}$

(a) $\frac{1}{3}(4x - x^2)^{-\frac{2}{3}}(4 - 2x)$	(b) $\frac{(x-3)(4)(3x+1)^3(3) - (3x+1)^4(1)}{(x-3)^2}$ $= \frac{(9x - 37)(2x+1)^3}{(x-3)^2}$
(c) $-\sin(\sin x) \cdot \cos x$	(d) $\frac{1}{\sqrt{1-(1-x)^2}}(-1) = \frac{-1}{\sqrt{2x - x^2}}$
(e) $\frac{d}{dx}\left(\frac{1}{2}\right)(\ln(x+3) - \ln(x-3))$ $= \frac{1}{2}\left(\frac{1}{x+3} - \frac{1}{x-3}\right)$ $= \frac{-3}{x^2 - 9}$	(f) $\frac{d}{dx}\left(\frac{e^{2x}}{e^{2x}+1}\right)$ $= \frac{(e^{2x}+1)(2e^{2x}) - e^{2x}(2e^{2x})}{(e^{2x}+1)^2}$ $= \frac{2e^{2x}}{(e^{2x}+1)^2}$

<u>Tutorial Review</u>

Tutorial 5A Questions 1, 4, 7 and 10.
Tutorial 5B Question 1 and 2.
Tutorial 5C Question 1 and 2.
Tutorial 5D Question 2 and 5.

<u>Revision Questions</u>

1. 2020/Promo/SAJC/Q3

(a) Differentiate $e^{2x}\cos^{-1}(3x)$ with respect to x, where $-\frac{1}{3} \le x \le \frac{1}{3}$. **[3]**

$\frac{d}{dx}\left[e^{2x}\cos^{-1}(3x)\right] = e^{2x}\left(\frac{-3}{\sqrt{1-(3x)^2}}\right) + 2e^{2x}\cos^{-1}(3x)$ $= e^{2x}\left[2\cos^{-1}(3x) - \frac{3}{\sqrt{1-9x^2}}\right]$

(b) Given that $\sqrt{x} + \sqrt{y} = \sqrt{a}$, where $x > 0$, $y > 0$ and a is a positive constant, show that $2x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right) - 1 = 0$. **[4]**

$\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentiating with respect to x,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$$

$$\sqrt{y}+\sqrt{x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$$

$$\sqrt{x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-\sqrt{y} \quad \text{(Note that } \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sqrt{y}}{\sqrt{x}}<0\text{)}$$

$$x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=y$$

Differentiating once more with respect to x,

$$x2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)=0$$

$\frac{\mathrm{d}y}{\mathrm{d}x}=0$ or $2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}-1=0$

(reject as $\frac{\mathrm{d}y}{\mathrm{d}x}<0$)

Hence, $2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}-1=0$

Alternate Solution

$$\sqrt{x}+\sqrt{y}=\sqrt{a}$$

$$y=x+a-2\sqrt{ax}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{\sqrt{a}}{\sqrt{x}}\ldots\ldots(1)$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\sqrt{a}}{2x\sqrt{x}}\ldots\ldots(2)$$

$$\begin{aligned}\text{LHS}&=2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}-1\\&=2x\left(\frac{\sqrt{a}}{2x\sqrt{x}}\right)+\left(1-\frac{\sqrt{a}}{\sqrt{x}}\right)-1,\ \text{Substitute (1) \& (2)}\\&=\frac{\sqrt{a}}{\sqrt{x}}+1-\frac{\sqrt{a}}{\sqrt{x}}-1\\&=0 \quad \text{(shown)}\end{aligned}$$

2. 2017/Prelim/VJC/P2/Q1

A curve C is defined by the parametric equations

$$x=\frac{t}{1+t}, \qquad y=\frac{t^2}{1+t},$$

where t takes all real values except -1.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, leaving your answer in terms of t. [3]

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$=\frac{(1+t)2t-t^2}{(1+t)^2}\div\frac{(1+t)(1)-t}{(1+t)^2}$$

$$=t^2+2t$$

(i) Show that the equation of the tangent to C at the point $\left(\frac{p}{1+p}, \frac{p^2}{1+p}\right)$ is

$$y=p(p+2)x-p^2.$$ [2]

(i) At point $\left(\frac{p}{1+p},\frac{p^2}{1+p}\right)$, $t=p$

Equation of tangent at point $\left(\frac{p}{1+p},\frac{p^2}{1+p}\right)$,

$$y-\frac{p^2}{1+p}=\left(p^2+2p\right)\left(x-\frac{p}{1+p}\right)$$

$$y=p(p+2)x+\frac{p^2}{1+p}-\frac{p^3}{1+p}-\frac{2p^2}{1+p}$$

$$y=p(p+2)x-\frac{p^2(p+1)}{1+p}$$

$$y=p(p+2)x-p^2$$

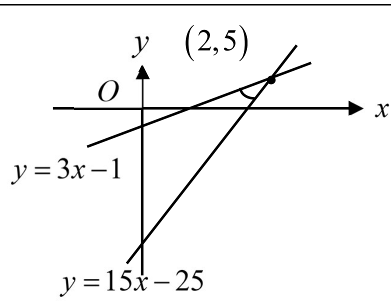
(ii) Find the acute angle between the two tangents to C which pass through the point $(2,5)$. [3]

(ii) Tangents pass through $(2,5)$

$\Rightarrow 5=p(p+2)(2)-p^2$

$p^2+4p-5=0$

$p=-5$ or $p=1$



Equations of tangents are

$y=3x-1$ and $y=15x-25$

Required acute angle between the 2 tangents

$=\tan^{-1}(15)-\tan^{-1}(3)$

$=0.255$ rad or $14.6°$

3. 2017/Prelim/NYJC/P1/Q2

The curve C has equation $2x-y^2=(x+y)^2$.

(i) Find the equations of the tangents to C which are parallel to the x-axis. [4]

Differentiating $2x-y^2=(x+y)^2$ _____(1)
implicitly with respect to x,

$$2-2y\frac{dy}{dx}=2(x+y)\left(1+\frac{dy}{dx}\right)$$

Where tangent is parallel to the x-axis, $\frac{dy}{dx}=0$.

$$2=2(x+y)$$

$$y=1-x \qquad \text{\_\_\_\_\_(2)}$$

Sub (2) in (1),

$$2x-(1-x)^2=(x+1-x)^2$$

$$x^2-4x+2=0$$

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)}}{2}=2\pm\sqrt{2}$$

When $x=2-\sqrt{2}$, $y=1-\left(2-\sqrt{2}\right)=-1+\sqrt{2}$

When $x=2+\sqrt{2}$, $y=1-\left(2+\sqrt{2}\right)=-1-\sqrt{2}$

The tangents are $y=-1+\sqrt{2}$ and $y=-1-\sqrt{2}$.

(ii) The line l is tangent to C at $A(2,-2)$. If the normal to C at the origin O meets l at the point B, find the area of triangle OAB. [4]

$$2-2y\frac{dy}{dx}=2(x+y)\left(1+\frac{dy}{dx}\right)$$

$$2=2(x+y)+2(x+2y)\frac{dy}{dx}$$

$$2=2(x+y)+2(x+2y)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{1-(x+y)}{x+2y}$$

When $x=0,\ y=0$, $-\frac{1}{\frac{dy}{dx}}=0$.

Hence normal to C at the origin is $y=0$.

When $x=2,\ y=-2$, $\frac{dy}{dx}=\frac{1}{-2}$

Tangent to C at $A(2,-2)$, $y-(-2)=-\frac{1}{2}(x-2)$

Where the normal and the tangent intersect,

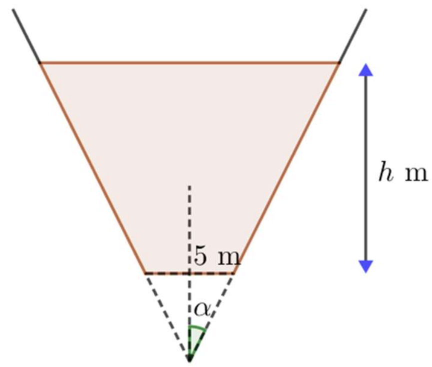
$$2=-\frac{1}{2}(x-2)$$

$$x=-2$$

Area of triangle $OAB=\frac{1}{2}(2)(2)=2$ units^2

4. **NJC JC1 Promo 9758/2019/Q6**

A vessel is formed by removing a smaller cone of radius 5 m from a bigger cone whose semi-vertical angle is α, where $\tan\alpha = 0.5$. Water flows out of the vessel at a rate of $k\sqrt{h}$ m^3 per minute, where k is a positive constant. At time t minutes, the height of the water surface from the hole is h m (see diagram).



(i) Show that the volume of the water V, in m^3, is given by $\frac{1}{12}\pi\left[(h+10)^3 - 1000\right]$. [4]

(ii) Find the rate of change of h, in terms of k, when $V = 120\pi$. [4]

(i)	Let h' be the height of the smaller cone $\tan\alpha = 0.5 = \frac{5}{h'}$ $\Rightarrow h' = 10$ Let R and H be the radius and height of the bigger cone respectively. $\tan\alpha = 0.5 = \frac{R}{H}$ $\Rightarrow H = 2R$ Also, we have $H = h + 10$ $V = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi(5)^2(10)$ $= \frac{1}{3}\pi\left(R^2 H - 250\right)$ $= \frac{1}{3}\pi\left[\left(\frac{H}{2}\right)^2 H - 250\right]$ $= \frac{1}{3}\pi\left(\frac{H^3}{4} - 250\right)$ $= \frac{1}{12}\pi\left(H^3 - 1000\right)$ $= \frac{1}{12}\pi\left[(h+10)^3 - 1000\right]$
(ii)	$\frac{dV}{dh} = \frac{1}{12}\pi\left[3(h+10)^2\right]$ $= \frac{\pi}{4}(h+10)^2$

$$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$$

$$-k\sqrt{h}=\frac{1}{4}\pi\left(h+10\right)^2\times\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{-4k\sqrt{h}}{\pi\left(h+10\right)^2}$$

When $V=120\pi$, $\frac{1}{12}\pi\left[\left(h+10\right)^3-1000\right]=120\pi$.

$$\left(h+10\right)^3-1000=1440$$

$$\left(h+10\right)^3=2440$$

$$h+10=\sqrt[3]{2440}$$

$$h=\sqrt[3]{2440}-10$$

When $h=\sqrt[3]{2440}-10$, $\frac{dh}{dt}=\frac{-4k\sqrt{\sqrt[3]{2440}-10}}{\pi\left(\sqrt[3]{2440}\right)^2}$

$=-0.0131k$ m/min (to 3 s.f.)

5. 2017/Prelim/HCI/P1/Q6

A particle moving along a path at time t, where $0<t<\frac{\pi}{3}$, is defined parametrically by

$$x=\cot 3t \quad \text{and} \quad y=2\operatorname{cosec}3t+1 .$$

(a) The tangent to the path at the point $P\left(\cot 3p,\ 2\operatorname{cosec}3p+1\right)$ meets the y-axis at the point Q. Show that the coordinates of Q is $\left(0,\ 2\sin 3p+1\right)$. [4]

$x=\cot 3t \Rightarrow \frac{dx}{dt}=-3\operatorname{cosec}^2 3t$

$y=2\operatorname{cosec}3t+1 \Rightarrow \frac{dy}{dt}=-6\operatorname{cosec}3t\cot 3t$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-6\operatorname{cosec}3t\cot 3t}{-3\operatorname{cosec}^2 3t}$$

$$=\frac{2\cot 3t}{\operatorname{cosec}3t}$$

$$=2\cos 3t$$

At point P, $\frac{dy}{dx}\Big|_{t=p}=2\cos 3p$

Equation of tangent at P:

$y-\left(2\operatorname{cosec}3p+1\right)=2\cos 3p\left(x-\cot 3p\right)$

When tangent meets y-axis, $x=0$.

Hence $y=-\left(2\cos 3p\right)\left(\cot 3p\right)+\left(2\operatorname{cosec}3p+1\right)$

$$y=\frac{-2\left(\cos^2 3p\right)}{\sin 3p}+\frac{2}{\sin 3p}+1$$

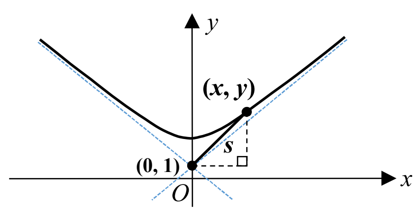
$y=\dfrac{-2(\cos^2 3p-1)}{\sin 3p}+1$

$y=\dfrac{-2(-\sin^2 3p)}{\sin 3p}+1$

$y=2\sin 3p+1$

Hence the coordinates of Q is $\left(0,\,2\sin 3p+1\right)$. (shown)

(b) The distance of the particle from the point $R(0,1)$ is denoted by s, where $s^2=x^2+(y-1)^2$. Find the exact rate of change of the particle's distance from R at time $t=\frac{\pi}{4}$. [4]



Method 1

$s^2=x^2+\left(y-1\right)^2$

$=\cot^2 3t+\left(2\operatorname{cosec}3t+1-1\right)^2$

$=(\operatorname{cosec}^2 3t-1)+4\operatorname{cosec}^2 3t$

$=5\operatorname{cosec}^2 3t-1$

Differentiate w.r.t. t,

$2s\dfrac{ds}{dt}=10\operatorname{cosec}3t(-\operatorname{cosec}3t\cot 3t)(3)$

$=-30\operatorname{cosec}^2 3t\cot 3t$

$s\dfrac{ds}{dt}=-15\operatorname{cosec}^2 3t\cot 3t$

When $t=\frac{\pi}{4}$, $s^2=(2\sqrt{2}+1-1)^2+(-1)^2=9$

$\therefore\ s=3$ (since $s>0$)

$\therefore\ \dfrac{ds}{dt}=-5\operatorname{cosec}^2 3\left(\frac{\pi}{4}\right)\cot 3\left(\frac{\pi}{4}\right)$

$=-5(2)(-1)$

$=10$ unit/s

Method 2

$$s^2 = x^2 + \left(y-1\right)^2$$
$$= \cot^2 3t + \left(2\operatorname{cosec}3t + 1 - 1\right)^2$$
$$= \cot^2 3t + 4\operatorname{cosec}^2 3t$$

Differentiate w.r.t. t ,

$$2s\frac{\mathrm{d}s}{\mathrm{d}t} = 2\cot 3t(-\operatorname{cosec}^2 3t)(3) + 8\operatorname{cosec}3t(-\operatorname{cosec}3t\cot 3t)(3)$$
$$= -6\operatorname{cosec}^2 3t\cot 3t - 24\operatorname{cosec}^2 3t\cot 3t$$
$$= -30\operatorname{cosec}^2 3t\cot 3t$$
$$s\frac{\mathrm{d}s}{\mathrm{d}t} = -15\operatorname{cosec}^2 3t\cot 3t$$

When $t = \frac{\pi}{4}$, $s^2 = (2\sqrt{2}+1-1)^2 + (-1)^2 = 9$

$\therefore\ s = 3$ (since $s > 0$)

$$\therefore\ \frac{\mathrm{d}s}{\mathrm{d}t} = -5\operatorname{cosec}^2 3\left(\frac{\pi}{4}\right)\cot 3\left(\frac{\pi}{4}\right)$$
$$= -5(2)(-1)$$
$$= 10 \text{ unit/s}$$

Method 3

$$s^2 = x^2 + (y-1)^2$$

Differentiate w.r.t. t ,

$$2s\frac{\mathrm{d}s}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2\left(y-1\right)\frac{\mathrm{d}y}{\mathrm{d}t}$$
$$s\frac{\mathrm{d}s}{\mathrm{d}t} = x\frac{\mathrm{d}x}{\mathrm{d}t} + \left(y-1\right)\frac{\mathrm{d}y}{\mathrm{d}t}$$

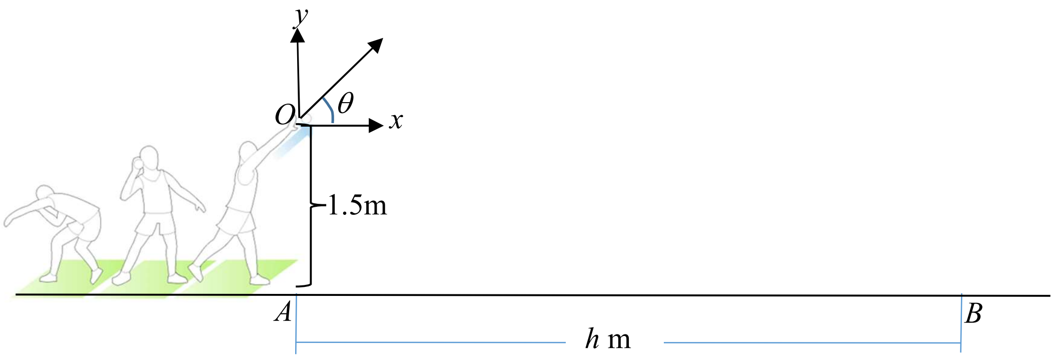
When $t = \frac{\pi}{4}$,

$$x = \frac{1}{\tan\left(\frac{3\pi}{4}\right)} = -1, \qquad y = \frac{2}{\sin\left(\frac{3\pi}{4}\right)} + 1 = 2\sqrt{2} + 1$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3\operatorname{cosec}^2 3t = \frac{-3}{\sin^2\left(\frac{3\pi}{4}\right)} = -6$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -6\cot 3t\operatorname{cosec}3t = \frac{-6}{\tan\left(\frac{3\pi}{4}\right)} \times \frac{1}{\sin\left(\frac{3\pi}{4}\right)} = 6\sqrt{2}$$
$$s^2 = (2\sqrt{2}+1-1)^2 + (-1)^2 = 9$$

$\therefore\ s = 3$ (since $s > 0$)

$$\text{Hence } \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{s}\left[x\frac{\mathrm{d}x}{\mathrm{d}t} + \left(y-1\right)\frac{\mathrm{d}y}{\mathrm{d}t}\right]$$
$$= \frac{1}{3}\left[(-1)(-6) + (2\sqrt{2})(6\sqrt{2})\right]$$
$$= 10 \text{ unit/s}$$

6. 2017/Prelim/TJC/P1/Q7



The diagram shows a shot put being projected with a velocity v ms^{-1} from the point O at an angle θ made with the horizontal. The point O is 1.5m above the point A on the ground. The x-y plane is taken to be the plane that contains the trajectory of this projectile motion with x-axis parallel to the horizontal and O being the origin. The equation of the trajectory of this projectile motion is known to be

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta},$$

where g ms^{-2} is the acceleration due to gravity.
The constant g is taken to be 10 and the distance between A and B is denoted by h m. Given that $v = 10$, show that h satisfies the equation

$$h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0.$$

[3]

As θ varies, h varies. Show that stationary value of h occurs when θ satisfies the following equation

$$3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0.$$

[5]

Hence find the stationary value of h. [2]

$$y = x\tan\theta - \frac{10x^2}{2(10)^2\cos^2\theta} \quad \Rightarrow \quad y = x\tan\theta - \frac{x^2}{20\cos^2\theta}$$

When $x = h$ $y = -1.5$

$$\therefore \quad -1.5 = h\tan\theta - \frac{h^2}{20\cos^2\theta}$$

$$\Rightarrow \quad -30\cos^2\theta = 20h\tan\theta\cos^2\theta - h^2$$

$$\Rightarrow \quad h^2 - 20h\sin\theta\cos\theta - 30\cos^2\theta = 0$$

$$\Rightarrow \quad h^2 - 10h\sin 2\theta - 15(1+\cos 2\theta) = 0$$

$$\Rightarrow \quad h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0 \quad (*)$$

Differentiate both sides w.r.t. θ, we have

$$2h\frac{dh}{d\theta} - 10\frac{dh}{d\theta}\sin 2\theta - 20h\cos 2\theta + 30\sin 2\theta = 0$$

At stationary value, $\frac{dh}{d\theta} = 0 \quad \Rightarrow \quad -20h\cos 2\theta + 30\sin 2\theta = 0 \quad \Rightarrow \quad h = \frac{3}{2}\tan 2\theta$

Sub into (*), we have

$$\left(\frac{3}{2}\tan 2\theta\right)^2 - 10\left(\frac{3}{2}\tan 2\theta\right)\sin 2\theta - 15\cos 2\theta - 15 = 0$$

$$\Rightarrow \quad \frac{9}{4}\tan^2 2\theta - 15\tan 2\theta\sin 2\theta - 15\cos 2\theta - 15 = 0$$

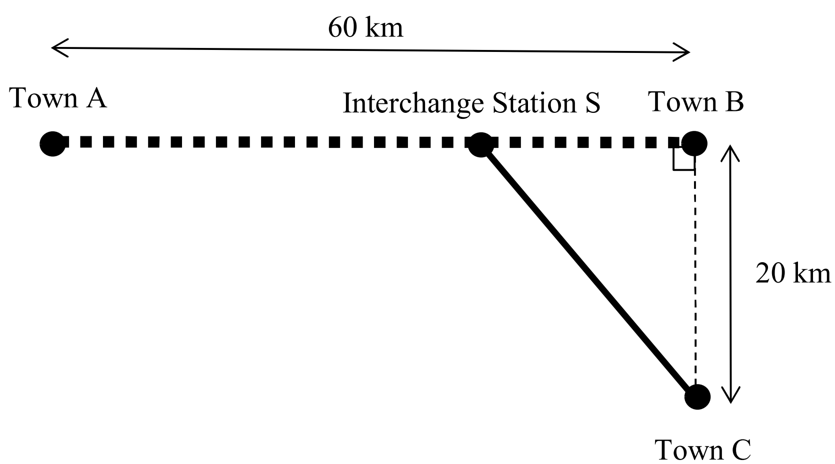
$$\Rightarrow \quad 3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0 \quad \text{(shown)}$$

Using GC, $\theta = 0.71999$ (5 sig fig)

Therefore, max $h = \frac{3}{2}\tan 2(0.71999) = 11.4$ (3 sig fig) □□□
□□

7. **MI PU2 Promo 9758/2019/02/Q1**

Town A, Town B and Town C are located in Wakandi Country. The distance between Town A and Town B is 60 km and the distance between Town B and Town C is 20 km. A railroad connects Town A to Town B (see diagram).



A manufacturer plans to deliver a certain number of containers of its goods daily from Town A to Town C. To support this plan, the Wakandi government decides to build Interchange Station S and a road connecting this station to Town C (see diagram). Once the road is built, the goods manufacturer can deliver its containers from Town A to Town C by a combination of rail and road via Interchange Station S.

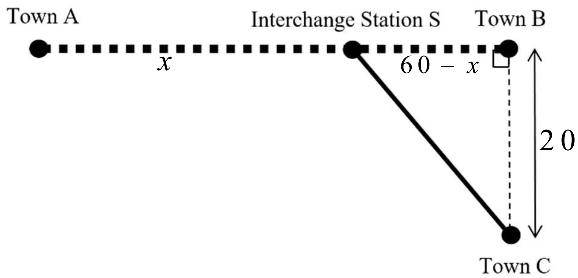
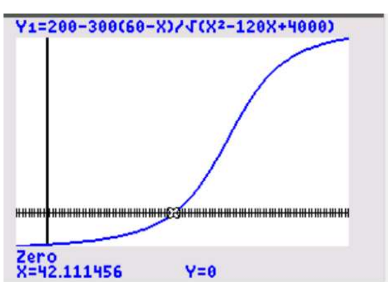
The cost to deliver the containers daily by rail is \$200 per km and the cost to deliver the containers daily by road is \$300 per km.

(i) Show that the daily total delivery cost, \$$T$, of the containers from Town A to Town C is given by

$$T = 200x + 300\sqrt{x^2 - 120x + 4000},$$

where x km is the distance between Town A and Interchange Station S. [2]

(ii) Hence use differentiation to find the value of x that gives a stationary value of T, giving your answer correct to 2 decimal places. Show that T is a minimum for this value of x. [4]

7(i)	 $SC^2 = (60-x)^2 + 20^2 \quad \Rightarrow \quad SC = \sqrt{x^2 - 120x + 4000}$ $T = 200x + 300\sqrt{x^2 - 120x + 4000}$ (shown)
(ii)	$T = 200x + 300\left[x^2 - 120x + 4000\right]^{\frac{1}{2}}$ $\frac{dT}{dx} = 200 + \frac{300}{2}\left[x^2 - 120x + 4000\right]^{-\frac{1}{2}}\left[2x - 120\right]$ $= 200 - \frac{300(60-x)}{\left[x^2 - 120x + 4000\right]^{\frac{1}{2}}}$ For stationary values of T, $\frac{dT}{dx} = 0$ $200 - \frac{300(60-x)}{\left[x^2 - 120x + 4000\right]^{\frac{1}{2}}} = 0$ **Method 1** Using GC (graph): $x = 42.111 = 42.11$ (2 d.p)  **Method 2** $200 = \frac{300(60-x)}{\left[x^2 - 120x + 4000\right]^{\frac{1}{2}}}$

$$2\left[x^2-120x+4000\right]^{\frac{1}{2}}=3(60-x)$$

$$4\left[x^2-120x+4000\right]=9(60-x)^2$$

$$=32400-1080x+9x^2$$

$$5x^2-600x+16400=0$$

Using GC,

$x=42.111$ or 77.889 (rej. as $0<x\le 60$)

$x=42.11$ (2 d.p)

or

$$x=\frac{-(-600)\pm\sqrt{(-600)^2-4(5)(16400)}}{2(5)}$$

$$x=\frac{600\pm\sqrt{32000}}{10}=\frac{600\pm 80\sqrt{5}}{10}=60\pm 8\sqrt{5}$$

$x=42.111$ or 77.889 (rej. as $0<x\le 60$)

$x=42.11$ (2 d.p)

For verification:

Method 1

Using the Second Derivative Test:

$$\left.\frac{\mathrm{d}^2T}{\mathrm{d}x^2}\right|_{x=42.111}=6.2111>0$$

$\Rightarrow T$ is minimum when $x=42.111$

Method 2

Using the First Derivative Test:

x	42.111^-	42.111	42.111^+
$\frac{\mathrm{d}T}{\mathrm{d}x}$	negative	0	positive
Shape	$\backslash$	$—$	$/$

$\Rightarrow T$ is minimum when $x=42.111$

8. 2017/Prelim/NYJC/P2/Q3

It is given that $y=\ln\left(\cos ax-\sin ax\right)$, where a is a non-zero constant.

(i) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+a^2=0.$ [3]

$y=\ln(\cos ax-\sin ax)$

$e^y=\cos ax-\sin ax$

$e^y\dfrac{dy}{dx}=-a\sin ax-a\cos ax$

$e^y\dfrac{d^2y}{dx^2}+e^y\left(\dfrac{dy}{dx}\right)^2=-a^2\cos ax+a^2\sin ax$

$e^y\dfrac{d^2y}{dx^2}+e^y\left(\dfrac{dy}{dx}\right)^2=-a^2(\cos ax-\sin ax)$

$e^y\dfrac{d^2y}{dx^2}+e^y\left(\dfrac{dy}{dx}\right)^2=-a^2e^y$

$\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2+a^2=0$

(ii) By further differentiation of the result in **(i)**, find, in terms of *a*, the Maclaurin series for *y*, up to and including the term in x^3. [3]

$\dfrac{d^3y}{dx^3}+2\left(\dfrac{dy}{dx}\right)\dfrac{d^2y}{dx^2}=0$

When $x=0$, $y=0$

$\dfrac{dy}{dx}=-a$, $\dfrac{d^2y}{dx^2}=-2a^2$, $\dfrac{d^3y}{dx^3}=-4a^3$

$y=-ax-a^2x^2-\dfrac{2}{3}a^3x^3+\cdots$

(iii) Hence show that when *x* is small enough for powers of *x* higher than 2 to be neglected and $a=2$, then $\cos 2x-\sin 2x\approx 1+kx+kx^2$ where *k* is a constant to be determined. [4]

$\ln(\cos 2x-\sin 2x)=-2x-4x^2-\dfrac{16}{3}x^3+\cdots$

$\cos 2x-\sin 2x\approx e^{-2x-4x^2}$

$\approx 1+(-2x-4x^2)+\dfrac{(-2x-4x^2)^2}{2!}$ (since $e^x\approx 1+x+\dfrac{x^2}{2!}$)

$\approx 1-2x-4x^2+\dfrac{(-2x)^2}{2}$

$=1-2x-2x^2$ where $k=-2$

(iv) Using appropriate expansions from the List of Formulae (MF26), verify the correctness of your answer in (iii). [2]

$\cos 2x-\sin 2x=1-\dfrac{(2x)^2}{2}-(2x)$

$=1-2x-2x^2$

9. 2016/Promo/PJC/Q4

(i) Given that $\mathrm{f}(x)=1+\cos 2x$, find $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$, $\mathrm{f}^{(3)}(0)$ and $\mathrm{f}^{(4)}(0)$. Write down the Maclaurin series for $\mathrm{f}(x)$ up to and including the term in x^4. [5]

$\mathrm{f}(x)=1+\cos 2x$ $\quad$ $\mathrm{f}(0)=2$ $\mathrm{f}'(x)=-2\sin 2x$ $\quad$ $\mathrm{f}'(0)=0$ $\mathrm{f}''(x)=-4\cos 2x$ $\quad$ $\mathrm{f}''(0)=-4$ $\mathrm{f}^{(3)}(x)=8\sin 2x$ $\quad$ $\mathrm{f}^{(3)}(0)=0$ $\mathrm{f}^{(4)}(x)=16\cos 2x$ $\quad$ $\mathrm{f}^{(4)}(0)=16$ Hence $\mathrm{f}(x)=1+\cos 2x=2+\frac{x^2}{2!}(-4)+\frac{x^4}{4!}(16)+...$ $\approx 2-2x^2+\frac{2}{3}x^4$	Alternative Method: $\mathrm{f}(x)=1+\cos 2x$ $\mathrm{f}'(x)=-2\sin 2x$ $\mathrm{f}''(x)=-4\cos 2x$ $=-4[\mathrm{f}(x)-1]$ $\mathrm{f}^{(3)}(x)=-4\mathrm{f}'(x)$ $\mathrm{f}^{(4)}(x)=-4\mathrm{f}''(x)$

(ii) Deduce the series for $\cos^2 x$ up to and including the term in x^4. [2]

$\cos 2x=2\cos^2 x-1$ $1+\cos 2x=2\cos^2 x$ $\therefore\ \cos^2 x\approx\frac{1}{2}(2-2x^2+\frac{2}{3}x^4)=1-x^2+\frac{1}{3}x^4$	Alternative Method $\cos x\approx 1-\frac{x^2}{2}+\frac{x^4}{24}$ $\cos^2 x=\frac{1}{2}(1+\cos 2x)$ $\approx\frac{1}{2}[1+1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}]$ $=1-x^2+\frac{1}{3}x^4$

(iii) Use appropriate expansions from the List of Formulae (MF26) to verify the correctness of your answer in part (ii). [2]

From MF26,

$$\cos x\approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}=1-\frac{x^2}{2}+\frac{x^4}{24}$$

$$\therefore\ \cos^2 x\approx\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)$$

$$\approx 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^4}{24}$$

$$\approx 1-x^2+\frac{1}{3}x^4 \quad \text{(Shown)}$$

10. **TMJC JC1 Promo 9758/2019/Q6**

(a) Given that $y=\ln\left(2-\mathrm{e}^{2x}\right)$, where $x<\frac{1}{2}\ln 2$, show that

$$\mathrm{e}^{y}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]=-4\mathrm{e}^{2x}.$$

Hence, find the Maclaurin series of y, up to and including the term in x^3.

[5]

(b) Find the expansion of $\frac{\sqrt{1-2x}}{2+3x}$ in ascending powers of x, up to and including the term in x^2. State the range of values of x for which this expansion is valid. [5]

10(a)	$y=\ln\left(2-\mathrm{e}^{2x}\right)$ Differentiate with respect to x: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2\mathrm{e}^{2x}}{2-\mathrm{e}^{2x}}$ $=\frac{-2\mathrm{e}^{2x}}{\mathrm{e}^{y}}$ $\mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x}=-2\mathrm{e}^{2x}$ Differentiate with respect to x: $\mathrm{e}^{y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\left(\mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-4\mathrm{e}^{2x}$ $\mathrm{e}^{y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}^{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-4\mathrm{e}^{2x}$ $\mathrm{e}^{y}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]=-4\mathrm{e}^{2x}$ Differentiate with respect to x: $\mathrm{e}^{y}\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]\left(\mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-8\mathrm{e}^{2x}$ $\mathrm{e}^{y}\left[\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3\right]=-8\mathrm{e}^{2x}$ When $x=0$, $y=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-2$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-8$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-48$ Maclaurin series for y is $y=-2x-8\left(\frac{x^2}{2!}\right)-48\left(\frac{x^3}{3!}\right)+\ldots$ $y=-2x-4x^2-8x^3+\ldots$
(b)	$\frac{\sqrt{1-2x}}{(2+3x)}$ $=(1-2x)^{\frac{1}{2}}(2+3x)^{-1}$

	$=\frac{1}{2}(1-2x)^{\frac{1}{2}}\left(1+\frac{3}{2}x\right)^{-1}$ $=\frac{1}{2}\left(1+\frac{1}{2}(-2x)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(-2x)^2+...\right)\left(1-\frac{3}{2}x+\left(\frac{3}{2}x\right)^2+...\right)$ $=\frac{1}{2}\left(1-x-\frac{1}{2}x^2+...\right)\left(1-\frac{3}{2}x+\frac{9}{4}x^2+...\right)$ $=\frac{1}{2}\left(1-\frac{3}{2}x+\frac{9}{4}x^2-x+\frac{3}{2}x^2-\frac{1}{2}x^2+...\right)$ $=\frac{1}{2}\left(1-\frac{5}{2}x+\frac{13}{4}x^2+...\right)$ $=\frac{1}{2}-\frac{5}{4}x+\frac{13}{8}x^2+...$ The expansion of $(1-2x)^{\frac{1}{2}}$ is valid for $\lvert 2x\rvert<1\Rightarrow\lvert x\rvert<\frac{1}{2}$ The expansion of $(2+3x)^{-1}$ is valid for $\left\lvert\frac{3x}{2}\right\rvert<1\Rightarrow\lvert x\rvert<\frac{2}{3}$ $-\frac{1}{2}<x<\frac{1}{2}$ and $-\frac{2}{3}<x<\frac{2}{3}$ $\Rightarrow -\frac{1}{2}<x<\frac{1}{2}$ Therefore, the expansion of $\frac{\sqrt{1-2x}}{(2+3x)}$ is valid for $-\frac{1}{2}<x<\frac{1}{2}$.

11. 2020/Promo/NYJC/Q10

A curve C has parametric equation

$$x=t^2-1, \qquad y=4t-t^3+15, \quad \text{where } t\in\mathbb{R}.$$

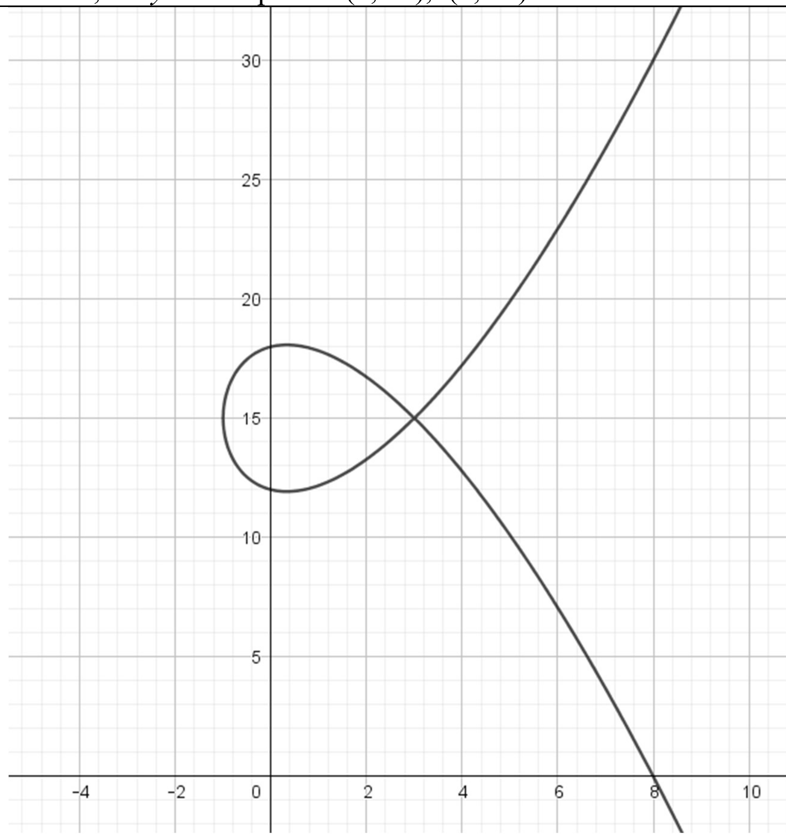
(i) Show that the curve C cuts the x-axis at the point (8, 0) and the y-axis at (0, 12) and (0, 18). [3]

(ii) Sketch C, giving the equation of the line of symmetry. [2]

(iii) Find the exact coordinates of the point where C crosses itself. [2]

(iv) Show that the equation of the tangent at the point (8, 0) is $23x+6y-184=0$. Hence, find the coordinates of the point at which the tangent cuts the curve again. [4]

11(i)	Show that the curve C cuts the x-axis at (8, 0). Let $y=0$. $y=4t-t^3+15=0 \quad\Rightarrow\quad t=3$. Thus $x=3^2-1=8$. Hence, the curve C cuts the x-axis at (8, 0). To find the y-intercepts, we let $x = 0$, then $x=t^2-1=0 \quad\Rightarrow\quad t=1,\quad -1.$

	When $t = 1$, $y = 4t - t^3 + 15 = 4 - 1 + 15 = 18$ When $t = -1$, $y = 4t - t^3 + 15 = -4 + 1 + 15 = 12$ Hence , the y-intercepts are (0, 18), (0, 12)
(ii)	 When $t = 0$, $y = 4t - t^3 + 15 = 15$. Equation of the symmetric axis is: $y = 15$.
(iii)	Observe that the point where C crosses itself lies on the symmetric axis with equation $y = 15$, thus its y-ordinate is 15. Substitute $y = 15$ into $y = 4t - t^3 + 15$, $15 = 4t - t^3 + 15 \Rightarrow t(4 - t^2) = 0 \Rightarrow t = 0, \quad 2, \quad -2$ When $t = 0$, $x = t^2 - 1 = -1$, $y = 15$, $(-1, 15)$ is not the required point. When $t_1 = \pm 2$, $x = t^2 - 1 \Rightarrow x = (\pm 2)^2 - 1 = 3$ Thus, the curve crosses itself at $(3, 15)$ **Alternately**, Let $x = t_1^2 - 1 = t_2^2 - 1$ ------------------------(1)

	and $y = 4t_1 - t_1^3 + 15 = 4t_2 - t_2^3 + 15$ -------------(2) where $t_1 \neq t_2$. From (1), $t_1 = t_2\ ,\ \ -t_2$ Obviously, we reject $t_1 = t_2$. Substitute $t_1 = -t_2$ into (2), $4(-t_2) - (-t_2)^3 = 4t_2 - t_2^3 \Rightarrow 8t_2 - 2t_2^3 = 0 \Rightarrow 2t_2(4 - t_2^2) = 0$ As $t_2 \neq 0$, $t_2^2 = 4 \Rightarrow t_2 = \pm 2$. Hence $t_1 = \mp 2$. From (1), $x = (\pm 2)^2 - 1 = 3$ From (2), $y = 4(2) - (2)^3 + 15 = 15$ or $y = 4(-2) - (-2)^3 + 15 = 15$ Thus, the curve crosses itself at $(3, 15)$
(iv)	$x = t^2 - 1t = \dfrac{13}{6} \Rightarrow \dfrac{dx}{dt} = 2t$ $y = 4t - t^3 + 15 \Rightarrow \dfrac{dy}{dt} = 4 - 3t^2$ $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{4-3t^2}{2t}$ At the point (8, 0), $t = 3$, $\dfrac{dy}{dx} = -\dfrac{4-3(3)^2}{2\times 3} = -\dfrac{23}{6}$ Equation of tangent at (8, 0) is : $\dfrac{y-0}{x-8} = -\dfrac{23}{6} \Rightarrow 23x + 6y - 184 = 0$ To find the point at which it cuts the curve again, we substitute $x = t^2 - 1$ and $y = 4t - t^3 + 15$ into the tangent equation, then $23(t^2 - 1) + 6(4t - t^3 + 15) - 184 = 0$ $\Rightarrow 23t^2 - 23 + 24t - 6t^3 + 90 - 184 = 0$ $\Rightarrow 6t^3 - 23t^2 - 24t + 117 = 0$ $\Rightarrow (t-3)^2(6t+13) = 0$ (can use GC to solve) $\Rightarrow t = 3$ (repeats), or $-\dfrac{13}{6}$ As $t = 3$ corresponds the point (8, 0), so $t = -\dfrac{13}{6}$. When $t = -\dfrac{13}{6}$, $x = t^2 - 1 = \left(\dfrac{13}{6}\right)^2 - 1 = \dfrac{133}{36}$ and $y = 4\left(-\dfrac{13}{6}\right) - \left(-\dfrac{13}{6}\right)^3 + 15 = \dfrac{3565}{216}$ It is $(\dfrac{133}{36}, \dfrac{3565}{216})$ or (3.69, 16.5)

12. 2021/Prelim/DHS/P1/Q2

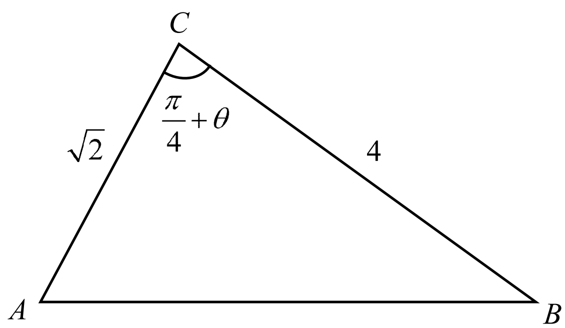
A triangle ABC is such that $AC=\sqrt{2}$, $BC=4$ and angle $ACB=\frac{1}{4}\pi+\theta$. Given that θ is sufficiently small for θ^3 and higher powers of θ to be neglected, show that

$$AB\approx\sqrt{10}\left[1+a\theta+b\theta^2\right],$$

where a and b are real constants. [5]

DHS Prelim 9758/2021/01/Q2

2



$$AB^2=\left(\sqrt{2}\right)^2+4^2-2\sqrt{2}(4)\cos\left(\frac{\pi}{4}+\theta\right)$$
$$=2+16-8\sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right)$$
$$=18-8\sqrt{2}\left(\cos\frac{\pi}{4}\cos\theta-\sin\frac{\pi}{4}\sin\theta\right)$$
$$=18-8\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta\right)$$
$$=18-8(\cos\theta-\sin\theta)$$
$$\approx18-8\left(1-\frac{\theta^2}{2}-\theta\right)\quad\text{(since }\theta\text{ is small)}$$
$$=18-8+4\theta^2+8\theta$$
$$=10+8\theta+4\theta^2$$

$$AB = \left(10 + 8\theta + 4\theta^2\right)^{\frac{1}{2}}$$

$$= \sqrt{10}\left[1 + \left(\frac{4}{5}\theta + \frac{2}{5}\theta^2\right)\right]^{\frac{1}{2}}$$

$$= \sqrt{10}\left[1 + \frac{1}{2}\left(\frac{4}{5}\theta + \frac{2}{5}\theta^2\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{4}{5}\theta + \frac{2}{5}\theta^2\right)^2 + \dots\right]$$

$$= \sqrt{10}\left[1 + \frac{2}{5}\theta + \frac{1}{5}\theta^2 - \frac{1}{8}\left(\frac{16}{25}\theta^2 + \dots\right) + \dots\right]$$

$$= \sqrt{10}\left(1 + \frac{2}{5}\theta + \frac{1}{5}\theta^2 - \frac{2}{25}\theta^2 + \dots\right)$$

$$\approx \sqrt{10}\left(1 + \frac{2}{5}\theta + \frac{3}{25}\theta^2\right)$$

13. 2021/Prelim/JPJC/P1/Q8

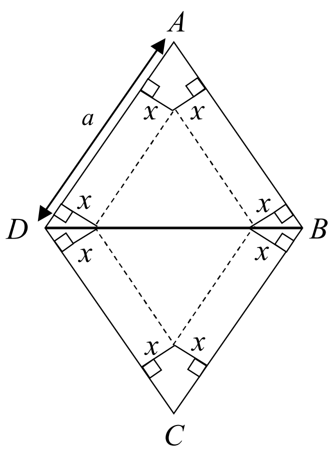


Fig. 1

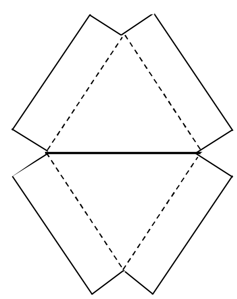


Fig. 2

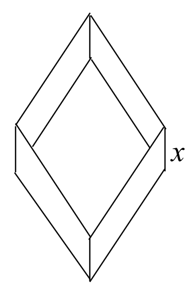


Fig. 3

Fig. 1 shows a piece of card, *ABCD*, in the form of a rhombus of sides *a* cm. The card is made up of 2 identical equilateral triangles *ABD* and *CBD*. Kite shapes are cut from each corner, to give the shape shown in Fig. 2. The remaining card in Fig. 2 is folded along the dotted lines, to form an open rhombus shaped prism of height *x* shown in Fig. 3.

(i) Show that the volume *V* of the prism is given by $V = \dfrac{\sqrt{3}x}{2}\left(a - \dfrac{4x}{\sqrt{3}}\right)^2$. [3]

(ii) Use differentiation to find, in terms of *a*, the maximum value of *V*, proving that it is a maximum. [6]

The prism is then filled with sand, evenly distributed through a sieve, at a constant rate of $\sqrt{3}$ cm^3/s. Find in terms of *a*, the rate of increase of the depth of the sand when the depth of the sand in the prism is $\sqrt{3}a$ cm. [3]

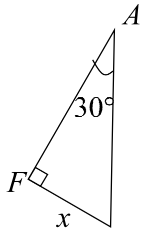
JPJC Prelim 9758/2021/01/Q8

Q8

Consider Vertex A

$\tan 30° = \dfrac{x}{AF}$

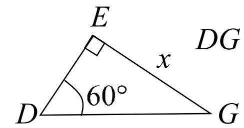
$AF = \dfrac{x}{\tan 30°}$



Consider Vertex D

$\tan 60° = \dfrac{x}{DE}$ or $\sin 60° = \dfrac{x}{DG}$

$DE = \dfrac{x}{\tan 60°}$ $DG = \dfrac{x}{\sin 60°}$



Length of box:

$$l = a - \frac{x}{\tan 30°} - \frac{x}{\tan 60°} = a - \sqrt{3}x - \frac{x}{\sqrt{3}} = a - \frac{4x}{\sqrt{3}} \quad \text{or} \quad l = a - 2\left(\frac{x}{\sin 60°}\right) = a - 2\left(\frac{2x}{\sqrt{3}}\right)$$

$$V = 2 \times \frac{1}{2} l^2 \sin 60° \times x = \frac{\sqrt{3}x}{2}\left(a - \frac{4x}{\sqrt{3}}\right)^2$$

$$\frac{dV}{dx} = \frac{\sqrt{3}}{2}\left(a - \frac{4x}{\sqrt{3}}\right)^2 - 4x\left(a - \frac{4x}{\sqrt{3}}\right) = \frac{1}{2}\left(a - \frac{4x}{\sqrt{3}}\right)\left(\sqrt{3}a - 12x\right)$$

At Stat pt, $\dfrac{dV}{dx} = 0$

$$\therefore a - \frac{4x}{\sqrt{3}} = 0 \quad \text{or} \quad \sqrt{3}a - 12x = 0$$

$$x = \frac{\sqrt{3}a}{4} \text{ or } \frac{\sqrt{3}a}{12}$$

$$\frac{d^2V}{dx^2} = \frac{1}{2}\left(-\frac{4}{\sqrt{3}}\left(\sqrt{3}a - 12x\right) - 12\left(a - \frac{4x}{\sqrt{3}}\right)\right) = \frac{48}{\sqrt{3}}x - 8a$$

$$x = \frac{\sqrt{3}a}{4}, \quad \frac{d^2V}{dx^2} = \frac{48}{\sqrt{3}}\left(\frac{\sqrt{3}a}{4}\right) - 8a = 4a > 0 \to \text{minimum, since } a > 0$$

$$x = \frac{\sqrt{3}a}{12}, \quad \frac{d^2V}{dx^2} = \frac{48}{\sqrt{3}}\left(\frac{\sqrt{3}a}{12}\right) - 8a = -4a < 0 \to \text{maximum, since } a > 0$$

$$\max V = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}a}{12}\right)\left(a - \frac{4}{\sqrt{3}}\left(\frac{a\sqrt{3}}{12}\right)\right)^2 = \frac{a^3}{18}$$

Let the volume and depth of the sand in the prism be V_s and x_s

$$\frac{dV_s}{dt} = \frac{dV_s}{dx_s} \times \frac{dx_s}{dt}$$

$$\frac{dx_s}{dt} = \frac{dV_s}{dt} \div \left.\frac{dV_s}{dx_s}\right|_{x_s = \sqrt{3}a}$$

$$\frac{dx_s}{dt} = \sqrt{3} \div \left(\frac{1}{2}\left(a - \frac{4\sqrt{3}a}{\sqrt{3}}\right)\left(\sqrt{3}a - 12\sqrt{3}a\right)\right)$$

$$\frac{dx_s}{dt} = \frac{2}{33a^2}$$

6 Integration & its Applications

<u>Basic Skills</u>

Integrate the following expressions with respect to x:

(a) $\int\left(6x^3-\frac{2}{\sqrt{x}}+1\right)\,\mathrm{d}x$ (b) $\int \sec 7x\,\mathrm{d}x$ (c) $\int \mathrm{e}^{4-5x}\,\mathrm{d}x$

(d) $\int \frac{\ln x}{x}\,\mathrm{d}x$ (e) $\int \mathrm{e}^{\cos^2 x}\sin 2x\,\mathrm{d}x$ (f) $\int \cos 5x \sin 3x\,\mathrm{d}x$

(g) $\int \frac{x^2-1}{1+x^2}\,\mathrm{d}x$ (h) $\int \frac{\mathrm{e}^x}{4+\mathrm{e}^{2x}}\,\mathrm{d}x$ (i) $\int \frac{x^3}{\sqrt{1-x^4}}\,\mathrm{d}x$

(j) $\int x^2\sin x\,\mathrm{d}x$ (k) $\int x^2\mathrm{e}^{2x}\,\mathrm{d}x$ (l) $\int \cos^{-1}x\,\mathrm{d}x$

(m) $\int \frac{1}{1+\sqrt{x}}\,\mathrm{d}x$ using $u^2=x$

(n) $\int \sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x$, using $x=\cos 2\theta$

(a) $\frac{4x^4}{2}-4\sqrt{x}+x+c$	(b) $\frac{\ln\lvert\sec 7x+\tan 7x\rvert}{7}+c$
(c) $\frac{\mathrm{e}^{4-5x}}{-5}+c$	(d) $\int\frac{\ln x}{x}\,\mathrm{d}x=\int\left(\frac{1}{x}\right)\ln x\,\mathrm{d}x=\frac{(\ln x)^2}{2}+c$
(e) $\int \mathrm{e}^{\cos^2 x}\sin 2x\,\mathrm{d}x$ $=-\int(-2\sin x\cos x)\mathrm{e}^{\cos^2 x}\mathrm{d}x$ $=-\mathrm{e}^{\cos^2 x}+c$	(f) $\int\cos 5x\sin 3x\,\mathrm{d}x=$ $\frac{1}{2}\int 2\cos 5x\sin 3x\,\mathrm{d}x$ $=\frac{1}{2}\int\sin 8x-\sin 2x\,\mathrm{d}x$ $=-\frac{\cos 8x}{16}+\frac{\cos 2x}{8}+c$
(g) $\int\frac{x^2-1}{1+x^2}\,\mathrm{d}x=\int 1-\frac{2}{1+x^2}\,\mathrm{d}x$ $=x-2\tan^{-1}x+c$	(h) $\int\frac{\mathrm{e}^x}{4+\mathrm{e}^{2x}}\,\mathrm{d}x=\int\frac{\mathrm{e}^x}{2^2+\mathrm{e}^{2x}}\,\mathrm{d}x$ $=\frac{1}{2}\tan^{-1}\left(\frac{\mathrm{e}^x}{2}\right)+c$
(i) $\int\frac{x^3}{\sqrt{1-x^4}}\,\mathrm{d}x=-\frac{1}{4}\int(-4x^3)\sqrt{1-x^4}\,\mathrm{d}x$ $=-\frac{\sqrt{1-x^4}}{2}+c$	(j) $\int x^2\sin x\,\mathrm{d}x=2\int x\cos x\,\mathrm{d}x-x^2\cos x$ $=2\left(x\sin x-\int\sin x\,\mathrm{d}x\right)-x^2\cos x$ $=2x\sin x+2\cos x-x^2\cos x+c$
(k) $\int x^2\mathrm{e}^{2x}\mathrm{d}x=\frac{1}{2}x^2\mathrm{e}^{2x}-\int x\mathrm{e}^{2x}\mathrm{d}x$ $=\frac{1}{2}x^2\mathrm{e}^{2x}-\left(\frac{1}{2}x\mathrm{e}^{2x}-\frac{1}{2}\int\mathrm{e}^{2x}\mathrm{d}x\right)$ $=\frac{1}{2}x^2\mathrm{e}^{2x}-\frac{1}{2}x\mathrm{e}^{2x}+\frac{1}{4}\mathrm{e}^{2x}+c$	(l) $\int\cos^{-1}x\,\mathrm{d}x=x\cos^{-1}x+\int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$ $=x\cos^{-1}x-\sqrt{1-x^2}+c$
(m) $\int\frac{1}{1+\sqrt{x}}\,\mathrm{d}x=\int\frac{1}{1+u}\cdot 2u\,\mathrm{d}u$ $=2\int 1-\frac{1}{1+u}\mathrm{d}u$ $=2(u-\ln\lvert 1+u\rvert)+c$ $=2\left(\sqrt{x}-\ln\left(1+\sqrt{x}\right)\right)+c$	(n) $\int\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x=\int\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}(-2\sin 2\theta)\mathrm{d}\theta$ $=-4\int\sin^2\theta\,\mathrm{d}\theta$ $=\sin 2\theta-2\theta+c$ $=\sqrt{1-x^2}-\cos^{-1}(x)+c$

Tutorial Review

Tutorial 6A Questions 15 and 16.
Tutorial 6B Questions 3, 10 and 13.

Revision Questions

1. 2020/Promo/MI/PU2/P1/Q1

(a) Find $\int \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x$. [2]

Method 1

$$\int \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1^2-(2x)^2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\sin^{-1}(2x)+c$$

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Method 2

$$\int \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x = \frac{1}{\sqrt{4}}\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\sin^{-1}\frac{x}{\sqrt{\frac{1}{4}}}+c$$

$$= \frac{1}{2}\sin^{-1}(2x)+c$$

(b) Express x^2+2x-1 in the form of $(x+a)^2+b$, where a and b are constants to be found.

Hence find $\int \frac{1}{x^2+2x-1}\,\mathrm{d}x$. [3]

$x^2+2x-1=(x+1)^2-2$

$a=1$ and $b=-2$

$$\int \frac{1}{x^2+2x-1}\,\mathrm{d}x = \int \frac{1}{(x+1)^2-2}\,\mathrm{d}x$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{x+1-\sqrt{2}}{x+1+\sqrt{2}}\right|+c$$

(c) Evaluate $\int_{-2}^{-1}\left(\frac{1}{x^2+2x-1}\right)^2\mathrm{d}x$, giving your answer correct to 3 decimal places. [1]

$$\int_{-2}^{-1}\left(\frac{1}{x^2+2x-1}\right)^2\mathrm{d}x = 0.40581 \approx 0.406$$

2. 2016/Promo/MJC/7

(a) Use the substitution $u=\sqrt{(x+1)}$ to find $\int \frac{x^2}{\sqrt{(x+1)}}\,\mathrm{d}x$. [4]

$$u=\sqrt{x+1} \Rightarrow u^2=x+1 \Rightarrow x=u^2-1$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x+1}}=\frac{1}{2u}$$

$$\begin{aligned}\int \frac{x^2}{\sqrt{x+1}}\,\mathrm{d}x &= \int \frac{\left(u^2-1\right)^2}{u}2u\,\mathrm{d}u \\ &= 2\int \left(u^2-1\right)^2\,\mathrm{d}u \\ &= 2\int u^4-2u^2+1\,\mathrm{d}u \\ &= \frac{2u^5}{5}-\frac{4u^3}{3}+2u+C \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{4}{3}(x+1)^{\frac{3}{2}}+2\sqrt{x+1}+C\end{aligned}$$

(b) Find $\int \frac{\mathrm{e}^{5x}}{\left(\mathrm{e}^{5x}-\mathrm{e}\right)^4}\,\mathrm{d}x$. [3]

$$\begin{aligned}\int \frac{\mathrm{e}^{5x}}{\left(\mathrm{e}^{5x}-\mathrm{e}\right)^4}\,\mathrm{d}x &= \int \mathrm{e}^{5x}\left(\mathrm{e}^{5x}-\mathrm{e}\right)^{-4}\mathrm{d}x \\ &= \frac{1}{5}\int 5\mathrm{e}^{5x}\left(\mathrm{e}^{5x}-\mathrm{e}\right)^{-4}\mathrm{d}x \\ &= \frac{\frac{1}{5}\left(\mathrm{e}^{5x}-\mathrm{e}\right)^{-3}}{-3}+C \\ &= -\frac{1}{15}\left(\mathrm{e}^{5x}-\mathrm{e}\right)^{-3}+C\end{aligned}$$

(c) Show that $\frac{x^2}{1+x^2}=1-\frac{1}{1+x^2}$. [1]

$$\text{RHS}=1-\frac{1}{1+x^2}=\frac{1+x^2-1}{1+x^2}=\frac{x^2}{1+x^2}=\text{LHS}$$

Hence find the exact value of $\int_0^{\sqrt{3}} x\tan^{-1} x \, dx$. [4]

$$\int_0^{\sqrt{3}} x\tan^{-1} x \, dx = \left[\frac{1}{2}x^2 \tan^{-1} x\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \left(\frac{1}{2}x^2\right)\left(\frac{1}{1+x^2}\right) dx$$

$$= \left[\frac{1}{2}x^2 \tan^{-1} x\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} 1 - \frac{1}{1+x^2} \, dx$$

$$= \left[\frac{1}{2}x^2 \tan^{-1} x\right]_0^{\sqrt{3}} - \frac{1}{2}\left[x - \tan^{-1} x\right]_0^{\sqrt{3}}$$

$$= \left[\frac{1}{2}(3)\frac{\pi}{3}\right] - \frac{1}{2}\left[\sqrt{3} - \frac{\pi}{3}\right]$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

3. 2020/Promo/YIJC/Q7

(a) Find $\int x\tan^{-1}\left(2x^2\right) dx$. [3]

Let $u = \tan^{-1}\left(2x^2\right) \Rightarrow \frac{du}{dx} = \frac{4x}{1+\left(2x^2\right)^2} = \frac{4x}{1+4x^4}$

Let $\frac{dv}{dx} = x \Rightarrow v = \frac{x^2}{2}$

$$\int x\tan^{-1}\left(2x^2\right) dx$$

$$= \left(\tan^{-1}\left(2x^2\right)\right)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\left(\frac{4x}{1+4x^4}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1}\left(2x^2\right) - \frac{1}{2}\int\left(\frac{4x^3}{1+4x^4}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1}\left(2x^2\right) - \frac{1}{2(4)}\int\left(\frac{16x^3}{1+4x^4}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1}\left(2x^2\right) - \frac{1}{8}\ln\left(1+4x^4\right) + C$$

(b) Use the substitution $x = \sin\theta$ to find the exact value of $\int_{0.5}^{1} x^2\sqrt{1-x^2} \, dx$. [4]

$$\int_{0.5}^{1} x^2\sqrt{1-x^2} \, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin\theta)^2 \sqrt{1-(\sin\theta)^2} \cos\theta \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin\theta\cos\theta)^2 \, d\theta$$

When $x = 1$, $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

When $x = 0.5$, $\sin\theta = 0.5 \Rightarrow \theta = \frac{\pi}{6}$

$\frac{dx}{d\theta} = \cos\theta$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4}\, \mathrm{d}\theta$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}\, \mathrm{d}\theta$$

$$= \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$$

$$= \frac{1}{8}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right)$$

4. 2017/Prelim/DHS/P2/Q1

(i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\tan^2 x$. Hence evaluate $\int_0^{\frac{1}{4}\pi} \sec^2 x \tan x\, \mathrm{e}^{\tan^2 x}\, \mathrm{d}x$, leaving your answer in exact form. [3]

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, \mathrm{e}^{\tan^2 x}\, \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sec^2 x \tan x\, \mathrm{e}^{\tan^2 x}\, \mathrm{d}x$$

$$= \frac{1}{2}\left[\mathrm{e}^{\tan^2 x}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\mathrm{e}^{\tan^2 \frac{\pi}{4}} - \mathrm{e}^{\tan^2 0}\right)$$

$$= \frac{1}{2}(\mathrm{e} - 1)$$

(ii) By expressing $1 + 72x - 32x^3$ as $1 + mx(9 - 4x^2)$ where m is a constant, find $\int \frac{1 + 72x - 32x^3}{\sqrt{(9 - 4x^2)}}\, \mathrm{d}x.$ [2]

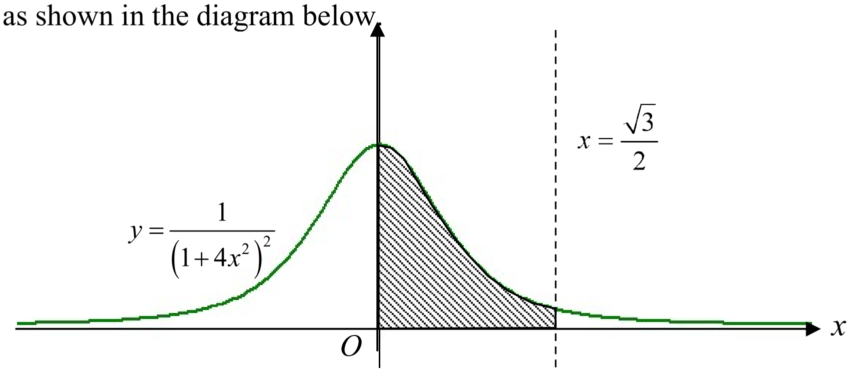
$$\int \frac{1 + 72x - 32x^3}{\sqrt{9 - 4x^2}}\, \mathrm{d}x = \int \frac{1 + 8x\left(9 - 4x^2\right)}{\sqrt{9 - 4x^2}}\, \mathrm{d}x$$

$$= \int \frac{1}{\sqrt{9 - 4x^2}} + 8x\left(9 - 4x^2\right)^{\frac{1}{2}}\, \mathrm{d}x$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) - \frac{2}{3}\left(9 - 4x^2\right)^{\frac{3}{2}} + C$$

5. 2011/Prelim/ACJC/P1/Q8

The region R is bounded by the curve $y=\dfrac{1}{\left(1+4x^2\right)^2}$, the line $x=\dfrac{\sqrt{3}}{2}$, the x-axis and the y-axis, as shown in the diagram below.



(i) Using the substitution $x=\dfrac{1}{2}\tan t$, find the exact area of R. **[6]**

$$x=\frac{1}{2}\tan t \;\Rightarrow\; \frac{dx}{dt}=\frac{1}{2}\sec^2 t$$

$$\text{Area } R=\int_0^{\frac{\sqrt{3}}{2}}\frac{1}{\left(1+4x^2\right)^2}\,\mathrm{d}x=\int_0^{\frac{\pi}{3}}\frac{1}{\left(1+\tan^2 t\right)^2}\left(\frac{1}{2}\sec^2 t\right)\mathrm{d}t$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{3}}\cos^2 t\ \mathrm{d}t=\frac{1}{2}\int_0^{\frac{\pi}{3}}\frac{1+\cos 2t}{2}\ \mathrm{d}t$$

$$=\frac{1}{4}\left[t+\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{3}}=\frac{\pi}{12}+\frac{\sqrt{3}}{16}\ \text{units}^2$$

(ii) Find the volume of the solid formed when R is rotated completely about the y-axis. **[3]**

$$\text{Volume}=\pi\int_{\frac{1}{16}}^{1}\frac{1}{4}\left(\frac{1}{\sqrt{y}}-1\right)\mathrm{d}y+\pi\left(\frac{\sqrt{3}}{2}\right)^2\frac{1}{16}$$

$$=\frac{3\pi}{16}=0.589\ \text{units}^3$$

6. 2020/Promo/HCI/Q8

(i) Find $\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x$. [4]

(ii) A curve has parametric equations

$$x=\cos\theta,\quad y=\mathrm{e}^{2\theta},\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}.$$

(a) Sketch the curve, giving the exact coordinates of any points of intersection with the axes. [2]

(b) Find the exact area of the region bounded by the curve and the y-axis. [4]

6(i)

$\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x$

$=\frac{1}{2}\mathrm{e}^{2x}\cos x+\frac{1}{2}\int \mathrm{e}^{2x}\sin x\ \mathrm{d}x$

$=\frac{1}{2}\mathrm{e}^{2x}\cos x+\frac{1}{2}\left(\frac{1}{2}\mathrm{e}^{2x}\sin x-\frac{1}{2}\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x\right)$

$\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x=\frac{1}{2}\mathrm{e}^{2x}\cos x+\frac{1}{4}\mathrm{e}^{2x}\sin x-\frac{1}{4}\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x$

$\frac{5}{4}\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x=\frac{1}{2}\mathrm{e}^{2x}\cos x+\frac{1}{4}\mathrm{e}^{2x}\sin x$

$\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x=\frac{1}{5}\mathrm{e}^{2x}\left(2\cos x+\sin x\right)+C$

Alternative Solution

$\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x$

$=\mathrm{e}^{2x}\sin x-\int 2\mathrm{e}^{2x}\sin x\ \mathrm{d}x$

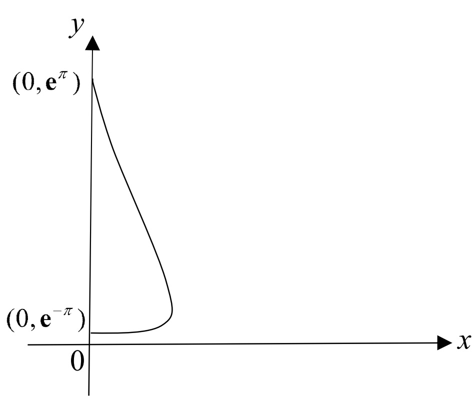
$=\mathrm{e}^{2x}\sin x-2\left(-\mathrm{e}^{2x}\cos x+\int 2\mathrm{e}^{2x}\cos x\ \mathrm{d}x\right)$

$\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x=\mathrm{e}^{2x}\sin x+2\mathrm{e}^{2x}\cos x-4\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x$

$5\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x=2\mathrm{e}^{2x}\cos x+\mathrm{e}^{2x}\sin x$

$\int \mathrm{e}^{2x}\cos x\ \mathrm{d}x=\frac{1}{5}\mathrm{e}^{2x}\left(2\cos x+\sin x\right)+C$

(ii)(a)



(ii)(b)

Area

$=\int_{\mathrm{e}^{-\pi}}^{\mathrm{e}^{\pi}} x\ \mathrm{d}y$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\mathrm{e}^{2\theta}\cos\theta\ \mathrm{d}\theta$

$=2\left[\frac{1}{5}\mathrm{e}^{2\theta}\left(2\cos\theta+\sin\theta\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$=\frac{2}{5}\left[\mathrm{e}^{\pi}\left(2\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\right)-\mathrm{e}^{-\pi}\left(2\cos\frac{\pi}{2}-\sin\frac{\pi}{2}\right)\right]$

	$=\dfrac{2}{5}\left(e^{\pi}+e^{-\pi}\right)$

7. **MI PU2 Promo 9758/2019/02/Q6**

The curve C_1 has equation $y=\sqrt{x+2}$.

(i) Find the exact volume of revolution when the region bounded by C_1, the *x*- and *y*-axes is rotated 2π radians about the *x*-axis. [2]

(ii) By considering $\sec^3 x$ as $\sec x \sec^2 x$, show that

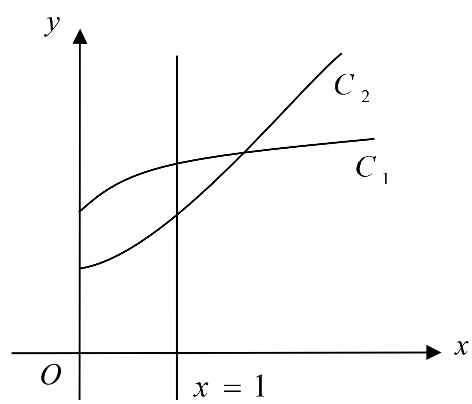
$$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + c,$$

where *c* is an arbitrary constant. [4]

(iii) A second curve C_2 has parametric equations

$$x=\tan\theta,\ \ y=\sec\theta,\ \ \text{for } -\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$

The diagram shows the curves C_1, C_2, and the line $x=1$, for $x \geq 0$.



Find the area bounded by C_1, C_2, the *y*-axis and the line $x=1$. Give your answer in the form $a\sqrt{3}+b\sqrt{2}+c\ln\left(\sqrt{2}+1\right)$, where *a*, *b* and *c* are constants to be determined. [4]

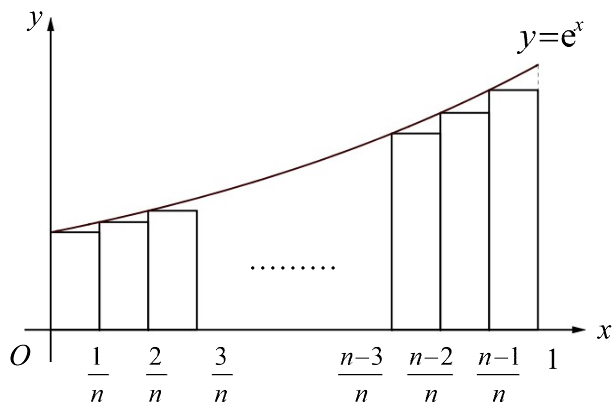
(i)	The required volume $=\pi\int_{-2}^{0} y^2 \ dx$ $=\pi\int_{-2}^{0} x+2 \ dx$ $=\pi\left[\dfrac{x^2}{2}+2x\right]_{-2}^{0}=\pi\left[0-\left(\dfrac{4}{2}+4\right)\right]=2\pi$ units^3

(ii)	$u = \sec x \quad , \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \sec x \tan x \quad , \quad v = \tan x$ $\int \sec^3 x \ \mathrm{d}x = \int \sec x \sec^2 x \ \mathrm{d}x$ $= \sec x \tan x - \int \sec x \tan^2 x \ \mathrm{d}x$ $= \sec x \tan x - \int \sec x(\sec^2 x - 1) \ \mathrm{d}x$ $= \sec x \tan x - \int \sec^3 x - \sec x \ \mathrm{d}x$ $= \sec x \tan x - \int \sec^3 x \ \mathrm{d}x + \int \sec x \ \mathrm{d}x$ $2\int \sec^3 x \ \mathrm{d}x = \sec x \tan x + \int \sec x \ \mathrm{d}x$ $\int \sec^3 x \ \mathrm{d}x = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\lvert\sec x + \tan x\rvert + c$
(iii)	Area under curve C_1 : $\int_0^1 (x+2)^{\frac{1}{2}} \ \mathrm{d}x = \left[\frac{2}{3}(x+2)^{\frac{3}{2}}\right]_0^1$ $= \frac{2}{3}\left[(1+2)^{\frac{3}{2}} - (0+2)^{\frac{3}{2}}\right]$ $= \frac{2}{3}\left[\sqrt{27} - \sqrt{8}\right]$ $= \frac{2}{3}\left[3\sqrt{3} - 2\sqrt{2}\right] = 2\sqrt{3} - \frac{4}{3}\sqrt{2}$ Area under curve C_2 : $x = \tan\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$ $\int_0^1 y \ \mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \sec\theta \sec^2\theta \ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \sec^3\theta \ \mathrm{d}\theta$ $= \left[\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\lvert\sec x + \tan x\rvert\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{2}\left[\sec x \tan x + \ln\lvert\sec x + \tan x\rvert\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{2}\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right) - 0\right]$ $= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\sqrt{2}+1\right)$

	The required area = Area under curve C_1 – Area under curve C_2 $= 2\sqrt{3} - \frac{4}{3}\sqrt{2} - \left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\sqrt{2}+1\right)\right]$ $= 2\sqrt{3} - \frac{11}{6}\sqrt{2} - \frac{1}{2}\ln\left(\sqrt{2}+1\right)$ units^2

8. 2017/Prelim/MJC/P2/Q4

The graph of $y=\mathrm{e}^x$, for $0 \le x \le 1$, is shown in the diagram below. Rectangles, each of width $\frac{1}{n}$ where n is an integer, are drawn under the curve.



(i) Show that the total area of all the n rectangles, A_n, is $\dfrac{c}{n\left(\mathrm{e}^{\frac{1}{n}}-1\right)}$, where c is an exact constant to be found. [3]

$$A_n = \frac{1}{n}\left(\mathrm{e}^0 + \mathrm{e}^{\frac{1}{n}} + \mathrm{e}^{\frac{2}{n}} + \mathrm{e}^{\frac{3}{n}} + \cdots + \mathrm{e}^{\frac{n-2}{n}} + \mathrm{e}^{\frac{n-1}{n}}\right)$$

$$= \frac{1}{n}\cdot\frac{\mathrm{e}^0\left(1-\left(\mathrm{e}^{\frac{1}{n}}\right)^n\right)}{1-\mathrm{e}^{\frac{1}{n}}}$$

$$= \frac{1}{n}\cdot\frac{1-\mathrm{e}}{1-\mathrm{e}^{\frac{1}{n}}} = \frac{\mathrm{e}-1}{n\left(\mathrm{e}^{\frac{1}{n}}-1\right)}$$

$\therefore c = \mathrm{e} - 1$

(ii) By considering the Maclaurin Series for $\mathrm{e}^x - 1$, or otherwise, find the value of $\lim_{x\to 0}\frac{1}{x}\left(\mathrm{e}^x - 1\right)$. [3]

$$\mathrm{e}^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\lim_{x \to 0} \frac{1}{x}\left(\mathrm{e}^x - 1\right) = \lim_{x \to 0}\left[\frac{1}{x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\right]$$

$$= \lim_{x \to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right)$$

$$= 1$$

(iii) Hence, without using integration, find the exact value of $\lim_{n \to \infty} A_n$. [2]

$$\lim_{n \to \infty} \frac{\mathrm{e} - 1}{n\left(\mathrm{e}^{\frac{1}{n}} - 1\right)} = \lim_{x \to 0} \frac{\mathrm{e} - 1}{\frac{1}{x}\left(\mathrm{e}^x - 1\right)}$$

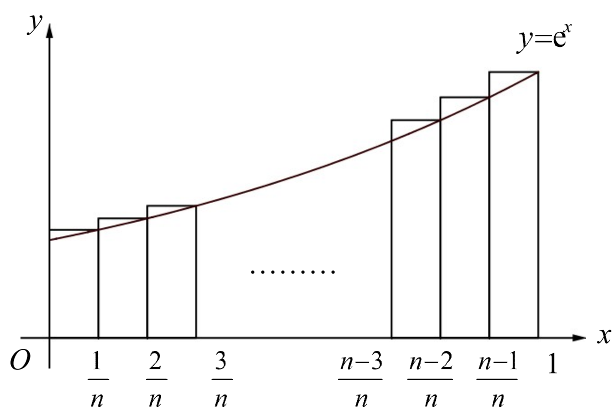
$$= \mathrm{e} - 1$$

(iv) Give a geometrical interpretation of the value you found in part **(iii)**, and verify your answer in part **(iii)** using integration. [2]

e – 1 is the exact area under the graph of $y = \mathrm{e}^x$ from $x = 0$ to $x = 1$.

area $= \int_0^1 \mathrm{e}^x \mathrm{d}x = \mathrm{e} - 1$.

Another set of n rectangles are drawn, as shown in the diagram below.

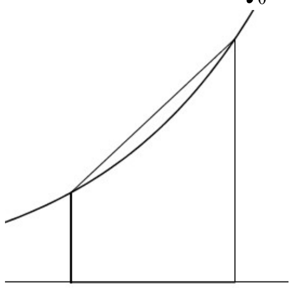


The total area of all the n rectangles in the second diagram is denoted by B_n. By considering the concavity of the graph of $y = \mathrm{e}^x$, or otherwise, show that

$$\frac{A_n + B_n}{2} > \int_0^1 \mathrm{e}^x \mathrm{d}x$$

for any positive integer n. [2]

Since the graph of $y = e^x$ is concave upwards, and $\frac{A_n + B_n}{2}$ is the sum of the area of n trapeziums each of width $\frac{1}{n}$, the area of all trapeziums will be greater than the exact area under the graph, which is $\int_0^1 e^x \, dx$.



9. **TJC JC1 Promo 9758/2019/Q5**

(a) Find $\int \sin x \cos 3x \, dx$. [2]

(b) Find $\int \frac{x-1}{\sqrt{1+2x-x^2}} \, dx$. Find the greatest integer value of b such that $\int_0^b \frac{x-1}{\sqrt{1+2x-x^2}} \, dx$ is defined. [5]

(c) Find $\int x \cos x \, dx$. Hence find the exact value of $\int_0^{2\pi} x|\cos x| \, dx$. [5]

(a)	$\int \sin x \cos 3x \, dx = \int \cos 3x \sin x \, dx$ $= \frac{1}{2}\int \sin 4x - \sin 2x \, dx$ $= -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + c$
(b)	$\int \frac{x-1}{\sqrt{1+2x-x^2}} \, dx = -\frac{1}{2}\int \frac{2(1-x)}{\sqrt{1+2x-x^2}} \, dx$ $= -\frac{1}{2}\frac{\left(1+2x-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c$ $= -\sqrt{1+2x-x^2} + c$
	For $\frac{x-1}{\sqrt{1+2x-x^2}}$ to be defined, $1+2x-x^2 > 0$ $(x-1)^2 - 2 < 0$ $\left(x-1+\sqrt{2}\right)\left(x-1-\sqrt{2}\right) < 0$ $\Rightarrow 1-\sqrt{2} < x < 1+\sqrt{2}$ Greatest integer value of *b* is 2

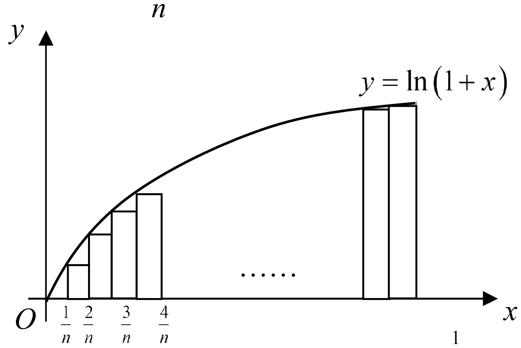
(c)	$\int x\cos x \, dx = x\sin x - \int \sin x \, dx$ $= x\sin x + \cos x + c$ $\int_0^{2\pi} x\lvert\cos x\rvert \, dx = \int_0^{\frac{\pi}{2}} x\cos x \, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos x \, dx + \int_{\frac{3\pi}{2}}^{2\pi} x\cos x \, dx$ $= \left[x\sin x + \cos x\right]_0^{\frac{\pi}{2}} - \left[x\sin x + \cos x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$ $+ \left[x\sin x + \cos x\right]_{\frac{3\pi}{2}}^{2\pi}$ (using above result) $= \left(\frac{\pi}{2} - 1\right) - \left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) + \left(1 + \frac{3\pi}{2}\right)$ $= 4\pi$

10. 2018/Prelim/HCI/P1/Q10

(a) (i) Find $\int x\sqrt{x-1} \, dx$ using integration by parts. [2]

(ii) The shape of a metal sculpture is formed by rotating the region bounded by the curve $y = \sqrt{a + x\sqrt{x-1}}$, where a is a positive integer, the lines $x = 1$ and $y = \sqrt{a+30}$, through 2π radians about the x-axis. Find the exact volume of the metal sculpture, giving your answer in terms of π. [4]

(b) (i) The diagram below shows a sketch of the graph of $y = \ln(1+x)$ for $0 \le x \le 1$. Rectangles each of width $\frac{1}{n}$ are drawn under the curve for $0 \le x \le 1$.



Show that A, the total area of all the rectangles, is given by

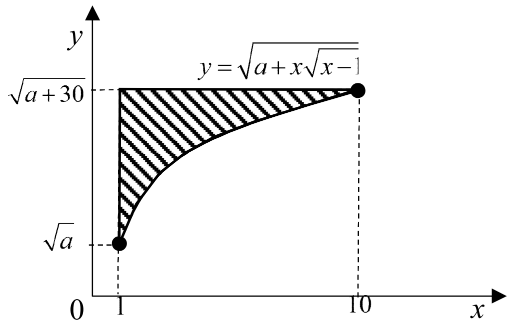
$$A = \frac{1}{n}\ln\left[\frac{(n+1)(n+2)(n+3)\ldots(2n-1)}{n^{n-1}}\right].$$ [2]

(ii) Find the exact value of $\lim_{n\to\infty} A$. [3]

10[14] (a)(i)

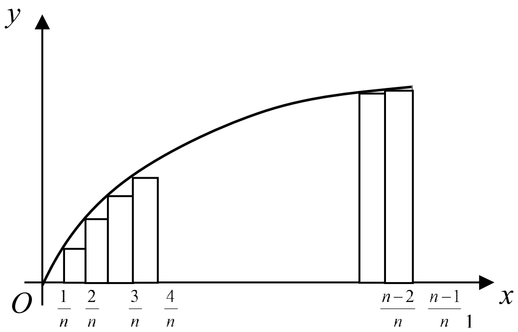
$$\int x\sqrt{x-1}\,\mathrm{d}x$$
$$=\frac{2x}{3}(x-1)^{\frac{3}{2}}-\int\frac{2}{3}(x-1)^{\frac{3}{2}}\,\mathrm{d}x$$
$$=\frac{2x}{3}(x-1)^{\frac{3}{2}}-\frac{4}{15}(x-1)^{\frac{5}{2}}+c$$
$$u=x\ \Rightarrow\ u'=1$$
$$v'=\sqrt{x-1}\ \Rightarrow\ v=\frac{2}{3}(x-1)^{\frac{3}{2}}$$

10 (a)(ii)



$$\sqrt{a+30}=\sqrt{a+x\sqrt{x-1}}$$
$$a+30=a+x\sqrt{x-1}$$
$$900=x^2(x-1)$$
$$x^3-x^2-900=0$$
$$x=10$$

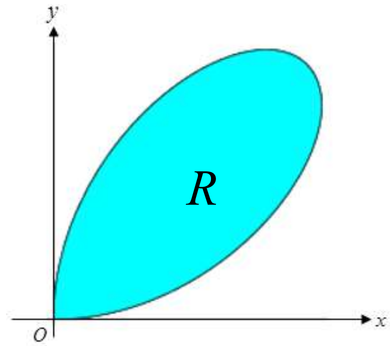
$$V_c=\int_1^{10}a+x\sqrt{x-1}\,\mathrm{d}x$$
$$=\pi\left[ax\right]_0^{10}+\int_1^{10}x\sqrt{x-1}\,\mathrm{d}x$$
$$=9a\pi+\int_1^{10}x\sqrt{x-1}\,\mathrm{d}x$$
$$=9a\pi+\left[\frac{2}{3}x(x-1)^{\frac{3}{2}}\right]_1^{10}-\frac{2}{3}\int_1^{10}(x-1)^{\frac{3}{2}}\,\mathrm{d}x$$
$$=9a\pi+\pi\left[\frac{2}{3}x(x-1)^{\frac{3}{2}}-\frac{4}{15}(x-1)^{\frac{5}{2}}\right]_1^{10}$$
$$=9a\pi+\pi\left[\frac{2700}{15}-\frac{972}{15}\right]$$
$$=\left(9a+\frac{1728}{15}\right)\pi\text{ units}^3$$
$$V=9\pi\left(\sqrt{a+30}\right)^2-V_c$$
$$=9\pi(a+30)-(9a+\frac{1728}{15})\pi$$
$$=\frac{774\pi}{5}\text{ units}^3$$

10 (b)(i)	 Area of 1st rectangle = $\frac{1}{n}\ln\left(1+\frac{1}{n}\right)$ Area of 2nd rectangle = $\frac{1}{n}\ln\left(1+\frac{2}{n}\right)$ Area of 3rd rectangle = $\frac{1}{n}\ln\left(1+\frac{3}{n}\right)$ ⋮ Area of $(n-1)$th rectangle = $\frac{1}{n}\ln\left(1+\frac{n-1}{n}\right)$ $A=\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\ln\left(1+\frac{3}{n}\right)+\cdots+\ln\left(1+\frac{n-1}{n}\right)\right]$ $=\frac{1}{n}\ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\cdots\left(1+\frac{n-1}{n}\right)\right]$ $=\frac{1}{n}\ln\left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\left(\frac{n+3}{n}\right)\cdots\left(\frac{2n-1}{n}\right)\right]$ $=\frac{1}{n}\ln\left[\frac{(n+1)(n+2)(n+3)\ldots(2n-1)}{n^{n-1}}\right]$
10 (b)(ii)	$u=\ln(1+x) \Rightarrow u'=\frac{1}{1+x}$ $v'=1\Rightarrow v=x$ $\lim_{n\to\infty} A=\int_0^1 \ln(1+x)\,\mathrm{d}x$ $=\left[x\ln(1+x)\right]_0^1-\int_0^1 1-\frac{1}{1+x}\,\mathrm{d}x$ $=\left[x\ln(1+x)\right]_0^1-[x-\ln(1+x)]_0^1$ $=\ln 2-[1-\ln 2]$ $=2\ln 2-1$ units ^{2}

11. 2017/JJC/P1/Q11

(a) The diagram below shows a section of *Folium of Descartes* curve which is defined parametrically by

$$x = \frac{3m}{1+m^3}, \quad y = \frac{3m^2}{1+m^3}, \quad m \geq 0 .$$



(i) It is known that the curve is symmetrical about the line $y = x$. Find the values of m where the curve meets the line $y = x$. [1]

$$x = \frac{3m}{1+m^3}, \quad y = \frac{3m^2}{1+m^3}, \quad m \geq 0$$

$$y = x$$

$$\frac{3m^2}{1+m^3} = \frac{3m}{1+m^3}$$

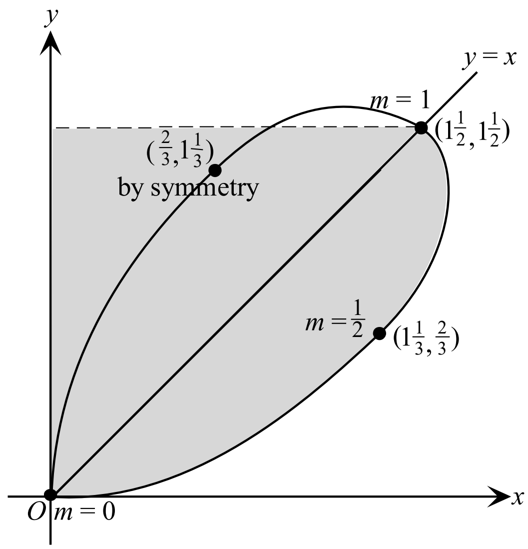
$$m(m-1) = 0$$

$$m = \underline{\underline{0 \text{ or } 1}}$$

(ii) Region R is the region enclosed by the curve in the first quadrant. Show that the area of R is given by $2\left(\int_0^{\frac{3}{2}} x \, \mathrm{d}y - \frac{9}{8}\right)$, and evaluate this integral. [5]

When $m = 0$, $y = 0$.

When $m = 1$, $y = \frac{3}{1+1} = \frac{3}{2}$.



Notes: Use GC to trace the path to see how m varies when the point moves along the path.

Area of (lower) half of the "leaf" is

$$\frac{1}{2}A = \int_0^{\frac{3}{2}} x \, \mathrm{d}y - \text{area of } \Delta \qquad \left(\text{Note: } \int_0^{\frac{3}{2}} x \, \mathrm{d}y = \text{shaded area}\right)$$

$$A = 2\left[\int_0^{\frac{3}{2}} x \, \mathrm{d}y - \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\right]$$

$$= 2\left(\int_0^{\frac{3}{2}} x \, \mathrm{d}y - \frac{9}{8}\right) \qquad \text{(Shown)}$$

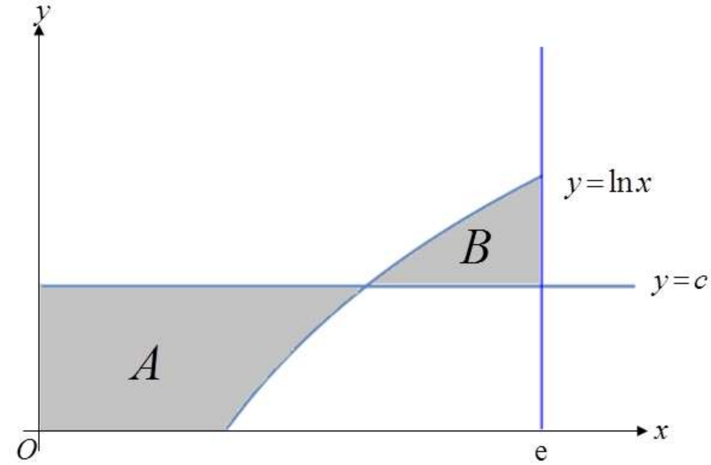
$$2\left(\int_0^{\frac{3}{2}} x \, \mathrm{d}y - \frac{9}{8}\right) = 2\int_0^1 \frac{3m}{1+m^3}\left[\frac{6m\left(1+m^3\right)-3m^2\left(3m^2\right)}{\left(1+m^3\right)^2}\right]\mathrm{d}m - \frac{9}{4}$$

$$= 2\int_0^1 \frac{3m\left(6m-3m^4\right)}{\left(1+m^3\right)^3}\mathrm{d}m - \frac{9}{4}$$

$$= \frac{15}{4} - \frac{9}{4} \qquad \text{(by GC)}$$

$$= \frac{3}{2}$$

(b) The diagram below shows a horizontal line $y = c$ intersecting the curve $y = \ln x$ at a point where the x-coordinate is such that $1 < x < e$.



The region A is bounded by the curve, the line $y = c$, the x-axis and the y-axis while the region B is bounded by the curve and the lines $x = e$ and $y = c$. Given that the volumes of revolution when A and B are rotated completely about the y-axis are equal, show that

$$c = \frac{e^2 + 1}{2e^2}.$$

[6]

$$\begin{aligned}
y &= \ln x \\
x &= e^y \\
V_A &= \pi \int_0^c \left(e^y\right)^2 \, dy \\
&= \pi \int_0^c e^{2y} \, dy \\
&= \pi \left[\frac{1}{2} e^{2y}\right]_0^c \\
&= \frac{\pi}{2}\left(e^{2c} - 1\right) \\
V_B &= (1-c)\pi e^2 - \pi \int_c^1 \left(e^y\right)^2 dy \quad \text{or} \quad \pi \int_c^1 \left[e^2 - \left(e^y\right)^2\right] dy \\
&= \pi(1-c)e^2 - \pi\left[\frac{1}{2}e^{2y}\right]_c^1 \\
&= \pi(1-c)e^2 - \frac{\pi}{2}\left(e^2 - e^{2c}\right) \\
V_A &= V_B \\
\frac{\pi}{2}\left(e^{2c} - 1\right) &= \pi(1-c)e^2 - \frac{\pi}{2}\left(e^2 - e^{2c}\right) \\
e^{2c} - 1 &= 2e^2(1-c) - e^2 + e^{2c} \\
&= 2e^2 - 2ce^2 - e^2 + e^{2c} \\
2ce^2 &= e^2 + 1 \\
c &= \frac{e^2 + 1}{2e^2} \quad \text{(Shown)}
\end{aligned}$$

12. 2021/Prelim/HCI/P1/Q4

(a) Find $\int \frac{6\tan 3x}{1+\cos 6x}\,\mathrm{d}x$. [3]

(b) It is given that $I_n = \int_1^{\mathrm{e}} (\ln x)^n \mathrm{d}x$, for $n \in \mathbb{Z}^+$.

(i) Use integration by parts to show that

$$I_n = \mathrm{e} - nI_{n-1}.$$

[2]

(ii) The region bounded by the curve $y = (\ln x)^2$, the x-axis and the line $x = \mathrm{e}$ is rotated through 2π radians about the x-axis. Using the result in **(b)(i)**, find the exact volume of the solid formed. [3]

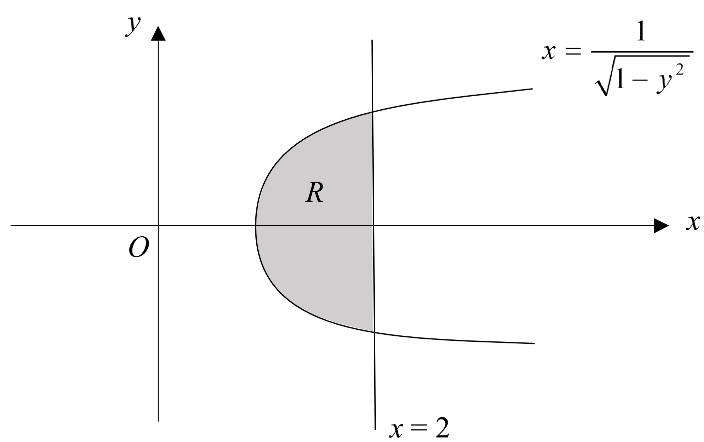
HCI Prelim 9758/2021/01/Q4

4(a)	$\int \frac{6\tan 3x}{1+\cos 6x}\mathrm{d}x = \int \frac{6\tan 3x}{1+(2\cos^2 3x - 1)}\mathrm{d}x$ $= \int \frac{6\tan 3x}{2\cos^2 3x}\mathrm{d}x$ $= \int 3\sec^2 3x \tan 3x \ \mathrm{d}x$ $= \frac{\tan^2 3x}{2} + C$ OR $\int \frac{3\sin 3x}{(\cos 3x)^3}\mathrm{d}x$ $= -\frac{(\cos 3x)^{-2}}{-2} + C$ $= \frac{1}{2}\sec^2 3x + C$	
4(b)(i)	$I_n = \int_1^{\mathrm{e}} (\ln x)^n \mathrm{d}x$ $= \left[x(\ln x)^n\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} xn(\ln x)^{n-1}\left(\frac{1}{x}\right)\mathrm{d}x$ $= [\mathrm{e}(\ln \mathrm{e})^n - 0] - n\int_1^{\mathrm{e}} (\ln x)^{n-1}\mathrm{d}x$ $= \mathrm{e} - nI_{n-1}$ (Shown)	
4(ii)	Required volume $= \pi\int_1^{\mathrm{e}} (\ln x)^4 \mathrm{d}x = \pi I_4$ <u>Method 1</u>: Using the result in (b)(i), $I_4 = \mathrm{e} - 4I_3$ $I_3 = \mathrm{e} - 3I_2$ $I_2 = \mathrm{e} - 2I_1$	

	where $I_1 = \int_1^{e} \ln x \, dx$ $= \left[x \ln x\right]_1^{e} - \int_1^{e} x\left(\frac{1}{x}\right) dx$ $= e - \int_1^{e} 1 \, dx$ $= e - \left[x\right]_1^{e}$ $= e - (e - 1)$ $= 1$ Using the result in (b)(i), $\therefore I_4 = e - 4I_3$ $= e - 4(e - 3I_2)$ $= e - 4e + 12(e - 2I_1)$ $= 9e - 24I_1$ $= 9e - 24$ Thus, the required volume is $(9e - 24)\pi$ units^3.	

13. 2021/Prelim/DHS/P1/Q4

The diagram shows the shaded region R bounded by curve C with equation $x = \frac{1}{\sqrt{1-y^2}}$ and the line $x = 2$.



(i) Find the exact volume of the solid generated when R is rotated through 2π radians about the y-axis. [5]

(ii) Write down the equation of the curve obtained when C is translated by 2 units in the negative x-direction. Hence, or otherwise, find the volume of the solid generated when R is rotated through 2π radians about the line $x = 2$. [3]

Qn	Solution
4(i)	When $x=2$, $2=\frac{1}{\sqrt{1-y^2}}$ $\sqrt{1-y^2}=\frac{1}{2}$ $1-y^2=\frac{1}{4}$ $y^2=\frac{3}{4}$ $y=\frac{\sqrt{3}}{2}$ or $y=-\frac{\sqrt{3}}{2}$ Volume of solid $=\pi(2^2)(\sqrt{3})-2\pi\int_0^{\frac{\sqrt{3}}{2}}\frac{1}{1-y^2}\,\mathrm{d}y$ $=4\sqrt{3}\pi-\frac{2\pi}{2}\left[\ln\left(\frac{1+y}{1-y}\right)\right]_0^{\frac{\sqrt{3}}{2}}$ $=4\sqrt{3}\pi-\pi\left[\ln\left(\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right)-\ln 1\right]$ $=4\sqrt{3}\pi-\pi\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$
(ii)	Eqn of new curve: $x+2=\frac{1}{\sqrt{1-y^2}}$ $\therefore x=\frac{1}{\sqrt{1-y^2}}-2$ Volume of solid $=2\pi\int_0^{\frac{\sqrt{3}}{2}}\left(\frac{1}{\sqrt{1-y^2}}-2\right)^2\,\mathrm{d}y$ $=3.7213$ $=3.72$ (3sf)

7 Differential Equations

Tutorial Review

Tutorial 7 Questions 3, 7 and 12.

Revision Questions

1. 2020/Prom0/MI/PU2/P1/Q5

(i) By substituting $y = vx$, where v is a function of x, show that the differential equation

$$\frac{dy}{dx} = \frac{y^2 + x^2}{y} + \frac{y}{x}$$

can be written as $\frac{dv}{dx} = \frac{v^2+1}{v}$. [3]

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{(vx)^2 + x^2}{vx} + v$$

$$x\frac{dv}{dx} = \frac{v^2x^2 + x^2}{vx} = \frac{v^2x + x}{v}$$

$$\therefore \frac{dv}{dx} = \frac{v^2x + x}{vx} = \frac{v^2+1}{v} \text{ (shown)}$$

(ii) Hence find the general solution of the differential equation, leaving your answer in the form $y^2 = f(x)$. [4]

$$\int \frac{v}{v^2+1}\,dv = \int 1\,dx$$

$$\frac{1}{2}\int \frac{2v}{v^2+1}\,dv = x + c$$

$$\frac{1}{2}\ln\left(v^2+1\right) = x + c \text{ (since } v^2 + 1 > 0 \text{ for all } v \in \mathbb{R})$$

$$\ln\left(v^2+1\right) = 2x + c'$$

$$v^2 + 1 = e^{2x+c'} = Ae^{2x}, A = e^{c'}$$

$$v^2 = Ae^{2x} - 1$$

$$\frac{y^2}{x^2} = Ae^{2x} - 1$$

$$\therefore y^2 = x^2\left(Ae^{2x} - 1\right)$$

2. 2020/Promo/DHS/Q7

(a) Find the particular solution of the differential equation $\frac{\mathrm{d}u}{\mathrm{d}t} = u + \frac{1}{u}$ given that $u = 0$ when $t = 2$. Give your answer in the form $u^2 = \mathrm{f}(t)$. [4]

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u + \frac{1}{u} = \frac{u^2+1}{u}$$

$$\frac{u}{u^2+1}\frac{\mathrm{d}u}{\mathrm{d}t} = 1$$

$$\int \frac{u}{u^2+1}\,\mathrm{d}u = \int 1\,\mathrm{d}t$$

$$\frac{1}{2}\int \frac{2u}{u^2+1}\,\mathrm{d}u = \int 1\,\mathrm{d}t$$

$$\frac{1}{2}\ln(u^2+1) = t + c, \quad \text{(note: } u^2+1>0)$$

Subt $u = 0,\ t = 2$:

$$\frac{1}{2}\ln(0+1) = 2 + c$$

$$c = -2$$

$$\ln(u^2+1) = 2(t-2)$$

$$u^2 = \mathrm{e}^{2t-4} - 1$$

(b) **(i)** Find $\int \frac{x}{\left(4-9x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x.$ [2]

$$\int \frac{x}{\left(4-9x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x = -\frac{1}{18}\int (-18x)\left(4-9x^2\right)^{-\frac{3}{2}}\,\mathrm{d}x$$

$$= -\frac{1}{18}\frac{\left(4-9x^2\right)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)} + C$$

$$= \frac{1}{9\sqrt{4-9x^2}} + C$$

(ii) Find the general solution of the differential equation $(4-9x^2)^{\frac{3}{2}}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=9x.$ [3]

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=9x(4-9x^2)^{-\frac{3}{2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\displaystyle\int\frac{9x}{(4-9x^2)^{\frac{3}{2}}}\,\mathrm{d}x$ $=9\displaystyle\int\frac{x}{(4-9x^2)^{\frac{3}{2}}}\,\mathrm{d}x$ $=\dfrac{1}{\sqrt{4-9x^2}}+C$ $y=\displaystyle\int\left(\frac{1}{\sqrt{4-9x^2}}+C\right)\mathrm{d}x$ $=\displaystyle\int\left(\frac{1}{3\sqrt{(\frac{2}{3})^2-x^2}}+C\right)\mathrm{d}x$ $=\dfrac{1}{3}\sin^{-1}\left(\dfrac{x}{(\frac{2}{3})}\right)+Cx+D$ $=\dfrac{1}{3}\sin^{-1}\dfrac{3x}{2}+Cx+D$

3. **NJC JC2 Prelim 9758/2019/02/Q2**

Find the general solution of the differential equation

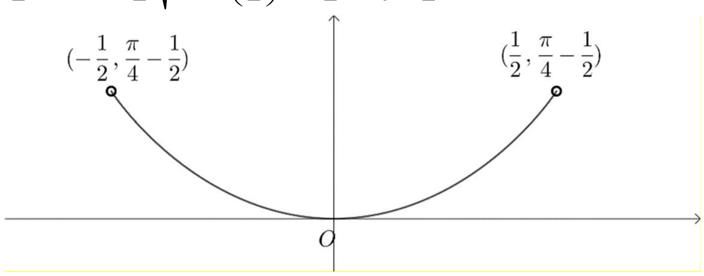
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{2}{\sqrt{1-4x^2}}\text{, where } -\frac{1}{2}<x<\frac{1}{2}.$$ [5]

The particular solution curve $y=\mathrm{f}(x)$ has a minimum point at the origin.

(i) Find $\mathrm{f}(x)$. [2]

(ii) Sketch the graph of this particular solution. [2]

(i)	$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{2}{\sqrt{1-4x^2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\displaystyle\int\frac{2}{\sqrt{1-4x^2}}\,\mathrm{d}x=\int\frac{2}{\sqrt{1-(2x)^2}}\,\mathrm{d}x=\sin^{-1}(2x)+C$

	$y=\int \sin^{-1}(2x)+C\ \mathrm{d}x$ $=\int 1\cdot\sin^{-1}(2x)\ \mathrm{d}x+Cx+D$ $=x\sin^{-1}(2x)-\int x\left(\frac{2}{\sqrt{1-4x^2}}\right)\mathrm{d}x+Cx+D$ $=x\sin^{-1}(2x)+\frac{1}{4}\int(-8x)(1-4x^2)^{-\frac{1}{2}}\ \mathrm{d}x+Cx+D$ $=x\sin^{-1}(2x)+\frac{1}{4}\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}}+Cx+D$ $=x\sin^{-1}(2x)+\frac{1}{2}\sqrt{1-4x^2}+Cx+D$
(b) (i)	When $x=0$, $y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=0$. $0=\sin^{-1}(0)+C\Rightarrow C=0$. $0=0\sin^{-1}(0)+\frac{1}{2}\sqrt{1-0}+C(0)+D\ \Rightarrow D=-\frac{1}{2}$ $\mathrm{f}(x)=x\sin^{-1}(2x)+\frac{1}{2}\sqrt{1-4x^2}-\frac{1}{2}$
(b) (ii)	$\frac{1}{2}\sin^{-1}1+\frac{1}{2}\sqrt{1-4\left(\frac{1}{2}\right)^2}-\frac{1}{2}=\frac{\pi}{4}-\frac{1}{2}$ 

4. 2017/HCI/Prelim/P1/Q10

Food energy taken in by a man goes partly to maintain the healthy functioning of his body and partly to increase body mass. The total food energy intake of the man per day is assumed to be a constant denoted by I (in joules). The food energy required to maintain the healthy functioning of his body is proportional to his body mass M (in kg). The increase of M with respect to time t (in days) is proportional to the energy not used by his body. If the man does not eat for one day, his body mass will be reduced by 1%.

(i) Show that I, M and t are related by the following differential equation:

$$\frac{\mathrm{d}M}{\mathrm{d}t}=\frac{I-aM}{100a}\ ,\ \text{where } a \text{ is a constant.}$$

State an assumption for this model to be valid. [3]

$\frac{dM}{dt} \propto I - kM$, where k is a positive constant.

$\frac{dM}{dt} = b(I - kM)$

If $I = 0$, $-\frac{1}{100}M = b(0 - kM)$

$-\frac{M}{100} = -bkM$

$b = \frac{1}{100k}$

$\frac{dM}{dt} = \frac{1}{100k}(I - kM) = \frac{I - kM}{100k}$

$= \frac{I - aM}{100a}$, where $a = k$ (shown)

Assumption (any 1 below):

- The man does not exercise so that no food energy is used up through exercising.
- The man does not fall sick so that no food energy is used up to help him recover from his illness.
- The man does not consume weight enhancing/loss supplements that affect his food energy gain/loss other than maintaining the healthy functioning of his body and increasing body mass.

(ii) Find the total food energy intake per day, I, of the man in terms of a and M if he wants to maintain a constant body mass. [1]

For $\frac{dM}{dt}$ to be zero, $I = aM$

It is given that the man's initial mass is 100kg.

(iii) Solve the differential equation in part **(i)**, giving M in terms of I, a and t. [3]

$\int \frac{a}{I - aM}\,dM = \int \frac{1}{100}\,dt$

$-\ln|I - aM| = \frac{t}{100} + C$

$\ln|I - aM| = \frac{-t}{100} - C$

$I - aM = \pm e^{\frac{-t}{100}} e^{-C} = Ae^{\frac{-t}{100}}$, where $A = \pm e^{-C}$

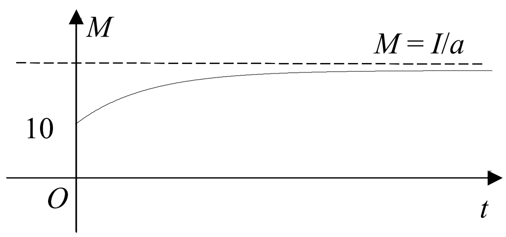
When $t = 0$, $M = 100 \Rightarrow A = I - 100a$

$I - aM = (I - 100a)e^{\frac{-t}{100}}$

$aM = I - (I - 100a)e^{\frac{-t}{100}}$

$M = \frac{I}{a} - \left(\frac{I}{a} - 100\right)e^{\frac{-t}{100}}$

(iv) Sketch the graph of M against t for the case where $I > 100a$. Interpret the shape of the graph with regard to the man's food energy intake. [3]



Explanation (any 1 below):

- The man consumes more food than is necessary for maintaining a healthy functioning body. Therefore the graph shows that his body mass will increase.
- Since $I > 100a$, hence $\frac{I}{a} > 100$. The man's body mass is always less than $\frac{I}{a}$.

  In the long run, the man's body mass will approach $\frac{I}{a}$.

(v) If the man's total food energy intake per day is $50a$, find the time taken in days for the man to reduce his body mass from 100kg to 90kg. [2]

Given $I = 50a$,

$$90 = 50 - (50 - 100)e^{\frac{-t}{100}}$$

Using equation found in **(iii)**

$$50e^{\frac{-t}{100}} = 40$$

$$e^{\frac{-t}{100}} = \frac{4}{5}$$

$$\frac{-t}{100} = \ln\frac{4}{5}$$

$$\therefore\ t = -100\ln\frac{4}{5} = 22.3 \text{ days} \quad (3 \text{ s.f.})$$

5. **HCI JC2 Prelim 9758/2019/02/Q4**

(a) When studying a colony of bugs, a scientist found that the birth rate of the bugs is inversely proportional to its population and the death rate is proportional to its population. The population of the bugs (in thousands) at time t days after they were first observed is denoted by P. It was found that when the population is 2000, it remains constant.

(i) Assuming that P and t are continuous variables, show that $\frac{dP}{dt} = k\left(\frac{4}{P} - P\right)$, where k is a constant. [3]

(ii) Given that the initial population of the bugs was 4000, and that the population was decreasing at the rate of 3000 per day at that instant, find P in terms of t. [4]

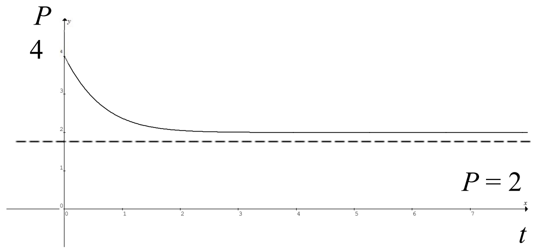
(iii) Sketch the graph of P against t, giving the equation of any asymptote(s). State what happens to the population of the bugs in the long run.

[2]

(b) Another population of bugs, N (in thousands) in time t days can be modelled by the differential equation $\frac{\mathrm{d}N}{\mathrm{d}t} = 4 + \frac{N}{t}$ for $t \geq 1$. Using the substitution $u = \frac{N}{t}$, solve this equation, given that the population was 1000 when $t = 1$.

[3]

5ai	$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{a}{P} - bP$, where a, b are constants When $P = 2$, $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{a}{2} - 2b = 0 \Rightarrow a = 4b$ $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{4b}{P} - bP$ $= b\left(\frac{4}{P} - P\right)$ $= k\left(\frac{4}{P} - P\right)$, where $k = b$
aii	When $P = 4$, $\frac{\mathrm{d}P}{\mathrm{d}t} = -3$. $\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(\frac{4}{4} - 4\right) = -3$ $\Rightarrow k = 1$ $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{4}{P} - P$ $= \frac{4 - P^2}{P}$ $\frac{\mathrm{d}t}{\mathrm{d}P} = \frac{P}{4 - P^2}$ $= -\frac{1}{2}\frac{-2P}{4 - P^2}$ $t = -\frac{1}{2}\ln\left\|4 - P^2\right\| + C$ $-2t = \ln\left\|4 - P^2\right\| - 2C$ $\ln\left\|4 - P^2\right\| = 2C - 2t$ $4 - P^2 = \pm e^{2C}e^{-2t}$ $= Ae^{-2t}$, where $A = \pm e^{2C}$ When $t = 0$, $P = 4$. $A = -12$

	$4-P^2=-12\mathrm{e}^{-2t}$ $P^2=4+12\mathrm{e}^{-2t}$ $P=2\sqrt{1+3\mathrm{e}^{-2t}}$
a iii	 The population of the bugs will decrease and approach 2000 in the long run.
b	**Method 1:** $u=\frac{N}{t}$ $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{t\frac{\mathrm{d}N}{\mathrm{d}t}-N}{t^2}$ $t\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\mathrm{d}N}{\mathrm{d}t}-\frac{N}{t}$ Substitute into $\frac{\mathrm{d}N}{\mathrm{d}t}=4+\frac{N}{t}$: $t\frac{\mathrm{d}u}{\mathrm{d}t}=4$ $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{4}{t}$ $u=4\ln t+C$ $\frac{N}{t}=4\ln t+C$ When $t=1$, $N=1\Rightarrow C=1$. $N=4t\ln t+t$ **Method 2:** $u=\frac{N}{t}$ $N=ut$ $\frac{\mathrm{d}N}{\mathrm{d}t}=\frac{\mathrm{d}u}{\mathrm{d}t}t+u$ Substitute into $\frac{\mathrm{d}N}{\mathrm{d}t}=4+\frac{N}{t}$:

	$\frac{\mathrm{d}u}{\mathrm{d}t}t+u=4+u$ $\frac{\mathrm{d}u}{\mathrm{d}t}t=4$ $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{4}{t}$ $u=4\ln t+C$ $\frac{N}{t}=4\ln t+C$ When $t=1$, $N=1\Rightarrow C=1$. $N=4t\ln t+t$

6. 2018/EJC Prelim/P1/Q10

An epidemiologist is studying the spread of a disease, dengue fever, which is spread by mosquitoes, in town A. P is defined as the number of infected people (in thousands) t years after the study begins. The epidemiologist predicts that the rate of increase of P is proportional to the product of the number of infected people and the number of uninfected people. It is known that town A has 10 thousand people of which a thousand were infected initially.

(i) Write down a differential equation that is satisfied by P. [1]

(ii) Given that the epidemiologist projects that it will take 2 years for half the town's population to be infected, solve the differential equation in **(i)** and express P in terms of t. [6]

(iii) Hence, sketch a graph of P against t [2]

A second epidemiologist proposes an alternative model for the spread of the disease with the following differential equation: $\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{2\cos t}{(2-\sin t)^2}$ (*).

(iv) Using the same initial condition, solve the differential equation (*) to find an expression of P in terms of t. [3]

(v) Find the greatest and least values of P predicted by the alternative model. [2]

(vi) The government of town A deems the alternative model as a more realistic model for the spread of the disease as it more closely follows the observed pattern of the spread of the disease. What could be a possible factor contributing to this? [1]

	(i) $\frac{\mathrm{d}P}{\mathrm{d}t}=kP(10-P)$
	(ii) $\frac{\mathrm{d}P}{\mathrm{d}t}=kP(10-P)$ **<u>Method 1 to integrate:</u>** $\int\frac{1}{P(10-P)}\mathrm{d}P=k\int\mathrm{d}t$ $\frac{1}{10}\int\frac{1}{P}+\frac{1}{10-P}\mathrm{d}P=k\int\mathrm{d}t$ $\frac{1}{10}\left[\ln\lvert P\rvert-\ln\lvert(10-P)\rvert\right]=kt+C$

$\frac{1}{10}\ln\left|\frac{P}{10-P}\right| = kt + c$

$\frac{1}{10}\ln\left(\frac{P}{10-P}\right) = kt + C$

$\ln\left(\frac{P}{10-P}\right) = 10kt + C_1$

$\frac{P}{10-P} = e^{10kt+C_1} = Ae^{10kt}$

Method 2 to integrate

$\int \frac{1}{P(10-P)}\,\mathrm{d}P = k\int \mathrm{d}t$

$\int \frac{1}{25-(P-5)^2}\,dP = k\int \mathrm{d}t$

$\frac{1}{10}\ln\left|\frac{5+(P-5)}{5-(P-5)}\right| = kt + c$

$\frac{1}{10}\ln\left|\frac{P}{10-P}\right| = kt + c$

From either Method 1 or 2,
since $P > 0, 10 - P \geq 0$

$\frac{1}{10}\ln\left(\frac{P}{10-P}\right) = kt + C$

$\ln\left(\frac{P}{10-P}\right) = 10kt + C_1$

$\frac{P}{10-P} = e^{10kt+C_1} = Ae^{10kt}$

Substitute in values into solution

Sub $t = 0, P = 1$

$\frac{P}{10-P} = e^{10kt+C_1} = Ae^{10kt}$

$\frac{1}{9} = Ae^0 \Rightarrow A = \frac{1}{9}$

$\frac{P}{10-P} = \frac{1}{9}e^{10kt}$

Sub $t = 2, P = 5$

$\frac{5}{10-5} = \frac{1}{9}e^{10(2)k}$

$1 = \frac{1}{9}e^{20k}$

$e^{20k} = 9 \Rightarrow k = \frac{1}{20}\ln(9) \approx 0.10986$

So we have

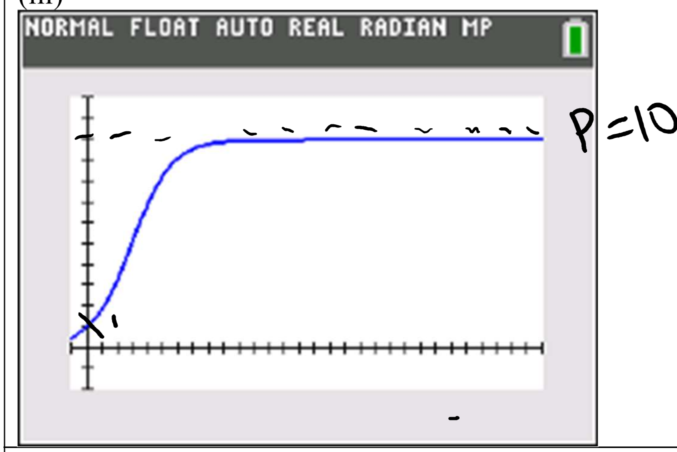
$$\frac{P}{10-P}=\frac{1}{9}e^{\frac{t}{2}\ln(9)}$$

$$9P=(10-P)e^{\frac{t}{2}\ln(9)}$$

$$P\left(9+e^{\frac{t}{2}\ln(9)}\right)=10e^{\frac{t}{2}\ln(9)}$$

$$P=\frac{10e^{\frac{t}{2}\ln(9)}}{9+e^{\frac{t}{2}\ln(9)}}$$

(iii)



(iv)

$$\frac{dP}{dt}=\frac{2\cos t}{(2-\sin t)^2}=(-2)(-\cos t)(2-\sin t)^{-2}$$

$$P=\frac{-2(2-\sin t)^{-1}}{-1}=\frac{2}{2-\sin t}+c$$

Sub $t=0, P=1$

$$1=\frac{2}{2-\sin 0}+c$$

$$c=1-1=0$$

Thus $P=\dfrac{2}{2-\sin t}$

(v)

Since $-1\le \sin t\le 1$,

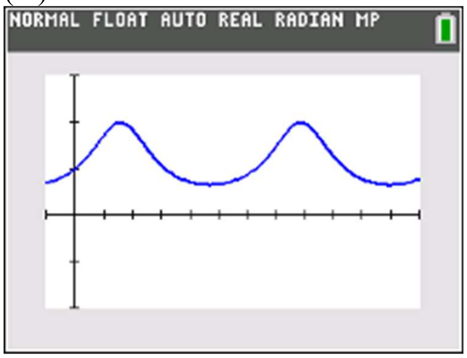
Largest value of P is when $\sin t=1$

Largest value of P = 2

Smallest value of P is when $\sin t=-1$

Smallest value of P = $\dfrac{2}{3}$

(vi)



We can use the GC to plot $P = \dfrac{2}{2-\sin t}$.

The second model could be deemed more suitable, as it shows oscillating values of P, which could correspond to the population of the mosquitoes which could vary seasonally. (For example, when the season is hot and rainy, the environment is more conducive for the breeding of mosquitoes.)

7. 2021/Prelim/NYJC/P2/Q2

Many industries use rectangular tanks to handle their water, wastewater and chemical storage and processing needs. Because of their shape, rectangular tanks can offer tremendous cost savings for shipping compared to cylindrical tanks.

In fabricating one such industrial-strength storage tank, a customer requires water to flow into the rectangular tank with a horizontal base area A, at a constant rate of n units of volume per unit time.

The water should flow out of the tank through a hole in the bottom, at a rate that is proportional to the square root of the depth of water x in the tank. It is also required that when the depth of the water in the tank is h, the level of water in the tank remains constant.

(i) Obtain a differential equation for the depth x at time t. [3]

(ii) It is known that the tank is filled to a depth of $4h$ initially. By using the substitution $x = hu^2$, show that u satisfies the differential equation $\dfrac{2Ah}{n}\dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{u-1}{u}$. [2]

(iii) By solving this differential equation, find, in terms of A, h and n, the time needed for the depth to reach $\dfrac{16}{9}h$. Describe how x varies with t. [6]

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2(i)	Let the volume of water in the tank be V. $\frac{\mathrm{d}V}{\mathrm{d}t}=n-k\sqrt{x}$ $\frac{\mathrm{d}(Ax)}{\mathrm{d}t}=n-k\sqrt{x}$ $A\frac{\mathrm{d}x}{\mathrm{d}t}=n-k\sqrt{x}$ When $x=h,\ \ \frac{\mathrm{d}x}{\mathrm{d}t}=0$, therefore $A\frac{\mathrm{d}x}{\mathrm{d}t}=n-k\sqrt{h}=0\Rightarrow k=\frac{n}{\sqrt{h}}$. $A\frac{\mathrm{d}x}{\mathrm{d}t}=n-k\sqrt{x}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{n}{A}\left(1-\sqrt{\frac{x}{h}}\right)$
2(ii)	Using $x=hu^2$, differentiating wrt t, $\frac{\mathrm{d}x}{\mathrm{d}t}=2hu\frac{\mathrm{d}u}{\mathrm{d}t}$ $A\frac{\mathrm{d}x}{\mathrm{d}t}=n-k\sqrt{x}\Rightarrow 2huA\frac{\mathrm{d}u}{\mathrm{d}t}=n-\frac{n}{\sqrt{h}}\sqrt{hu^2}$ $2huA\frac{\mathrm{d}u}{\mathrm{d}t}=n-nu=-n(u-1)$ $\frac{2Ah}{n}\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{u-1}{u}$
2(iii)	$\frac{2Ah}{n}\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{u-1}{u}$ Separating the variables, $\int\frac{u}{u-1}\mathrm{d}u=-\frac{n}{2Ah}\int\mathrm{d}t$ $\int 1+\frac{1}{u-1}\mathrm{d}u=-\frac{n}{2Ah}t+C$ $u+\ln(u-1)=-\frac{n}{2Ah}t+C$ When $x=4h,\ \ u=2$ When $x=\frac{16}{9}h,\ \ u=\frac{4}{3}$ When $x=h,\ \ u=1$ When $t=0,\ x=4h,\ \ u=2$, we have $C=2$ When $x=\frac{16}{9}h,\ \ u=\frac{4}{3}$, we have $\frac{4}{3}+\ln\left(\frac{1}{3}\right)=-\frac{n}{2Ah}t+2$ Therefore, $t=\frac{2Ah}{n}\left(\frac{2}{3}+\ln 3\right)$ As t increases, x decreases and approaches a depth of h.

8. 2021/Prelim/EJC/P1/Q5

(a) **(i)** It is given that $y\frac{\mathrm{d}y}{\mathrm{d}x}+x=\sqrt{x^2+y^2}$. Using the substitution $w=x^2+y^2$, show that the differential equation can be transformed to $\frac{\mathrm{d}w}{\mathrm{d}x}=\mathrm{f}(w)$, where the function $\mathrm{f}(w)$ is to be found. [2]

(ii) Hence, given that $y=4$ when $x=3$, solve the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x}+x=\sqrt{x^2+y^2}$. [3]

(b) Solve the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{x}$, where $x>0$. [4]

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5	(a)(i) $w=x^2+y^2$ Differentiating implicitly w.r.t. *x*, $\frac{\mathrm{d}w}{\mathrm{d}x}=2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=2\left(x+y\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ Thus, we have $x+y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\frac{\mathrm{d}w}{\mathrm{d}x}$ Substitute into the given DE: $\frac{1}{2}\frac{\mathrm{d}w}{\mathrm{d}x}=\sqrt{w}\Rightarrow\frac{\mathrm{d}w}{\mathrm{d}x}=2\sqrt{w}$
	(a)(ii) Separating variables: $\frac{1}{\sqrt{w}}\frac{\mathrm{d}w}{\mathrm{d}x}=2$ Integrating w.r.t *x*, $\int\frac{1}{\sqrt{w}}\frac{\mathrm{d}w}{\mathrm{d}x}\mathrm{d}x=\int 2\,\mathrm{d}x$ $\Rightarrow\int w^{-\frac{1}{2}}\mathrm{d}w=\int 2\,\mathrm{d}x\Rightarrow 2w^{\frac{1}{2}}=2x+C$ Substituting back, $2\sqrt{x^2+y^2}=2x+C$ When $x=3$, $y=4$: $2\sqrt{3^2+4^2}=2(3)+C\Rightarrow C=4$ The particular solution is $\sqrt{x^2+y^2}=x+2$ (or $x^2+y^2=(x+2)^2$)
	(b) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{x}$ Integrate w.r.t. *x*, $\frac{\mathrm{d}y}{\mathrm{d}x}=\ln x+C$ (note: no need $\|x\|$ because $x>0$) Integrate w.r.t. *x*, $y=x\ln x-\int x\cdot\frac{1}{x}\,\mathrm{d}x+Cx$ $=x\ln x-x+Cx+D$ $=x\ln x+C_1x+D$ (where $C_1=C-1$)