Solutions (Complex Numbers)

1(i)	$z = k + i$ $z^2 = (k+i)^2 = k^2 + 2(k)(i) + (i)^2 = (k^2-1) + (2k)i$ $z^3 = (k+i)^3 = k^3 + 3(k)^2(i) + 3(k)(i)^2 + (i)^3$ $= (k^3 - 3k) + (3k^2 - 1)i$ $z^3 - iz^2 - 2z - 4i = 0$ $[(k^3-3k) + (3k^2-1)i] - i[(k^2-1) + (2k)i] - 2[k+i] - 4i = 0$ $[(k^3-3k) + 2k - 2k] + i[(3k^2-1) - (k^2-1) - 2 - 4] = 0$ $(k^3-3k) + i(2k^2-6) = 0$ $k(k^2-3) = 0$ and $2k^2 - 6 = 0$ $(k = 0 \text{ or } k = \pm\sqrt{3})$ and $k = \pm\sqrt{3}$ Hence, $k = \pm\sqrt{3}$
(ii)	$z = \sqrt{3} + i \quad (\because k > 0)$ $\lvert z \rvert = \sqrt{1+3} = 2$ $\arg(z) = \frac{\pi}{6}$ **<u>Method 1: By Polar Form & Trigonometry</u>** $z = 2e^{i\pi/6} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ $z^n = 2^n e^{in\pi/6} = 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$ z^n is real $\Leftrightarrow \sin\frac{n\pi}{6} = 0$ $\Leftrightarrow \frac{n\pi}{6} = k\pi$, where $k \in \mathbb{Z}$

	$\Leftrightarrow n = 6k$, where $k \in \mathbb{Z}$ Hence, $n = 0, \pm 6, \pm 12, \pm 18, \ldots$ **<u>Method 2: By Properties of arg(z)</u>** $\arg(z^n) = n\arg(z) = \frac{n\pi}{6}$ z^n is real, the point representing z^n on the Argand diagram is on the *x*-axis. Thus, $\arg(z^n) = \frac{n\pi}{6} = k\pi$, where $k \in \mathbb{Z}$ $\therefore n = 6k$, where $k \in \mathbb{Z}$ i.e. $n = 0, \pm 6, \pm 12, \pm 18, \ldots$ Given $\left\|z^n\right\| > 100$. $\left\|z^n\right\| = \left\|z\right\|^n = 2^n$ Hence, $2^n > 100$ But *n* is a multiple of 6. We then have $2^6 = 64 < 100$ $2^{12} = 4096 > 100$ The least value of *n* is then 12.

2	$iz + 2w = 1 \Rightarrow -z + 2iw = i \Rightarrow z = 2iw - i$ ---(1) $4z + (2-i)w^* = -6$ ---(2) Sub (1) into (2) $4(2iw - i) + (3-i)w^* = -6$ Let $w = x + yi$ $8i(x+yi) + (3-i)(x-yi) = -6+4i$ $8xi - 8y + 3x - 3yi - xi - y = -6 + 4i$ $(-8y + 3x - y) + (8x - x - 3y)i = -6 + 4i$ Comparing: $-9y + 3x = -6 \Rightarrow -3y + x = -2$ ---(3) $7x - 3y = 4$ ----(4) *Solving* (3) & (4) $7(3y-2) - 3y = 4 \Rightarrow 18y = 18$ $\Rightarrow y = 1 \Rightarrow x = 1$ So $w = 1 + i \Rightarrow z = 2i(1+i) - i = -2 + i$

3	$z+(2+i)w=-9+16i$ (1) $z^*+w=3i$ (2) Substitute $w=3i-z^*$ into equation (1) $z+(2+i)(3i-z^*)=-9+16i$ $z+(-2-i)z^*+(-3+6i)=-9+16i$ $z+(-2-i)z^*=-6+10i$ Let $z=x+iy$ $(x+iy)+(-2-i)(x-iy)=-6+10i$ $(-x-y)+i(-x+3y)=-6+10i$ Equating real parts: $-x-y=-6 \Leftrightarrow x+y=6$ (3) Equating imaginary parts: $-x+3y=10$ (4) Solving equations (3) and (4): $x=2$ and $y=4$ $z=2+4i$ $w=3i-(2-4i)=-2+7i$

4(i)	$z=\dfrac{-(-2\mathrm{i})\pm\sqrt{(-2\mathrm{i})^2-4(1)(-2)}}{2(1)}$ $=\dfrac{2\mathrm{i}\pm\sqrt{4\mathrm{i}^2+8}}{2}$ $=\dfrac{2\mathrm{i}\pm 2}{2}$ $=\mathrm{i}\pm 1$ Note that $\arg(\mathrm{i}+1)=\dfrac{\pi}{4}$ and $\arg(\mathrm{i}-1)=\dfrac{3\pi}{4}$ Since $\arg(z_1)<\arg(z_2)$, $\therefore z_1=1+\mathrm{i}$ (shown)
(ii)	$x^2=(1+\mathrm{i})^2=1+2\mathrm{i}+\mathrm{i}^2=2\mathrm{i}$ $x^3=(2\mathrm{i})(1+\mathrm{i})=2\mathrm{i}+2\mathrm{i}^2=-2+2\mathrm{i}$ $x^4=(2\mathrm{i})^2=4\mathrm{i}^2=-4$ $(1+\mathrm{i})^4-6(1+\mathrm{i})^3+s(1+\mathrm{i})^2-18(1+\mathrm{i})+10=0$ $-4-6(-2+2\mathrm{i})+s(2\mathrm{i})-18-18\mathrm{i}+10=0$ By comparing imaginary parts, $-12+2s-18=0$ $\therefore s=15$ Since the coefficients of the equation are all real, and $1+\mathrm{i}$ is a root of the equation, $1-\mathrm{i}$ is also a root of the equation.

	$\left[x-(1+\mathrm{i})\right]\left[x-(1-\mathrm{i})\right]=(x-1)^2-\mathrm{i}^2$ $=x^2-2x+2$ By long division, $x^4-6x^3+15x^2-18x+10=\left(x^2-2x+2\right)\left(x^2-4x+5\right)$ Solving $x^2-4x+5=0$, $x=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}$ $=\dfrac{4\pm\sqrt{-4}}{2}$ $=2\pm\mathrm{i}$ The other roots are $1-\mathrm{i}$, $2+\mathrm{i}$ and $2-\mathrm{i}$.
(iii)	$\arg\left(\dfrac{{z_1}^n}{{z_1}^*}\right)=n\arg(z_1)-\arg\left({z_1}^*\right)$ $=n\arg(z_1)+\arg(z_1)$ $=(n+1)\dfrac{\pi}{4}$ Since $\dfrac{{z_1}^n}{{z_1}^*}$ is purely imaginary, $(n+1)\dfrac{\pi}{4}=\dfrac{\pi}{2}+k\pi$, where $k\in\mathbb{Z}$ $\dfrac{1}{4}(n+1)=\dfrac{1}{2}+k$ $n+1=2+4k$ $n=1+4k$ The two smallest positive integers of *n* are 1 and 5.

5(i)	The assumption is that *a*, *b* and *c* are all real.
(ii)	Let $x^3+ax^2+bx+c=(x-(3+\mathrm{i}))(x-(3-\mathrm{i}))(x-2)$ $=(x^2-6x+10)(x-2)$ $=x^3-8x^2+22x-20$ By comparing coefficients, we have *a* = –8, *b* = 22 and *c* = –20.

6	$(1-4\text{i})^2=-15-8\text{i}$ $\left(\frac{z}{2}+3\right)^2=(-1)(-15-8\text{i})=\text{i}^2(1-4\text{i})^2$ $\left(\frac{z}{2}+3\right)^2=(4+\text{i})^2$ $\frac{z}{2}+3=4+\text{i}$ or $\frac{z}{2}+3=-4-\text{i}$ $z=2+2\text{i}$ or $z=-14-2\text{i}$

7(i)	$z^2-6z+36=0\Rightarrow z=\frac{6\pm\sqrt{36-4(1)(36)}}{2}=3\pm3\sqrt{3}\text{i}$ Thus, $z_1=6e^{\text{i}\frac{\pi}{3}}$ and $z_2=6e^{-\text{i}\frac{\pi}{3}}$
(ii)	$\frac{\left(6e^{i\frac{\pi}{3}}\right)^4}{\left(6e^{i\left(\frac{\pi}{2}-\frac{\pi}{3}\right)}\right)}=6^3e^{i(\frac{7\pi}{6})}$ $=6^3\left[\cos\left(\frac{-5\pi}{6}\right)+i\sin\left(\frac{-5\pi}{6}\right)\right]$
(iii)	$z_2=6e^{-\text{i}\frac{\pi}{3}}\Rightarrow\quad z_2{}^n=6^ne^{\text{i}\left(-\frac{n\pi}{3}\right)}$ Since $z_2{}^n\in\mathbb{R}^+$, $-\frac{n\pi}{3}=2k\pi$ for some integer *k*. $n=-6k$. $n=\ldots,\ 12,\ 6,\ 0,\ -6,\ -12,\ \ldots$ Smallest positive integer n = 6.

8(i)	$kw^2+kww^*+\text{i}w-\text{i}w^*-1=0$ $kw(w+w^*)+\text{i}(w-w^*)-1=0$ $k(a+b\text{i})(2a)+\text{i}(2b\text{i})-1=0$ $\left(2ka^2-2b\right)+2abk\text{i}=1----(+)$ <u>Real part</u> $2ka^2-2b=1\Rightarrow b=\frac{2ka^2-1}{2}$ ---(1) <u>Im part</u>

	$ab=0 \quad \because k\neq 0$ $\Rightarrow b=0$ **or** $a=0$ ie, w is either purely real or imaginary.
(ii)	Hence Since w is real, $b=0$. Using $k=2$ and $b=0$ From part **(i)**: $\frac{2(2)a^2-1}{2}=0$ $4a^2=1\Rightarrow a=\pm\sqrt{\frac{1}{4}}$ ie, $w=-\frac{1}{2}$ or $w=\frac{1}{2}$ Otherwise Since w is real, $b=0$, ie, $w=a$ Using $k=2$ and $w=a$ eqn becomes: $2a^2+2a^2+\mathrm{i}a-\mathrm{i}a-1=0$ $4a^2=1\Rightarrow a=\pm\sqrt{\frac{1}{4}}$ ie, $w=-\frac{1}{2}$ or $w=\frac{1}{2}$

9	The statement is only true if p is real. (i) Using GC, p = 5. (ii) We have $z^4-z^3+4z^2+3z+5=(z-(1-2i))(z-(1+2i))(z^2+az+b)$, $= (z^2-2z+5)(z^2+az+b)$ Comparing coefficients of similar terms, we have $a = b = 1$ For $z^2+z+1=0$, we have $z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$
	$\left\|\frac{a^3}{a^*}\right\|=\frac{\|a\|^3}{\|a\|}=R^2$ arg(q) = arg (a^3) - arg (a^*) = 3 arg(a) + arg(a) = 4 α Thus, $q=R^2\left[\cos(4\alpha)+i\sin(4\alpha)\right]$ $\Rightarrow q^{\frac{1}{6}}= R^{\frac{1}{3}}\left[\cos\left(\frac{2}{3}\alpha\right)+i\sin\left(\frac{2}{3}\alpha\right)\right]$

	Given that $\cos\left(\frac{2}{3}\alpha\right)=0$, $\frac{2}{3}\alpha=\frac{\pi}{2}\Rightarrow\alpha=\frac{3\pi}{4}$ or 2.36 radians

10a	$\arg\left(w^5\right)=5\arg\left(w\right)=0,\pm\pi,\pm2\pi...$ $\arg\left(w\right)=0,\frac{\pi}{5},-\frac{\pi}{5},\frac{2\pi}{5},\frac{-2\pi}{5},...$ Since $k<0$, $\arg\left(w\right)=-\frac{\pi}{5}$ or $-\frac{2\pi}{5}$. $\frac{k}{\sqrt{3}}=\tan\left(-\frac{\pi}{5}\right)$ or $\frac{k}{\sqrt{3}}=\tan\left(-\frac{2\pi}{5}\right)$ $k=\sqrt{3}\tan\left(-\frac{\pi}{5}\right)$ or $k=\sqrt{3}\tan\left(-\frac{2\pi}{5}\right)$ $n=-\frac{1}{5}$ or $-\frac{2}{5}$
bi	**Method 1** $1-z^2=1-\left(\cos\theta+\mathrm{i}\sin\theta\right)^2$ $=1-\left(\cos^2\theta+2\mathrm{i}\cos\theta\sin\theta+\left(\mathrm{i}\sin\theta\right)^2\right)$ $=1-\left(1-\sin^2\theta+2\mathrm{i}\sin\theta\cos\theta-\sin^2\theta\right)$ $=1-1+2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$ **Method 2** $1-z^2=1-\left(\cos\theta+\mathrm{i}\sin\theta\right)^2$ $=1-\left(\cos2\theta+\mathrm{i}\sin2\theta\right)$ $=1-\cos2\theta-\mathrm{i}\sin2\theta$ $=1-\left(1-2\sin^2\theta\right)-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$
bii	**Method 1**

$$\left|1-z^2\right| = \left|2\sin\theta\left(\sin\theta - \mathrm{i}\cos\theta\right)\right|$$
$$= 2\sin\theta\sqrt{\sin^2\theta+\cos^2\theta}$$
$$= 2\sin\theta$$

Given that $0 \le \theta \le \frac{\pi}{2}$

$$\arg\left(1-z^2\right) = \arg\left[2\sin\theta\left(\sin\theta - \mathrm{i}\cos\theta\right)\right]$$
$$= \arg\left(2\sin\theta\right) + \arg\left(\sin\theta - \mathrm{i}\cos\theta\right)$$
$$= 0 - \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$
$$= -\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)$$
$$= -\left(\frac{\pi}{2}-\theta\right)$$
$$= \theta - \frac{\pi}{2}$$

Method 2

$$1-z^2 = 2\sin\theta\left(\sin\theta - \mathrm{i}\cos\theta\right)$$
$$= 2\sin\theta(-\mathrm{i})\left(\cos\theta + \mathrm{i}\sin\theta\right)$$
$$= \left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}$$
$$\left|1-z^2\right| = \left|\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}\right|$$
$$= 2\sin\theta$$

$$\arg\left(1-z^2\right) = \arg\left(\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}\right)$$
$$= \arg\left(-2\mathrm{i}\sin\theta\right) + \arg\left(\mathrm{e}^{\mathrm{i}\theta}\right)$$
$$= -\frac{\pi}{2} + \theta$$

Method 3

$$1-z^2 = 2\sin\theta\left(\sin\theta - \mathrm{i}\cos\theta\right)$$
$$= 2\sin\theta\left(\cos\left(\frac{\pi}{2}-\theta\right) - \mathrm{i}\sin\left(\frac{\pi}{2}-\theta\right)\right)$$
$$= 2\sin\theta\left(\cos\left(\theta-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta-\frac{\pi}{2}\right)\right)$$

	$\left\|1-z^2\right\| = 2\sin\theta$ $\arg\left(1-z^2\right) = \theta - \frac{\pi}{2}$

11i	$a^2b = \frac{1}{2}(1+\sqrt{3}\mathrm{i})^2(1-\mathrm{i})$ $= \frac{1}{2}(1+2\sqrt{3}\mathrm{i}-3)(1-\mathrm{i})$ $= (-1+\sqrt{3}\mathrm{i})(1-\mathrm{i})$ $= (\sqrt{3}-1)+(\sqrt{3}+1)\mathrm{i}$
ii	$\left\|a^2b\right\| = \left\|a\right\|^2\left\|b\right\|$ $= 2^2\left(\frac{1}{\sqrt{2}}\right)$ $= 2\sqrt{2}$ $\arg(a^2b) = 2\arg(a)+\arg(b)$ $= 2\left(-\frac{2\pi}{3}\right)-\frac{\pi}{4}$ $= -\frac{19\pi}{12}$ $\therefore \quad \arg(a^2b) = -\frac{19\pi}{12}+2\pi = \frac{5\pi}{12}$
iii	Considering the imaginary part of a^2b, we have $2\sqrt{2}\sin\frac{5\pi}{12} = \sqrt{3}+1$ $\Rightarrow \sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$
iv	*BA* can be obtained by rotating *BC* through 90^0 in the anticlockwise direction about *B*. $\mathrm{i}(c-b) = a-b$ $\Rightarrow c = -\mathrm{i}(a-b)+b$ $= -\mathrm{i}a+b(1+\mathrm{i})$ $= \mathrm{i}(1+\sqrt{3}\mathrm{i})+\frac{1}{2}(2)$

	$=(1-\sqrt{3})+\mathrm{i}$

12	Given that $z=1+k\mathrm{i}$ is a root, so substitute into the given equation $(1+k\mathrm{i})^4-(1+k\mathrm{i})^3-9(1+k\mathrm{i})^2+29(1+k\mathrm{i})-60=0$ $1+4k\mathrm{i}-6k^2-4k^3\mathrm{i}+k^4-(1+3k\mathrm{i}-3k^2-k^3\mathrm{i})-(9+18k\mathrm{i}-9k^2)+29+29k\mathrm{i}-60=0$ Comparing the real **or** imaginary parts on both sides, $4k-4k^3-3k+k^3-18k+29k=-3k^3+12k=0$ $\Rightarrow$ $\underline{\underline{k=\pm 2}}$ or $k=0$ (N.A.) **OR,** $-6k^2+k^4+3k^2-9+9k^2-31=k^4+6k^2-40=0$ By GC, $\Rightarrow$ $\underline{\underline{k=\pm 2}}$ Hence, $(z-(1-2\mathrm{i}))(z-(1+2\mathrm{i}))=z^2-2z+5$ is a factor of the given equation $z^4-z^3-9z^2+29z-60=(z^2-2z+5)(z^2+z-12)=0$ $(z^2-2z+5)=0$ or $(z^2+z-12)=0$ $\therefore \underline{\underline{z=1\pm 2\mathrm{i}\,,\ z=-4 \text{ or } z=3}}$

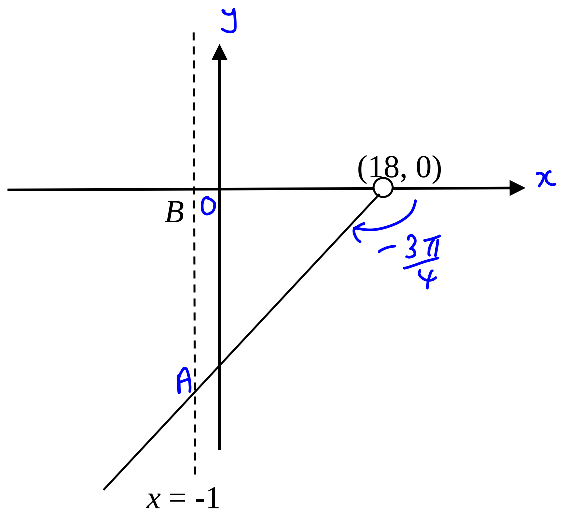
13i	$az^3-9z^2+bz-5=0$ $a(2-\mathrm{i})^3-9(2-\mathrm{i})^2+b(2-\mathrm{i})-5=0$ Using GC, $(2-11\mathrm{i})a-27+36\mathrm{i}+b(2-\mathrm{i})-5=0$ $(2a+2b-32)+(-11a-b+36)\mathrm{i}=0$ Comparing real parts, $2a+2b-32=0 \qquad ---(1)$ Comparing imaginary parts, $-11a-b+36=0 \qquad ---(2)$ Solving, $a=2, b=14$
ii	As a and b are real numbers, and $2-\mathrm{i}$ is a root, $2+\mathrm{i}$ is also a root. The third root must be a real number. A quadratic factor is

	$\left[z-(2-\mathrm{i})\right]\left[z-(2+\mathrm{i})\right]$ $=\left[(z-2)+\mathrm{i}\right]\left[(z-2)-\mathrm{i}\right]$ $=(z-2)^2-\mathrm{i}^2$ $=z^2-4z+4-(-1)$ $=z^2-4z+5$ $2z^3-9z^2+14z-5=\left(z^2-4z+5\right)(2z+c)$ Comparing constants, $5c=-5\Rightarrow c=-1$ $2z-1=0\Rightarrow z=\frac{1}{2}$ $\therefore$ The roots are $2-\mathrm{i}$, $2+\mathrm{i}$ and $\frac{1}{2}$.

14i	$p^*+10i=qi+5$ --------- (1) $\lvert p\rvert^2-q-5+2i=0 \Rightarrow q=\lvert p\rvert^2-5+2i$ Substitute into (1): $p^*+10i=\left(\lvert p\rvert^2-5+2i\right)i+5$ Let $p=x+yi$ $x-yi+10i=\left(x^2+y^2-5+2i\right)i+5$ Equating real parts: $x=-2+5=3$ Equating imaginary parts: $-y+10=x^2+y^2-5$ $\Rightarrow -y+10=9+y^2-5$ $\Rightarrow y^2+y-6=0$ $\Rightarrow y=-3$ or 2 (rejected as Im(p) < 0) Therefore $p=3-3i$. $\lvert p\rvert=\sqrt{3^2+3^2}=\sqrt{18}$ and $\arg(p)=-\frac{\pi}{4}$ $p^{2n}=\left(\sqrt{18}e^{-i\frac{\pi}{4}}\right)^{2n}$ $=\left(\sqrt{18}\right)^{2n}\left(\cos\frac{2n\pi}{4}-i\sin\frac{2n\pi}{4}\right)$ p^{2n} is purely imaginary $\Rightarrow \cos\frac{n\pi}{2}=0$ $\Rightarrow n=2k+1$, where $k\in\mathbb{Z}$
ii	$\arg\left(\frac{w}{p}-p^*\right)=-\frac{\pi}{2} \Rightarrow \arg\left(\frac{w-pp^*}{p}\right)=-\frac{\pi}{2}$ $\Rightarrow \arg\left(w-pp^*\right)-\arg(p)=-\frac{\pi}{2}$

$$\Rightarrow \arg(w-18)-\left(-\frac{\pi}{4}\right)=-\frac{\pi}{2}$$

$$\Rightarrow \arg(w-18)=-\frac{3\pi}{4}$$



$w+w^*=-2$

$\Rightarrow x=-1$

$\Rightarrow w$ is represented by point A

$\tan\frac{\pi}{4}=\frac{BA}{19} \quad \Rightarrow BA=19$

$\Rightarrow w=-1-19i$

15a

$z=w+2\mathrm{i}-1$ --- (1)

$z^2-\mathrm{i}w+\frac{5}{2}=0$ --- (2)

Method 1

From (1): $w=z-2\mathrm{i}+1$ --- (3)

Substitute (3) into (2):

$$z^2-\mathrm{i}(z-2\mathrm{i}+1)+\frac{5}{2}=0$$

$$z^2-\mathrm{i}z-\mathrm{i}+\frac{1}{2}=0$$

$$z=\frac{-(-\mathrm{i})\pm\sqrt{(-\mathrm{i})^2-4(1)\left(-\mathrm{i}+\frac{1}{2}\right)}}{2(1)}$$

$$=\frac{\mathrm{i}\pm\sqrt{-3+4\mathrm{i}}}{2}$$

$$=\frac{\mathrm{i}\pm(1+2\mathrm{i})}{2}$$

$z=\frac{1}{2}+\frac{3}{2}\mathrm{i},\ w=\frac{3}{2}-\frac{1}{2}\mathrm{i}$, or $z=-\frac{1}{2}-\frac{1}{2}\mathrm{i},\ w=\frac{1}{2}-\frac{5}{2}\mathrm{i}$,

	---- **Method 2** Substitute (1) into (2): $\left(w+2\mathrm{i}-1\right)^2-\mathrm{i}w+\frac{5}{2}=0$ $w^2+\left(2\mathrm{i}-1\right)^2+2\left(2\mathrm{i}-1\right)w-\mathrm{i}w+\frac{5}{2}=0$ $w^2+w\left(3\mathrm{i}-2\right)-\frac{1}{2}-4\mathrm{i}=0$ $w=\frac{-\left(3\mathrm{i}-2\right)\pm\sqrt{\left(3\mathrm{i}-2\right)^2-4\left(1\right)\left(-\frac{1}{2}-4\mathrm{i}\right)}}{2(1)}$ $w=\frac{-\left(3\mathrm{i}-2\right)\pm\left(1+2\mathrm{i}\right)}{2}$ $w=\frac{3}{2}-\frac{1}{2}\mathrm{i}\,,\ z=\frac{1}{2}+\frac{3}{2}\mathrm{i}$ or $w=\frac{1}{2}-\frac{5}{2}\mathrm{i}\,,\ z=-\frac{1}{2}-\frac{1}{2}\mathrm{i}$
b	$z=w-\frac{1}{w}=2\cos\theta+2\mathrm{i}\sin\theta-\left(\frac{1}{2}\cos\theta-\frac{1}{2}\mathrm{i}\sin\theta\right)=\frac{3}{2}\cos\theta+\frac{5}{2}\mathrm{i}\sin\theta$ $\mathrm{Re}(z)=\frac{3}{2}\cos\theta,\quad \mathrm{Im}(z)=\frac{5}{2}\sin\theta$

16i	$1+\mathrm{e}^{\mathrm{i}\theta}=\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}+\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right)$ $=\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(\cos\frac{\theta}{2}-\mathrm{i}\sin\frac{\theta}{2}+\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)$ $=\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(2\cos\frac{\theta}{2}\right)=2\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\cos\frac{\theta}{2}$

ii	$w=\dfrac{e^{i\theta}}{1+e^{i\theta}}$ $=\dfrac{e^{i\theta}}{2e^{i\frac{\theta}{2}}\cos\frac{\theta}{2}}=\dfrac{e^{i\frac{\theta}{2}}}{2\cos\frac{\theta}{2}}$ $=\dfrac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}=\frac{1}{2}+\frac{1}{2}i\tan\frac{\theta}{2}$ $\therefore \operatorname{Im}(w)=\frac{1}{2}\tan\frac{\theta}{2}$
17a	$z^3-2(2-i)z^2+(8-3i)z-5+i=0$ Let $z=x$ be the real root. $x^3-2(2-i)x^2+(8-3i)x-5+i=0$ $x^3-4x^2+2ix^2+8x-3ix-5+i=0$ $(x^3-4x^2+8x-5)+(2x^2-3x+1)i=0$ Since $z=x$ is a root, $x^3-4x^2+8x-5=0$ and $2x^2-3x+1=0$ From GC: $x=1$ Therefore, the real root is $z=1$ $z^3-2(2-i)z^2+(8-3i)z-5+i=0$ $(z-1)(z^2+Az+(5-i))=0$ $(z-1)(z^2+(-3+2i)z+(5-i))=0$ $z=1$ or $z^2+(-3+2i)z+(5-i)=0$ $z=\dfrac{-(-3+2i)\pm\sqrt{(-3+2i)^2-4(5-i)}}{2}$ $=\dfrac{-(-3+2i)\pm(1-4i)}{2}$ $=2-3i$ or $1+i$ Roots: 1, $2-3i$, $1+i$

bi	$1-u^2 = 1-(\cos\theta+\mathrm{i}\sin\theta)^2$ $= 1-\cos^2\theta+\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $= 2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $= 2\sin\theta(\sin\theta-2\mathrm{i}\cos\theta)$ $= -2\mathrm{i}\sin\theta(\cos\theta+\mathrm{i}\sin\theta)$ $= -2\mathrm{i}u\sin\theta$ **Alternative** $u = \cos\theta+\mathrm{i}\sin\theta = \mathrm{e}^{\theta\mathrm{i}}$ $1-u^2 = 1-\mathrm{e}^{2\theta\mathrm{i}}$ $= \mathrm{e}^{\theta\mathrm{i}}\left(\mathrm{e}^{-\theta\mathrm{i}}-\mathrm{e}^{\theta\mathrm{i}}\right)$ $= u(\cos\theta-\mathrm{i}\sin\theta-\mathrm{i}\sin\theta-\cos\theta)$ $= u(-2\mathrm{i}\sin\theta)$ $= -2\mathrm{i}u\sin\theta$ $\left\|1-u^2\right\| = \left\|-2\mathrm{i}u\sin\theta\right\| = \left\|-2\sin\theta\right\|\left\|\mathrm{i}\right\|\left\|u\right\|$ $= 2\sin\theta$ $\arg\left(1-u^2\right) = \arg(-2\mathrm{i}u\sin\theta)$ $= \arg(-2\mathrm{i}\sin\theta)+\arg(u)$ $= -\frac{\pi}{2}+\theta$
bii	$\left(1-u^2\right)^{10}$ is real and negative: $\arg\left(1-u^2\right)^{10} = 10\arg\left(1-u^2\right) = (2k+1)\pi, k\in\mathbb{Z}$ $10\left(-\frac{\pi}{2}+\theta\right) = (2k+1)\pi$ $-5\pi+10\theta = (2k+1)\pi$ $\theta = \frac{(2k+6)\pi}{10}$ $0<\theta<\frac{\pi}{2}$: $\theta = \frac{1}{5}\pi, \frac{2}{5}\pi$ **Alternative** $\left(1-u^2\right)^{10} = \left(2\sin\theta\mathrm{e}^{-\frac{\pi}{2}+\theta}\right)^{10}$ $= \left(2^{10}\sin^{10}\theta\right)\left(\cos(-5\pi+10\theta)+\mathrm{i}\sin(-5\pi+10\theta)\right)$

	Since $\left(1-u^2\right)^{10}$ is real and negative, and $2^{10}\sin^{10}\theta>0$, $\sin\left(-5\pi+10\theta\right)=0$ and $\cos\left(-5\pi+10\theta\right)<0$ $-5\pi+10\theta=k\pi \quad, k\in\mathbb{Z}$ $\theta=\dfrac{\left(k+5\right)\pi}{10}$ $0<\theta<\dfrac{\pi}{2}$: $\theta=\dfrac{1}{10}\pi,\dfrac{1}{5}\pi,\dfrac{3}{10}\pi,\dfrac{2}{5}\pi$ Only when $\theta=\dfrac{1}{5}\pi,\dfrac{2}{5}\pi$ will $\cos\left(-5\pi+10\theta\right)<0$. Therefore, $\theta=\dfrac{1}{5}\pi,\dfrac{2}{5}\pi$.

18	$\dfrac{z-8i}{z+6}=\dfrac{x+i(y-8)}{(x+6)+iy}=\dfrac{x(x+6)+y(y-8)+i(y-8)(x+6)-ixy}{(x+6)^2+y^2}$ $\mathrm{Re}(w)=0\Rightarrow\mathrm{Re}(\dfrac{z-8i}{z+6})=0$ $\dfrac{x(x+6)+y(y-8)}{(x+6)^2+y^2}=0\Rightarrow x^2+6x+y^2-8y=0$ $\Rightarrow(x+3)^2+(y-4)^2=5^2$ Therefore, locus is a circle of centre (-3,4) and radius 5. If w is real, Im(w)=0, ie $(y-8)(x+6)-xy=0\Rightarrow xy+6y-8x-48-xy=0$ $\Rightarrow 3y-4x=24$ which is a straight line.

19(a)	(i) $z=-1+2\mathrm{i}$, $w=1+b\mathrm{i}$ $\dfrac{w}{z}=\dfrac{1+b\mathrm{i}}{-1+2\mathrm{i}}\times\dfrac{-1-2\mathrm{i}}{-1-2\mathrm{i}}$ $=\dfrac{(1+b\mathrm{i})(-1-2\mathrm{i})}{1^2+2^2}$ $=\dfrac{-1+2b+\mathrm{i}(-2-b)}{5}$

	$\text{Im}(\frac{w}{z}) = \frac{-2-b}{5} = -\frac{3}{5}$ $b+2=3 \Rightarrow b=1$ (ii) $\arg(zw) = \arg((-1+2\text{i})(1-\text{i})) = 2.82$ (GC)
(b)	$\frac{1}{\text{e}^{\text{i}z}} = 2+\text{i}$ $\text{e}^{-\text{i}(a+\text{i}b)} = \sqrt{5}\text{e}^{\text{i}\tan^{-1}\frac{1}{2}}$ $\text{e}^{-\text{i}a+b} = \sqrt{5}\text{e}^{\text{i}\tan^{-1}\frac{1}{2}}$ $\text{e}^{b}\text{e}^{-\text{i}a} = \sqrt{5}\text{e}^{\text{i}\tan^{-1}\frac{1}{2}}$ Comparing : $\text{e}^{b} = \sqrt{5}$, $b = \frac{1}{2}\ln 5$ and $a = -\tan^{-1}\frac{1}{2}$

20	$1-z^2 = 1-(\cos 2\theta + \text{i}\sin 2\theta)$ $= 1-\cos 2\theta - \text{i}(2\sin\theta\cos\theta)$ $= 2\sin^2\theta - \text{i}(2\sin\theta\cos\theta)$ $= (-2i\sin\theta)(\cos\theta + \text{i}\sin\theta)$ $= (-2\text{i}\sin\theta)z$ (shown) Alternatively : $1-z^2 = 1-(e^{i2\theta})$ $= e^{i\theta}(e^{-i\theta} - e^{i\theta})$ $= e^{i\theta}(\cos\theta - i\sin\theta - \cos\theta - i\sin\theta)$ $= z(-2i\sin\theta)$ (Shown) $\left\|1-z^2\right\| = \left\|-2i\sin\theta\right\|\left\|z\right\| = 2\sin\theta$ $\arg(1-z^2) = \arg(-2\text{i}\sin\theta) + \arg(z)$ $= \arg(2\sin\theta) + \arg(-\text{i}) + \arg(z)$ $= \theta - \frac{\pi}{2}$

21	(i) $z = r\text{e}^{\text{i}\theta}$ is a root, $z = r\text{e}^{-\text{i}\theta}$ is another root. A quadratic factor of $\text{P}(z)$

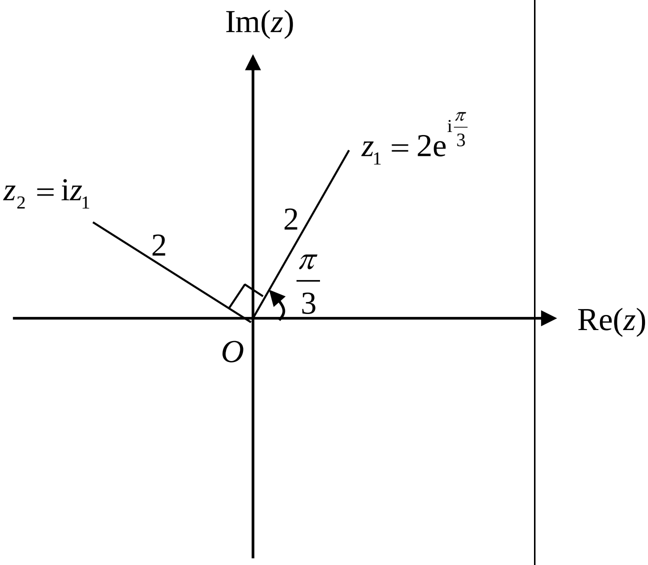
Note :
$z + z^* = r\text{e}^{\text{i}\theta} + r\text{e}^{-\text{i}\theta}$
$= 2r\cos\theta = 2x = 2\,\text{Re}(z)$
is a standard result that you may

$= \left(z - re^{i\theta}\right)\left(z - re^{-i\theta}\right)$

$= z^2 - zre^{-i\theta} - zre^{i\theta} + r^2$

$= z^2 - zr\left(e^{i\theta} + e^{-i\theta}\right) + r^2$

$= z^2 - 2rz\cos\theta + r^2$ (shown)



(ii) $z_2 = iz_1$

$|z_2| = |iz_1| = |i||z_1| = 2$

$\arg(z_2) = \arg(i) + \arg(z_1) = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

z_2 is an anti-clockwise rotation of z_1 about the origin by $\frac{\pi}{2}$.

(iii) $P(z) = \left[z^2 - 2(2)z\cos\frac{\pi}{3} + 2^2\right]\left[z^2 - 2(2)z\cos\frac{5\pi}{6} + 2^2\right]$

$= \left[z^2 - 4z\left(\frac{1}{2}\right) + 4\right]\left[z^2 - 4z\left(-\frac{\sqrt{3}}{2}\right) + 4\right]$

$= \left(z^2 - 2z + 4\right)\left(z^2 + 2\sqrt{3}z + 4\right)$

22(i) Since $1+i$ is a root of the equation $2w^3 + aw^2 + bw - 2 = 0$,

$2(1+i)^3 + a(1+i)^2 + b(1+i) - 2 = 0$

$2(-2+2i) + a(2i) + b(1+i) - 2 = 0$

$(b-6) + (4+2a+b)i = 0+0i$

Comparing real terms,

$b - 6 = 0$

$b = 6$

Comparing imaginary terms,

$4 + 2a + b = 0$

$a = \frac{-b-4}{2}$

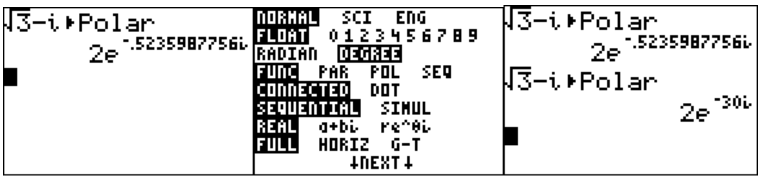
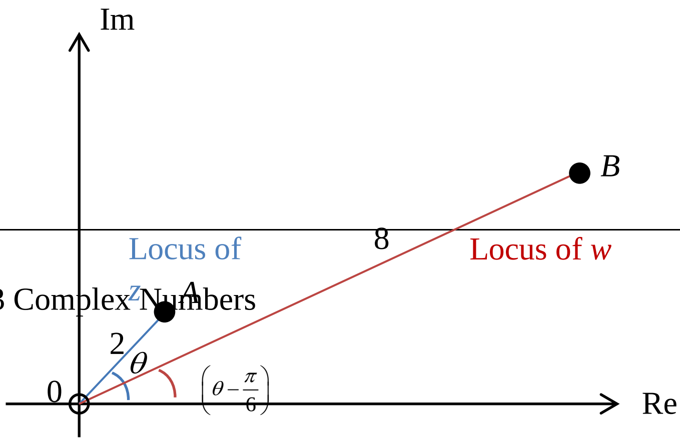
$\therefore a = \frac{-6-4}{2} = -5$

(ii) Since polynomial equation has real coefficients, $1+i$ and $1-i$ are roots to the equation.

$2w^3 - 5w^2 + 6w - 2 = \left(w - (1+i)\right)\left(w - (1-i)\right)(2w - A)$

Comparing constants,

	$-A(1+\mathrm{i})(1-\mathrm{i})=-2$ $A(1-\mathrm{i}^2)=2$ $A(1-(-1))=2$ $A=1$ $2w^3-5w^2+6w-2=0$ $(w-(1+\mathrm{i}))(w-(1-\mathrm{i}))(2w-1)=0$ $w=1+\mathrm{i},\quad 1-\mathrm{i},\quad \frac{1}{2}.$
	Alternative to parts **(ii)** and **(iii)** Since coefficients are real, if first root is 1 + I, then second root is **<u>1 – i</u>** Quadratic factor is $(w-1-\mathrm{i})(w-1+\mathrm{i})$ $= \boldsymbol{w^2-2w+2}$ $2w^3+aw^2+bw-2=(w^2-2w+2)(2w-1)$ $=(2w^3-4w^2+4w)+(-w^2+2w-2)$ $=2w^3-5w^2+6w-2$ giving $\boldsymbol{a=-5}$ and $\boldsymbol{b=6}$ And third root is $\boldsymbol{w=1/2}$

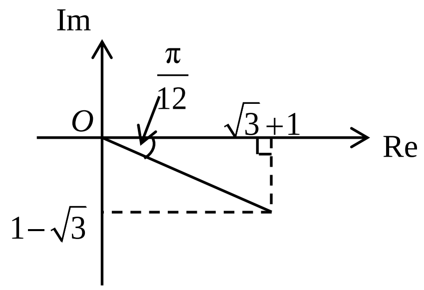
23(i)	$\sqrt{3}-\mathrm{i}=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$ $w=2\left(\sqrt{3}-\mathrm{i}\right)z$ $=2\left(2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}\right)r\mathrm{e}^{\mathrm{i}\theta}$ $=4r\mathrm{e}^{\mathrm{i}\left(\theta-\frac{\pi}{6}\right)}$ $\lvert w\rvert=4r$ $\arg w=\theta-\frac{\pi}{6}\quad\left(\because \frac{\pi}{6}<\theta\le\frac{\pi}{2}\right)$ **Useful screenshots:** 
(ii)	

	Remark: Locus of z could also be drawn along the positive Im-axis as values of θ include $\frac{\pi}{2}$.
(iii)	$\left\|\frac{w^2}{2z^*}\right\| = \frac{\|w\|^2}{2\|z\|} = \frac{16r^2}{2r} = 8r$ Since $0 < r \le 2$, $\therefore 0 < \left\|\frac{w^2}{2z^*}\right\| \le 16.$

24(ai)	$\frac{(w^*)^2}{w} = 3 - \mathrm{i}b \Rightarrow \frac{(a-\mathrm{i}b)^2}{(a+\mathrm{i}b)} = 3 - \mathrm{i}b$ $a^2 - b^2 - 2\mathrm{i}ab = (3-\mathrm{i}b)(a+\mathrm{i}b) = 3a + b^2 + \mathrm{i}(-ab+3b)$ Equating the real and the imaginary parts: $a^2 - b^2 = 3a + b^2$ **.....**(1) and $-2ab = -ab + 3b$ …(2) From (2) $a = -3$ since $b \ne 0$ From (1), $9 - b^2 = -9 + b^2$ $b^2 = 9$ $b = \pm 3$ Possible values of w are $-3 \pm 3\mathrm{i}$
(bi)	$z^2 - 2z + 4 = 0$ $z = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}\mathrm{i}$ $\alpha = 1 + \sqrt{3}\mathrm{i} = 2e^{\mathrm{i}\left(\frac{\pi}{3}\right)}$ and $\beta = 1 - \sqrt{3}\mathrm{i} = 2e^{-\mathrm{i}\left(\frac{\pi}{3}\right)}$
(bii)	$\alpha^{10} - \beta^{10} = 2^{10}\left(e^{\mathrm{i}\left(\frac{10\pi}{3}\right)} - e^{-\mathrm{i}\left(\frac{10\pi}{3}\right)}\right)$

	$= 2^{10}\left(2\mathrm{i}\sin\frac{10\pi}{3}\right)$ $= 2^{10}\left(2\mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right)$ $= 2^{10}\left(2\left(-\frac{\sqrt{3}}{2}\right)\right)\mathrm{i}$ $= -1024\sqrt{3}\mathrm{i}$ $\left\|\alpha^{10}-\beta^{10}\right\| = 1024\sqrt{3}$ So $\arg\left(\alpha^{10}-\beta^{10}\right) = -\frac{\pi}{2}$

25 (a)	$2z_1 + \mathrm{i}\,z_2^* = 7-6\mathrm{i}$ --- (1) $z_1 - \mathrm{i}\,z_2 = 6-6\mathrm{i}$ --- (2) (1) – (2) x 2: $\mathrm{i}\,z_2^* + 2\mathrm{i}\,z_2 = 7-6\mathrm{i}-2(6-6\mathrm{i}) = -5+6\mathrm{i}$ $z_2^* + 2z_2 = 6+5\mathrm{i}$ Since $z_2^* + 2z_2 = 3\mathrm{Re}(z_2) + \mathrm{Im}(z_2)\mathrm{i} = 6+5\mathrm{i}$, $z_2 = 2+5\mathrm{i}$ Sub $z_2 = 2+5\mathrm{i}$ into (2): $z_1 = 6-6\mathrm{i}+\mathrm{i}(2+5\mathrm{i}) = 1-4\mathrm{i}$
(bi)	$\|w\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$ $\arg(w) = \tan^{-1}(-1) = -\frac{\pi}{4}$ $\left\|\frac{v}{w^*}\right\| = \frac{\|v\|}{\|w^*\|} = \frac{\|v\|}{\|w\|} = \frac{2}{\left(\frac{1}{\sqrt{2}}\right)} = 2\sqrt{2}$ $\arg\left(\frac{v}{w^*}\right) = \arg(v) - \arg(w^*) = \arg(v) + \arg(w) = \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$
(bii)	

	$v=2\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)=\sqrt{3}+\mathrm{i}$ $\frac{v}{w^*}=\frac{\sqrt{3}+\mathrm{i}}{\frac{1}{2}+\frac{1}{2}\mathrm{i}}=\frac{2\left(\sqrt{3}+\mathrm{i}\right)}{1+\mathrm{i}}\times\frac{1-\mathrm{i}}{1-\mathrm{i}}$ $=\left(\sqrt{3}+1\right)+\left(1-\sqrt{3}\right)\mathrm{i}$ $\therefore\ \mathrm{Re}\left(\frac{v}{w^*}\right)=\sqrt{3}+1$ and $\mathrm{Im}\left(\frac{v}{w^*}\right)=1-\sqrt{3}$ **Alternative solution** $\frac{1}{w^*}=\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]=1-\mathrm{i}$ $\frac{v}{w^*}=\left(\sqrt{3}+\mathrm{i}\right)\left(1-\mathrm{i}\right)=\sqrt{3}-\sqrt{3}\mathrm{i}+\mathrm{i}+1=\left(\sqrt{3}+1\right)+\left(1-\sqrt{3}\right)\mathrm{i}$
(biii)	Using results in **(i)** and **(ii)**,  From the Argand diagram, $\tan\left(\frac{\pi}{12}\right)=\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}=2-\sqrt{3}$

26	$\lvert z\rvert=\left\lvert\frac{(1+i)^3}{\sqrt{3}-i}\right\rvert=\frac{\sqrt{2}^3}{2}$ $=\sqrt{2}$ $\arg\frac{(1+i)^3}{\sqrt{3}-i}=3\arg(1+i)-\arg(\sqrt{3}-i)$ $=3\left(\frac{\pi}{4}\right)-\left(-\frac{\pi}{6}\right)=\frac{11\pi}{12}$

$$\therefore z=\frac{(1+i)^3}{\sqrt{3}-i}=\sqrt{2}\,e^{i\left(\frac{11\pi}{12}\right)}=\sqrt{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$$

$$\frac{(1+i)^3}{\sqrt{3}-i}=\frac{2(-1+i)}{\sqrt{3}-i}=\frac{2(-1+i)}{\sqrt{3}-i}\times\frac{\sqrt{3}+i}{\sqrt{3}+i}$$

$$=\frac{-(1+\sqrt{3})}{2}+\frac{(\sqrt{3}-1)i}{2}$$

$$\therefore\quad \frac{-(1+\sqrt{3})}{2}+\frac{(\sqrt{3}-1)i}{2}=\sqrt{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$$

$$\Rightarrow \sqrt{2}\cos\frac{11\pi}{12}=-\frac{-(1+\sqrt{3})}{2}$$

$$\sqrt{2}\sin\frac{11\pi}{12}=\frac{(\sqrt{3}-1)}{2}$$

$$\therefore \tan\frac{11\pi}{12}=-\frac{\sqrt{3}-1}{\sqrt{3}+1}=\sqrt{3}-2$$

27

From the diagram,

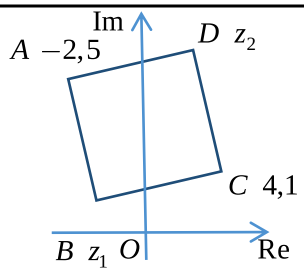
$$\arg\left(4+\mathrm{i}-z_1\right)+\frac{\pi}{2}=\arg\left(-2+5\mathrm{i}-z_1\right)$$

$$\mathrm{i}\ \ 4+\mathrm{i}-z_1\ \ =\ -2+5\mathrm{i}-z_1$$

$$4\mathrm{i}-1-\mathrm{i}z_1=-2+5\mathrm{i}-z_1$$

$$1-\mathrm{i}\ \ z_1=-1+\mathrm{i}$$

$$z_1=-1$$



Midpoint of AC is $\left(\frac{-2+4}{2},\frac{5+1}{2}\right)=\ 1,3$

Let $z_2=x+\mathrm{i}y$

Since the diagonals of a square bisect other,

Midpoint of BD is $\ 1,3$

$$\left(\frac{x-1}{2},\frac{y+0}{2}\right)=\ 1,3$$

$$\therefore x=3,\ y=6$$

$$z_2=3+6\mathrm{i}$$

28

Let f(x) = $x^4+ax^3+5x^2-x-10$.

Since the coefficients of f(x) are real, and 1 + 2i is a root of f(x) = 0, therefore 1 – 2i

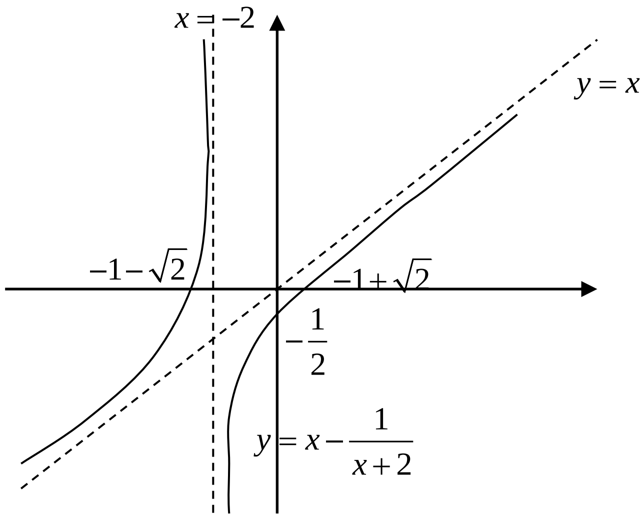
	is also a root. $\mathrm{f}(x)=\left(x-(1+2\mathrm{i})\right)\left(x-(1-2\mathrm{i})\right)\left(x^2+bx+c\right)$ $=\left((x-1)-2\mathrm{i}\right)\left((x-1)+2\mathrm{i}\right)\left(x^2+bx+c\right)$ $=\left((x-1)^2-(2\mathrm{i})^2\right)\left(x^2+bx+c\right)$ $=\left(x^2-2x+5\right)\left(x^2+bx+c\right)$ Comparing coefficients of constant: c = –2 x: 5b – 2c = –1 $\Rightarrow$ b = –1 x3: –2 – 1 = a $\Rightarrow$ a = –3 $\therefore \mathrm{f}(x)=\left(x^2-2x+5\right)\left(x^2-x-2\right)=\left(x^2-2x+5\right)(x-2)(x+1)$ The other roots are 1 – 2i, –1 and 2.
	$5\left(x^2-2x^4\right)=x^3+3x-1$ $-10x^4-x^3+5x^2-3x+1=0$ $\frac{1}{x^4}-\frac{3}{x^3}+\frac{5}{x^2}-\frac{1}{x}-10=0$ Replace x by $\frac{1}{x}$, $\frac{1}{x}=-1$ or $\frac{1}{x}=2$ $x=-1$ or $x=\frac{1}{2}$

29 (i)	$z=\frac{3+\mathrm{i}}{2-\mathrm{i}}=\frac{(3+\mathrm{i})(2+\mathrm{i})}{2^2+1}=\frac{1}{5}(5+5\mathrm{i})=1+\mathrm{i}$ Therefore, | z | = $\sqrt{2}$ [Or $\lvert z\rvert=\left\lvert\frac{3+\mathrm{i}}{2-\mathrm{i}}\right\rvert=\frac{\sqrt{10}}{\sqrt{5}}=\sqrt{2}$] arg z = $\frac{\pi}{4}$
(ii)	$\mathrm{e}^{x+\mathrm{i}2y}=z$ $\mathrm{e}^{x}\mathrm{e}^{\mathrm{i}2y}=\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ $\Rightarrow \mathrm{e}^{x}=\sqrt{2}$ or $\mathrm{e}^{\mathrm{i}2y}=\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ or $\mathrm{e}^{\mathrm{i}\left(-\frac{7\pi}{4}\right)}$

	$\Rightarrow x=\ln\sqrt{2}=\frac{1}{2}\ln 2$ or $y=\frac{\pi}{8}$ or $-\frac{7\pi}{8}$
(iii)	For $\left(\frac{z^2}{z^*}\right)^n$ to be purely imaginary, $\arg\left(\frac{z^2}{z^*}\right)^n = (2k+1)\frac{\pi}{2}, k\in\mathbb{Z}$ $n\left[2\arg z-\arg z^*\right] =(2k+1)\frac{\pi}{2}$ $n\left[\frac{\pi}{2}+\frac{\pi}{4}\right] =(2k+1)\frac{\pi}{2}$ $n=\frac{2}{3}(2k+1)$ Hence, the smallest positive integer n = 2

30	$w^2=\left(z^2-z\right)^2$ $=z^4-2z^3+z^2$ $z^4-2z^3-2z^2+3z-10=0$ $\left(z^4-2z^3+z^2\right)-3z^2+3z-10=0$ $\left(z^4-2z^3+z^2\right)-3\left(z^2-z\right)-10=0$ $w^2-3w-10=0$ $(w-5)(w+2)=0$ $w=5$ or $w=-2$ $z^2-z=5$ $z^2-z=-2$ $z^2-z-5=0$ $z^2-z+2=0$ $z=\frac{1\pm\sqrt{1-4(-5)}}{2}$ $z=\frac{1\pm\sqrt{1-4(2)}}{2}$ $=\frac{1\pm\sqrt{21}}{2}$ $=\frac{1\pm\sqrt{-7}}{2}$ $=\frac{1\pm\sqrt{7}\mathrm{i}}{2}$

31 (i)	$z=\left(1+i\right)(t-2)+\frac{1-i}{t\left(1+i\right)}$

	$=(1+i)(t-2)+\frac{1-i}{t(1+i)}\frac{1-i}{1-i}$ $=(1+i)(t-2)-\frac{1}{t}i$ $\text{Im}(z)=t-2-\frac{1}{t}$
(ii)	$x=\text{Re}(z)=t-2 \quad \Rightarrow t=x+2$ Therefore $\quad y=x-\frac{1}{x+2}$
(iii)	

32 (a)	<u>Method 1: Expressing *z* in the form *x* + *y*i</u> Let $z=x+y\text{i}$. $\left\|\frac{2\text{i}-z^*}{z}-1\right\|^2-z=\text{i}$ $\left\|\frac{2\text{i}-(x+y\text{i})^*-(x+y\text{i})}{x+y\text{i}}\right\|^2-(x+y\text{i})=\text{i}$ $\left\|\frac{2\text{i}-2x}{x+y\text{i}}\right\|^2-(x+y\text{i})=\text{i}$ $\frac{4x^2+4}{x^2+y^2}-x-y\text{i}=\text{i}$ Comparing real and imaginary parts, $\frac{4x^2+4}{x^2+y^2}-x=0$ and $-y=1$.

	$\therefore y=-1$ $\frac{4\left(x^2+1\right)}{x^2+1}-x=0 \Rightarrow 4-x=0$ $x=4$. Thus, $z=4-\text{i}$.
	<u>Method 2: Observing that the modulus of a complex number is real</u> Let $z=x+y\text{i}$. Since $\left\|\frac{2\text{i}-z^*}{z}-1\right\|^2 \in \mathbb{R}, \quad -y=1 \Rightarrow y=-1.$ Therefore $z=x-\text{i}$. Hence, $\left\|\frac{2\text{i}-(x+\text{i})}{x-\text{i}}-1\right\|^2-x+\text{i}=\text{i}$ $\left\|\frac{\text{i}-x}{x-\text{i}}-1\right\|^2-x=0$ $\|-1-1\|^2-x=0$ $x=4$ Hence $z=4-\text{i}$.

(b) (i)	$p=-\sqrt{3}+\text{i}=2\text{e}^{\text{i}\frac{5\pi}{6}}$ $q=-4\text{i}=4\text{e}^{-\text{i}\frac{\pi}{2}}$
(b) (ii)	$\frac{p^{10}}{q^5}=\frac{2^{10}\text{e}^{\text{i}\frac{50\pi}{6}}}{4^5\text{e}^{-\text{i}\frac{5\pi}{2}}}=\text{e}^{\text{i}\left(\frac{50\pi}{6}+\frac{5\pi}{2}\right)}=\text{e}^{\text{i}\left(\frac{65\pi}{6}\right)}$ $\frac{p^{10}}{q^5}+\frac{q^5}{p^{10}}=\text{e}^{\text{i}\frac{65\pi}{6}}+\frac{1}{\text{e}^{\text{i}\frac{65\pi}{6}}}$ $=\text{e}^{\text{i}\frac{65\pi}{6}}+\text{e}^{-\text{i}\frac{65\pi}{6}}$ $=\cos\frac{65\pi}{6}+\text{i}\sin\frac{65\pi}{6}+\cos\left(-\frac{65\pi}{6}\right)+\text{i}\sin\left(-\frac{65\pi}{6}\right)$ $=2\cos\left(\frac{65\pi}{6}\right)$ $=-2\cos\left(\frac{\pi}{6}\right)=-\sqrt{3}$

33	$w^2+aw^*+b=0$ $\left(w^2+aw^*+b\right)^*=0^*$ $\left(w^2\right)^*+\left(aw^*\right)^*+b^*=0$ $\left(w^*\right)^2+a\left(w^*\right)^*+b=0$, $a^*=a$ and $b^*=b$ since a and b are real. Hence, w^* is a root of $z^2+az^*+b=0$. $z^2+6z^*+9=0$ $\left(x+\mathrm{i}y\right)^2+6\left(x-\mathrm{i}y\right)+9=0$ $x^2-y^2+2\mathrm{i}xy+6x-6\mathrm{i}y+9=0$ $x^2-y^2+6x+9+2y\left(x-3\right)\mathrm{i}=0$ Compare imaginary parts, $y=0$ or $x=3$. Consider real parts: When $y=0$, $x^2+6x+9=0$ which gives $x=-3$ When $x=3$, $3^2-y^2+18+9=0$ giving $y=\pm 6$ Hence $z=-3$, $3+6\mathrm{i}$, $3-6\mathrm{i}$

34(a)	$\dfrac{\mathrm{i}z}{z-2z^*-2}=-1$ $\mathrm{i}z=-z+2z^*+2$ Let $z = x + y\mathrm{i}$ $\mathrm{i}(x+y\mathrm{i})=-(x+y\mathrm{i})+2(x-y\mathrm{i})+2$ $-y+x\mathrm{i}=(x+2)-3y\mathrm{i}$ Equating real & imaginary parts, $y=-(x+2)$ ----------- (1) $x=-3y$ ----------- (2) Solving (1) & (2), $x = -3$, $y = 1$ Hence, $z = -3 + \mathrm{i}$
(b)(i)	$\dfrac{z}{z-r}=\dfrac{r\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{\mathrm{i}\theta}-r}$ $=\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}-1}$ $=\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}(\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}-\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}})}$

	$= \dfrac{e^{i\frac{\theta}{2}}}{2i\sin(\frac{\theta}{2})}$ $= \dfrac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{2i\sin\frac{\theta}{2}}$ (Note: $\frac{1}{i} = -i$) $= \frac{1}{2} - \frac{1}{2}i\cot(\frac{\theta}{2})$.

DHS Prelim 9758/2018/01/Q7	
35 **(a)**	$z^* = \dfrac{(2i)^3}{(\sqrt{3}+i)^4} = \dfrac{-8i}{(\sqrt{3}+i)^4}$ $\lvert z\rvert = \lvert z^*\rvert = \left\lvert \dfrac{-8i}{(\sqrt{3}+i)^4} \right\rvert = \dfrac{8}{\left(\sqrt{(\sqrt{3})^2+1^2}\right)^4} = \dfrac{8}{16} = \dfrac{1}{2}$ $\arg(z) = -\arg(z^*)$ $= -\arg\left(\dfrac{-8i}{(\sqrt{3}+i)^4}\right)$ $= -\left[\arg(-8i) - 4\arg(\sqrt{3}+i)\right]$ $= -\left[-\frac{1}{2}\pi - 4\left(\frac{1}{6}\pi\right)\right]$ $= \frac{7}{6}\pi$ $\therefore \arg(z) = \frac{7}{6}\pi - 2\pi = -\frac{5}{6}\pi$ $\arg(z^n) = n\arg(z) = -\frac{5}{6}n\pi$ Since z^n is purely imaginary, $-\frac{5}{6}n\pi = (2k+1)\left(\frac{1}{2}\pi\right),\ k \in \mathbb{Z}$ $\Rightarrow n = -\dfrac{3}{5}(2k+1)$ $\therefore$ smallest positive integer $n = 3$ (when $k = -3$) **<u>Alternative</u>** $z^* = \dfrac{(2i)^3}{(\sqrt{3}+i)^4} = \dfrac{\left(2e^{i\frac{\pi}{2}}\right)^3}{\left(2e^{i\frac{\pi}{6}}\right)^4}$ $= \dfrac{8e^{i\frac{3\pi}{2}}}{16e^{i\frac{4\pi}{6}}} = \dfrac{1}{2}e^{i\left(\frac{3\pi}{2}-\frac{4\pi}{6}\right)}$ $= \dfrac{1}{2}e^{i\frac{5\pi}{6}}$ $\therefore z = \dfrac{1}{2}e^{-i\frac{5\pi}{6}}$ $\Rightarrow \lvert z\rvert = \dfrac{1}{2},\ \arg(z) - \frac{5}{6}\pi$ $z = re^{-i\theta}$ where $r = \lvert z\rvert = \frac{1}{2}$ $\theta = \arg(z) = -\frac{5}{6}\pi$

(b)(i)	Let $\mathrm{f}(x) = ax^4 + bx^3 + cx^2 + 24x - 44$ $\mathrm{f}(1) = -18 \Rightarrow a + b + c = 2$ $\mathrm{f}(-1) = -54 \Rightarrow a - b + c = 14$ $\mathrm{f}(2) = 0 \Rightarrow 16a + 8b + 4c = -4$ From GC : $a = 1, b = -6, c = 7$
(ii)	$x^4 - 6x^3 + 7x^2 + 24x - 44 = 0$ If $3 - (\sqrt{2})\mathrm{i}$ is a root, $3 + (\sqrt{2})\mathrm{i}$ is also a root (since <u>equation has all real coefficients</u> OR by <u>conjugate root theorem</u>) **<u>Method 1</u>** Compare product of last erms, $\left[x - (3 - (\sqrt{2})\mathrm{i})\right]\left[x - (3 + (\sqrt{2})\mathrm{i})\right](x - 2)(x + a) = x^4 - 6x^3 + 7x^2 + 24x - 44$ $(3 - (\sqrt{2})\mathrm{i})(3 + (\sqrt{2})\mathrm{i})(-2)(a) = -44$ $(3^2 + (\sqrt{2})^2)(-2)a = -44$ $a = 2$ **<u>Method 2</u>** $\left[x - (3 - (\sqrt{2})\mathrm{i})\right]\left[x - (3 + (\sqrt{2})\mathrm{i})\right] = \left[(x - 3) + (\sqrt{2})\mathrm{i}\right]\left[(x - 3) - (\sqrt{2})\mathrm{i}\right]$ $= \left[(x - 3)^2 + 2\right] = x^2 - 6x + 11$ Since $(x - 2)$ is a factor of the polynomial equation, $x^4 - 6x^3 + 7x^2 + 24x - 44 = 0$ $\Rightarrow (x^2 - 6x + 11)(x - 2)(x + 2) = 0$ (by inspection) $\therefore$ the other roots are $3 + (\sqrt{2})\mathrm{i}$, 2 and -2

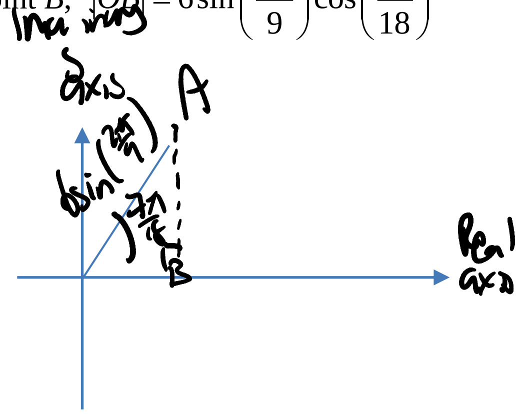
36 (i)	$\text{LHS} = a\left(\frac{1}{z_0}\right)^2 + b\left(\frac{1}{z_0}\right) + a = \left(\frac{1}{z_0}\right)^2\left(a + bz_0 + az_0^{\ 2}\right) = 0$ $\because a + bz_0 + az_0^{\ 2} = 0$ Thus $z = \frac{1}{z_0}$ is a solution. Since *a* and *b* are real constants, $\frac{1}{z_0} = z_0^{\ *}$ $z_0 z_0^{\ *} = 1$ $\lvert z_0 \rvert^2 = 1$ Since $\lvert z_0 \rvert > 0$, $\lvert z_0 \rvert = 1$ **Alternative for first part:** Let second root be z_1 product of roots $z_0 z_1 = \frac{a}{a} = 1$ $\therefore z_1 = \frac{1}{z_0}$
(ii)	Let $z_0 = x_0 + \mathrm{i}y_0$ Since $\mathrm{Im}(z_0) = \frac{1}{2}$, $y_0 = \frac{1}{2}$. From part (i), $\lvert z_0 \rvert = 1$ $\sqrt{x_0^{\ 2} + y_0^{\ 2}} = 1$ $\sqrt{x_0^{\ 2} + \left(\frac{1}{2}\right)^2} = 1$ $x_0 = \pm\frac{\sqrt{3}}{2}$ $z_0 = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$ or $-\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$
(iii)	Since $\mathrm{Re}(z_0) > 0$, $z_0 = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$. Subst into $az_0^{\ 2} + bz_0 + a = 0$,

	$a\left(\frac{\sqrt{3}}{2}+\mathrm{i}\frac{1}{2}\right)^2+b\left(\frac{\sqrt{3}}{2}+\mathrm{i}\frac{1}{2}\right)+a=0$ $a\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)+b\left(\frac{\sqrt{3}}{2}+\mathrm{i}\frac{1}{2}\right)+a=0$ $\left(\frac{3}{2}a+\frac{\sqrt{3}}{2}b\right)+\mathrm{i}\left(\frac{1}{2}b+\frac{\sqrt{3}}{2}a\right)=0$ $\therefore b=-\sqrt{3}a$

37(i)	Since $z^2-3z+9=0$ has all real coefficients, given that $z=3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ is a root of the equation, $z=3\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$ is the other root of the equation.
(ii)	$\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}=(\cos\theta+\mathrm{i}\sin\theta)-[\cos(-\theta)+\mathrm{i}\sin(-\theta)]$ $=(\cos\theta+\mathrm{i}\sin\theta)-(\cos\theta-\mathrm{i}\sin\theta)$ $=2\mathrm{i}\sin\theta$
(iii)	Since $w_1=3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$, $w_2=3\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}$ $w_2-w_1=3\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{9}\right)}-3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$ $=3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{9}\right)}\left[\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{9}\right)}-\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{9}\right)}\right]$ $=3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{9}\right)}\left[2\mathrm{i}\sin\left(\frac{2\pi}{9}\right)\right]$ $=6\sin\left(\frac{2\pi}{9}\right)\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{9}+\frac{\pi}{2}\right)}$ $=6\sin\left(\frac{2\pi}{9}\right)\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{18}\right)}$

(iv)

At point B, $|OB| = 6\sin\left(\frac{2\pi}{9}\right)\cos\left(\frac{7\pi}{18}\right)$



Hence,

Area of triangle OAB

$$= \frac{1}{2}|OB||OA|\sin\left(\frac{7\pi}{18}\right)$$

$$= \frac{1}{2}\left[6\sin\left(\frac{2\pi}{9}\right)\cos\left(\frac{7\pi}{18}\right)\right]\left[6\sin\left(\frac{2\pi}{9}\right)\right]\sin\left(\frac{7\pi}{18}\right)$$

$$= \frac{36}{2}\sin^2\left(\frac{2\pi}{9}\right)\sin\left(\frac{7\pi}{18}\right)\cos\left(\frac{7\pi}{18}\right)$$

$$= \frac{36}{2}\sin^2\left(\frac{2\pi}{9}\right)\left[\frac{\sin\left(\frac{14\pi}{18}\right)}{2}\right]$$

$$= 9\sin^2\left(\frac{2\pi}{9}\right)\sin\left(\frac{7\pi}{9}\right)$$

38. Suggested solution

(a)(i)

Since $z_1 = -1 + \mathrm{i}$ is a root,

$$(-1+\mathrm{i})^2 + a(-1+\mathrm{i}) + (1-\sqrt{3}) + b\mathrm{i} = 0$$

$-2\text{i}+a\left(-1+\text{i}\right)+\left(1-\sqrt{3}\right)+b\text{i}=0$

$-a+\left(1-\sqrt{3}\right)+\left(a+b-2\right)\text{i}=0$

Comparing Re and Im parts

$-a+\left(1-\sqrt{3}\right)=0\Rightarrow a=1-\sqrt{3}$

$a+b-2=0\Rightarrow b=1+\sqrt{3}$

(ii)

$z^2+\left(1-\sqrt{3}\right)z+\left(1-\sqrt{3}\right)+\left(1+\sqrt{3}\right)\text{i}=0$

$z^2+\left(1-\sqrt{3}\right)z+\left(1-\sqrt{3}\right)+\left(1+\sqrt{3}\right)\text{i}=\left[z-\left(-1+\text{i}\right)\right]\left(z-z_2\right)$

Method 1: Comparing z

$1-\sqrt{3}=-z_2-\left(-1+\text{i}\right)\Rightarrow z_2=\sqrt{3}-\text{i}$

Method 2: Comparing "constant"

$\left(1-\sqrt{3}\right)+\left(1+\sqrt{3}\right)\text{i} = z_2\left(-1+\text{i}\right)$

$\Rightarrow z_2=\dfrac{\left(1-\sqrt{3}\right)+\left(1+\sqrt{3}\right)\text{i}}{\left(-1+\text{i}\right)}=\dfrac{\left[\left(1-\sqrt{3}\right)+\left(1+\sqrt{3}\right)\text{i}\right]\left[-1-\text{i}\right]}{2}$

$=\dfrac{-\left[\left(1-\sqrt{3}\right)+\left(1+\sqrt{3}\right)\text{i}\right]\left[1+\text{i}\right]}{2}=\sqrt{3}-\text{i}$

Method 3: Sum of roots

Sum of roots = $-\left(1-\sqrt{3}\right)$

$-1+\text{i}+z_2=-\left(1-\sqrt{3}\right)$

$z_2=\sqrt{3}-\text{i}$

Method 4: General formula

$$z_2 = \frac{-\left(1-\sqrt{3}\right) \pm \sqrt{\left(1-\sqrt{3}\right)^2 - 4(1)\left[\left(1-\sqrt{3}\right)+\left(1+\sqrt{3}\right)\mathrm{i}\right]}}{2}$$

$$= \frac{-\left(1-\sqrt{3}\right) \pm \sqrt{1-2\sqrt{3}+3-4+4\sqrt{3}-4\mathrm{i}-4\sqrt{3}\mathrm{i}}}{2}$$

$$= \frac{-\left(1-\sqrt{3}\right) \pm \sqrt{2\sqrt{3}-4\sqrt{3}\mathrm{i}-4\mathrm{i}}}{2}$$

$$= \frac{-\left(1-\sqrt{3}\right) \pm \sqrt{\left(1+\sqrt{3}-2\mathrm{i}\right)^2}}{2}$$

$$= \frac{-\left(1-\sqrt{3}\right) \pm \left(1+\sqrt{3}-2\mathrm{i}\right)}{2}$$

$= -1+\mathrm{i}$ (rej) or $\sqrt{3}-\mathrm{i}$

(b)**(i)**

Method 1:

$$w_1 = 2-2\mathrm{i} = 2\sqrt{2}e^{-\frac{\pi}{4}\mathrm{i}} \text{ or } 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$$

$$w_2 = -\sqrt{3}+\mathrm{i} = 2e^{\frac{5\pi}{6}\mathrm{i}} \text{ or } 2\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)$$

$$w_1w_2 = 4\sqrt{2}e^{\left(-\frac{\pi}{4}+\frac{5\pi}{6}\right)\mathrm{i}} = 4\sqrt{2}e^{\frac{7\pi}{12}\mathrm{i}}$$

$|w_1w_2| = 4\sqrt{2}$ and $\arg\left(w_1w_2\right) = \frac{7\pi}{12}$

Method 2:

$$w_1w_2 = 2\left(1-\sqrt{3}\right)+2\left(1+\sqrt{3}\right)\mathrm{i}$$

$$|w_1w_2| = \sqrt{4\left(1-\sqrt{3}\right)^2+4\left(1+\sqrt{3}\right)^2} = \sqrt{32} = 4\sqrt{2}$$

$$\arg\left(w_1w_2\right) = \pi - \tan^{-1}\frac{\left(1+\sqrt{3}\right)}{\left(\sqrt{3}-1\right)} = \frac{7}{12}\pi$$

(ii)

Method 1:
From (ii),

$$w_1w_2 = 4\sqrt{2}e^{\frac{7\pi}{12}\mathrm{i}} \text{ or } 4\sqrt{2}\left(\cos\left(\frac{7\pi}{12}\right)+\mathrm{i}\sin\left(\frac{7\pi}{12}\right)\right)$$

$$w_1w_2 = 2\left(1-\sqrt{3}\right)+2\left(1+\sqrt{3}\right)\mathrm{i}$$

Hence

$$4\sqrt{2}\cos\frac{7}{12}\pi = 2\left(1-\sqrt{3}\right) \Rightarrow \cos\frac{7}{12}\pi = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

Otherwise
Method 2:
Student using geometry approach on

$$w_1w_2 = 2\left(1-\sqrt{3}\right)+2\left(1+\sqrt{3}\right)\mathrm{i}$$

Method 3:
Student using special angles and addition formula

39. ACJC Prelim/2022/01/Q5

Do not use a calculator in answering this question.

Two complex numbers are $z_1 = 2\left(\cos\frac{\pi}{18} - \mathrm{i}\sin\frac{\pi}{18}\right)$ and $z_2 = 2\mathrm{i}$.

(i) Show that $\frac{z_1^{\,2}}{z_1^{\,*}} + z_2$ is $\sqrt{3}+\mathrm{i}$. [3]

(ii) A third complex number, z_3, is such that $\left(\frac{z_1^{\,2}}{z_1^{\,*}} + z_2\right)z_3$ is real and $\left|\left(\frac{z_1^{\,2}}{z_1^{\,*}} + z_2\right)z_3\right| = \frac{2}{3}$.

Find the possible values of z_3 in the form of $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

ACJC Prelim 9758/2022/01/Q5

(i)	$z_1 = 2\left(\cos\frac{\pi}{18} - \mathrm{i}\sin\frac{\pi}{18}\right) = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{18}}$

	$\frac{z_1^{\ 2}}{z_1^{\ *}}+z_2$ $=\frac{4\mathrm{e}^{-\mathrm{i}\frac{\pi}{9}}}{2\mathrm{e}^{\mathrm{i}\frac{\pi}{18}}}+2\mathrm{i}$ $=2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}+2\mathrm{i}$ $=2\left(\cos\frac{\pi}{6}-\mathrm{i}\sin\frac{\pi}{6}\right)+2\mathrm{i}$ $=2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\mathrm{i}\right)+2\mathrm{i}$ $=\sqrt{3}+\mathrm{i}$
(ii)	$\left(\frac{z_1^{\ 2}}{z_1^{\ *}}+z_2\right)z_3$ is real and $\left\|\left(\frac{z_1^{\ 2}}{z_1^{\ *}}+z_2\right)z_3\right\|=\frac{2}{3}$ $\left(\frac{z_1^{\ 2}}{z_1^{\ *}}+z_2\right)z_3=\frac{2}{3}$ or $-\frac{2}{3}$ $\left(\sqrt{3}+\mathrm{i}\right)z_3=\frac{2}{3}$ or $-\frac{2}{3}$ $z_3=\frac{2}{3\left(\sqrt{3}+\mathrm{i}\right)}$ or $-\frac{2}{3\left(\sqrt{3}+\mathrm{i}\right)}$ $=\frac{2}{3\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)}$ or $\mathrm{e}^{\mathrm{i}\pi}\frac{2}{3\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)}$ $=\frac{1}{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$ or $\frac{1}{3}\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$ $=\frac{1}{3}\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$ or $\frac{1}{3}\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)$
	<u>**Method 2**</u> $\frac{z_1^{\ 2}}{z_1^{\ *}}+z_2=\sqrt{3}+\mathrm{i}=2\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$

	$\left\|\left(\frac{z_1^{\,2}}{z_1^{\,*}}+z_2\right)z_3\right\|=\frac{2}{3}$ $\left\|\frac{z_1^{\,2}}{z_1^{\,*}}+z_2\right\|\left\|z_3\right\|=\frac{2}{3}$ $\left\|\frac{z_1^{\,2}}{z_1^{\,*}}+z_2\right\|2=\frac{2}{3}$ $\left\|z_3\right\|=\frac{1}{3}$ $\left(\frac{z_1^{\,2}}{z_1^{\,*}}+z_2\right)z_3$ is real $\Rightarrow \arg\left(\frac{z_1^{\,2}}{z_1^{\,*}}+z_2\right)z_3=0 \quad \text{or} \quad \pi$ $\Rightarrow \arg\left(\frac{z_1^{\,2}}{z_1^{\,*}}+z_2\right)+\arg z_3=0 \quad \text{or} \quad \pi$ $\Rightarrow \frac{\pi}{6}+\arg z_3=0 \quad \text{or} \quad \pi$ $\Rightarrow \arg z_3=-\frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6}$ $z_3=\frac{1}{3}\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right) \quad \text{or} \quad \frac{1}{3}\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)$

40. ACJC Prelim/2022/02/Q3

(i) Find the roots of the equation $\mathrm{i}z^2-(5+\mathrm{i})z+2-6\mathrm{i}=0$, giving your answers in cartesian form $a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. [2]

(ii) Hence find the roots of the equation $-\mathrm{i}w^2-(1-5\mathrm{i})w+2-6\mathrm{i}=0$, giving your answers in cartesian form $a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. [2]

(iii) Given that the roots found in part **(i)** are also roots of the equation $P(z)=0$, where $P(z)$ is a polynomial of degree 4 with real coefficients, find $P(z)$. [3]

ACJC Prelim 9758/2022/02/Q3

(i)	$\mathrm{i}z^2-(5+\mathrm{i})z+2-6\mathrm{i}=0$ $z=\dfrac{5+\mathrm{i}\pm\sqrt{[-(5+\mathrm{i})]^2-4(\mathrm{i})(2-6\mathrm{i})}}{2\mathrm{i}}$ $=\dfrac{5+\mathrm{i}\pm\sqrt{2\mathrm{i}}}{2\mathrm{i}}$ $=\dfrac{5+\mathrm{i}\pm(1+\mathrm{i})}{2\mathrm{i}}$ $=\dfrac{6+2\mathrm{i}}{2\mathrm{i}}$ or $\dfrac{4}{2\mathrm{i}}$ $=1-3\mathrm{i}$ or $-2\mathrm{i}$
(a) (ii)	$-\mathrm{i}w^2-(1-5\mathrm{i})w+2-6\mathrm{i}=0$ Since $w=\mathrm{i}z$, $w=\mathrm{i}(1-3\mathrm{i})$ or $\mathrm{i}(-2\mathrm{i})$ $=3+\mathrm{i}$ or 2
(a) (iii)	Since $P(z)$ is a polynomial of degree 4 with real coefficient, hence $1+3\mathrm{i}$ and $2\mathrm{i}$ are also the roots. $P(z)=(z+2\mathrm{i})(z-2\mathrm{i})(z-1-3\mathrm{i})(z-1+3\mathrm{i})$ $=(z^2+4)((z-1)^2+9)$ $=(z^2+4)(z^2-2z+10)$ $=z^4-2z^3+14z^2-8z+40$