Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9758/01

Paper 1

10 September 2024
3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	**5**	**5**	**7**	**8**	**7**	**9**	**9**	**14**	**12**	**12**	**12**	**100**

1 **(a)** Without the use of a calculator, solve the inequality $\frac{2x^2+2x-11}{x^2-2x+1} \geq 1$. [3]

(b) Hence solve the inequality $\frac{11x^2-2x-2}{x^2-2x+1} \leq -1$. [2]

Do not use a calculator in answering this question.

2 The complex number z is given by $z = \dfrac{\left(\cos\dfrac{\pi}{12} - \mathrm{i}\sin\dfrac{\pi}{12}\right)^2}{\cos\dfrac{\pi}{6} + \mathrm{i}\sin\dfrac{\pi}{6}}$.

(a) Find z in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$. [2]

(b) Show that $(1+z)^3 = p\mathrm{i}$, where p is a real constant to be determined. Hence or otherwise, find $(1+z)^3 + (1+z^*)^3$. Show your working clearly. [3]

3 It is given that $x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\ln x}{x}$ where $x>0$, and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1$ at $x=1$. Use the substitution $z=x\dfrac{\mathrm{d}y}{\mathrm{d}x}$ to show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2}\dfrac{(\ln x)^2}{x}+\dfrac{1}{x}$. Hence find the exact equation of the tangent to the curve $y=\mathrm{f}(x)$ at $\left(\mathrm{e},\frac{7}{6}\right)$. [7]

4 **(a)** Find $\displaystyle\int \frac{14+3x}{\sqrt{9-8x-x^2}}\,\mathrm{d}x$. [4]

(b) Find $\displaystyle\int_0^2 3x^2\mathrm{e}^{kx}\ \mathrm{d}x$ in terms of k, where k is a positive constant. Explain whether there exist solutions for k satisfying the equation $\displaystyle\int_0^2 3x^2\mathrm{e}^{kx}\ \mathrm{d}x = -\frac{6}{k^3}$. [4]

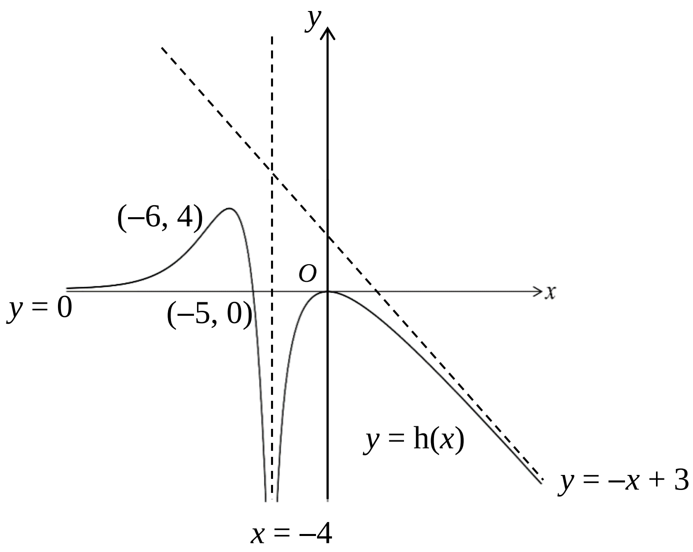
5 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{x^2}{x-4}, \ x \in \mathbb{R}, \ 4 < x \le 8.$$

(a) Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. On the same axes, sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$. [4]

(b) The region R is bounded by the curve $y = \mathrm{f}^{-1}(x)$, the lines $y = 5$, $y = 8$ and the y-axis. Find the exact area of R. [3]

6



(a) The diagram shows the curve $y = \mathrm{h}(x)$. The curve has maximum points at $(-6,\ 4)$ and the origin, and crosses the x-axis at $(-5, 0)$. The lines $y = 0$, $x = -4$ and $y = -x + 3$ are the horizontal, vertical and oblique asymptotes to the curve respectively.

(i) On the diagram given above, sketch the graph of $(x+6)^2 + (y-9)^2 = r^2$, where r is a positive constant. State the range of values of r for the equation $(x+6)^2 + (\mathrm{h}(x)-9)^2 = r^2$ to have at least one real root. [3]

(ii) On a separate diagram, sketch the graph of $y = \dfrac{1}{\mathrm{h}(x)}$. [3]

(b) The graph of $y = 10 - |x+1|$ undergoes a sequence of transformations which transform its equation into $y = |x-1|$. Describe and write down the transformations. [3]

7 The points A, B and C represent the complex numbers a, b and c respectively, such that $a=0$, $b=3$ and $c=-2+\mathrm{i}$. The three complex numbers are roots to the equation $\mathrm{f}(z)=0$ where $\mathrm{f}(z)$ is a quartic polynomial with real coefficients and z is a complex variable.

(a) Express $\mathrm{f}(z)$ as a product of two quadratic factors with real coefficients. [3]

(b) Sketch an Argand diagram showing the roots of the equation $\mathrm{f}(z)=0$. [2]

(c) The point W represents the complex number w, such that $c = \mathrm{i}\,w$. Find the value of $\overrightarrow{AC} \bullet \overrightarrow{AW}$ and the area of the triangle APW where P represents the complex number $c - b$. [4]

8 The line l_1 passes through the point A (1, 2, 4) and point B (−1, −1, 3). The line l_2 has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 6 \\ -7 \\ 2 \end{pmatrix},\ t \in \mathbb{R}.$$

(a) Explain why l_1 and l_2 are skew lines. [2]

(b) Find an equation of the plane, in scalar product form, that includes the midpoint of AB and the line l_2. [3]

The plane π_1 has equation $2x+7y+5z=24$. The point C lies on l_1 such that the foot of perpendicular of C onto π_1 has coordinates (3, 1, 1).

(c) Find the coordinates of C. [3]

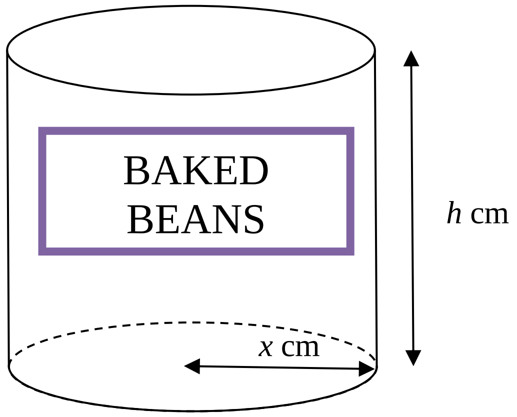
The plane π_2 has equation $3x-4y+\lambda z=\mu$ and line l_1 does not intersect π_2.

(d) Find the value of λ. Hence find the acute angle between the planes π_1 and π_2. [3]

8 [Continued]

(e) If the distance between π_2 and l_1 is 2 units, find the exact values of μ. [3]

9



A food company produces cans of baked beans. Each can is in the form of a closed right cylinder with a base radius of x cm and a height of h cm (see diagram) and its capacity is V cm^3, where V is a fixed constant. The cans are made of steel metal sheets with negligible thickness. The cost to make the curved surface of the can is 1 cent per cm^2 and the cost to make the top and bottom surfaces is k cents per cm^2. Let C cents be the production cost of a can. For economic reasons, the value of C is minimised by varying the value of x.

(a) Express h in terms of π, x and V. [1]

(b) Using differentiation, show that C is a minimum when $\dfrac{x}{h} = \dfrac{1}{2k}$. [6]

9 [Continued]

(c) (i) By using the relation in part **(a)**, show that, as h varies with time t, $\dfrac{\mathrm{d}h}{\mathrm{d}t}=-\dfrac{2h}{x}\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)$. [2]

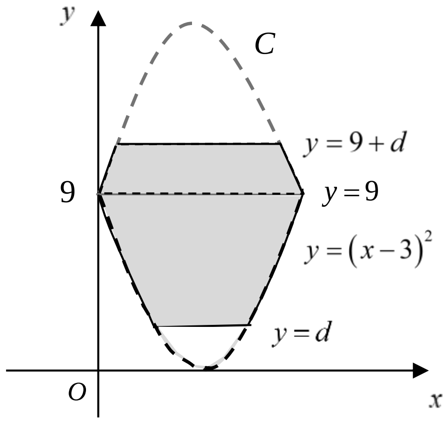
(ii) Due to inflation, k increases at a constant rate of 0.1 units per month. Use the relation in part **(b)** or otherwise, to find the rate of change of x at the instant when $k = 2$ and $x = 1$. [3]

10 **(a)** A curve is defined by the parametric equations

$$x = 2t^2 - t, \quad y = \frac{4}{t^3 - t}, \qquad \text{where } t > 1.$$

Find the area bounded by the curve, the x-axis and the lines $x = 3$ and $x = 6$. Leave your answer correct to 3 decimal places. [3]

(b) A chef plans to create an ornament for his master dish. The ornament is made by rotating the shaded region as shown in the diagram completely about the y-axis. The region is bounded by the parabola $y=\left(x-3\right)^2$, the curve C and the lines $y=9+d$ and $y=d$, where $0<d<9$. The curve C is the reflection of the parabola along the line $y = 9$.



(i) Find the equation of the curve C. [2]

(ii) Find the volume of the ornament in terms of d. [5]

10 [Continued]

(iii) By varying the value of d, find the maximum volume for the ornament correct to the nearest integer. State the value of d corresponding to this maximum volume. [2]

11 Taylor decides to manage her caffeine consumption by following a regime. Before starting the regime, there is no caffeine in her body. On the first day, she drinks two cups of coffee at 9 am and only one cup of coffee at 9 am on each subsequent day. Each cup of coffee contains 100 mg of caffeine. The caffeine level in her body decreases by 80% in 24 hours. You may assume that the time taken for her to drink coffee is negligible.

(a) Find the amount of caffeine in her body after consuming a cup of coffee at 9 am on the second day. [2]

(b) Find the amount of caffeine in her body after consuming a cup of coffee at 9 am on the nth day. [4]

11 [Continued]

(c) On which day at 9 am after consuming the coffee, will the amount of caffeine in her body to first go below 125.1 mg? [2]

Taylor's friend, Travis, follows her regime. Due to different body conditions, the caffeine level in his body decreases by q% in 24 hours.

(d) If the caffeine level in his body exceeds 400 mg at any time, it will be harmful to his body. Explain why this situation will never happen when $25 < q < 50$. [4]

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Section A: Pure Mathematics [40 marks]

1 It is given that $\mathrm{f}(r)=r^3$.

(a) Using the method of differences, find $\sum_{r=1}^{n}\left(\mathrm{f}(r+1)-\mathrm{f}(r)\right)$, leaving your answer in terms of n.

[2]

(b) By evaluating $\mathrm{f}(r+1)-\mathrm{f}(r)$ and using the result in part **(a)**, show $\sum_{r=1}^{n} r^2=\frac{n}{6}(n+1)(2n+1)$.

[4]

2 It is given that $y = \mathrm{f}(x)$, where $y^3 + 8 = 3xy$.

Find the Maclaurin series for $\mathrm{f}(x)$ up to and including the term in x^3. [6]

Hence write down the equation of the normal at the point on the curve of $y = \mathrm{f}(x)$ at $x = 0$. [1]

3 (a) (i) Write down the derivative of $\tan^{-1} x$. [1]

(ii) Hence use a suitable standard series from the List of Formulae (MF26) to find the Maclaurin expansion of $\tan^{-1} x$ in ascending powers of x, up to and including the term x^7. Write down, in terms of n, the coefficient of x^{2n-1} in the expansion of $\tan^{-1} x$. [3]

The complex number z_n is given by $z_n = \mathrm{e}^{\mathrm{i}\frac{(-1)^{n+1}a^{2n+1}}{2n-1}}$ for some real constant a.

(b) Find the argument of $z_1 z_2 z_3$, leaving your answer in the form $k\left(a - \frac{a^3}{b} + \frac{a^5}{c}\right)$ where k, b and c are constants to be determined. [2]

(c) Deduce $\lim_{n\to\infty} \arg(z_1 z_2 \ldots z_n)$ when ~~$a = \sqrt{3}$~~. $a = \frac{1}{\sqrt{3}}$ [1]

4 The function f is defined by

$$f(x)=\begin{cases} 2x & \text{for } 1 \le x < 8, \\ \dfrac{x}{2}+12 & \text{for } 8 \le x \le 30. \end{cases}$$

(a) Sketch the graph of $y=\mathrm{f}(x)$. [1]

(b) Solve the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [2]

(c) Show that the composite function f^2 exists and find its range. [2]

(d) Given that the composite function f^n exist for $n \geq 3$, find the range of f^3, f^4 and hence find the range of f^n as $n \to \infty$. [3]

5 Two chemicals X and Y react to form a chemical Z without any loss of mass. It is known that one part of X combines with two parts of Y to give three parts of Z. For example, 1.5 g of X combines with 3.0 g of Y to give 4.5 g of Z. Let z g be the mass of Z formed t minutes after the reaction started. The rate of change of z with respect to t, is proportional to the product of the remaining masses of X and Y not reacted at any time t. Initially, there are 10 g of X, 15 g of Y and 0 g of Z.

(a) Show that

$$\frac{dz}{dt} = k(30 - z)(45 - 2z),$$

where k is a positive constant. [2]

(b) It is observed that the mass of Z is 10 g after 5 min. Solve the differential equation in part **(a)** and find z in terms of t. [8]

(c) State, with justification, the mass of Z that can be formed after a long time. Hence, or otherwise, find the corresponding remaining masses of X and Y. [2]

Section B: Probability and Statistics [60 marks]

6 For events A and B, it is given that $\mathrm{P}(A)=\frac{2}{5}$, $\mathrm{P}\left(A|B'\right)=\frac{3}{5}$ and $\mathrm{P}\left(B|A'\right)=\frac{5}{6}$.

Find

(a) $\mathrm{P}(A'\cap B)$ [1]

(b) $\mathrm{P}(A'\cap B')$, [2]

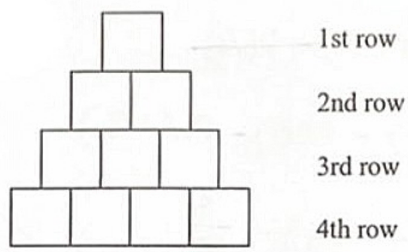
(c) $\mathrm{P}(A \cap B)$. [3]

Determine if events A and B are independent. [1]

7 A bakery packs a box of 10 cupcakes comprising 2 red, 3 blue and 5 green ones. The cupcakes are indistinguishable except for their colour.

(a) Find the number of ways they can be given to 10 children, if each child is given exactly one cupcake. [1]

(b) The bakery packs the 10 cupcakes in a triangular arrangement as shown below.



Find the number of ways to arrange all the cupcakes such that all blue cupcakes are next to each other in the same row. [2]

For another box containing 10 brown identical cupcakes, the bakery decides to spell out the word DELECTABLE by using whipped cream to write one letter on each cupcake.

(c) Find the number of ways the cupcakes can be arranged in a row in which the letters are **not** in alphabetical order. For example : the three letters 'BGZ' is arranged in alphabetical order while 'GBZ' is not. [2]

(d) How many three-letter code word can be formed from the word DELECTABLE? [3]

8 **(a)** A student uses only the product moment correlation coefficient to interpret the linear correlation for a sample drawn from a bivariate distribution. Give a reason why he should also draw a scatter diagram to support his answer. [1]

(b) A study is conducted to track the population of a city over a period of time. The population size, y thousands, in x years after Year 2000, are as follows.

x	3	5	7	9	11	13	15	17	19
y	32.3	33.1	35.8	36.1	39.5	41.7	46.4	50.8	57.2

(i) Draw a scatter diagram for these values. [1]

(ii) It is found that the inclusion of a 10^{th} point (x_{10}, y_{10}) will not affect the equations of the regression line y on x and x on y. Find the point (x_{10}, y_{10}). [1]

Omit the 10^{th} point (x_{10}, y_{10}) for the rest of this question.

(iii) Without calculating the product moment correlation coefficient, explain which of the following equations, where a and b are positive constants, is more appropriate to model the relationship between x and y.

$$\text{(A) } y = a + bx^2 \qquad \text{(B) } y = a + b\sqrt{x}$$

[1]

(iv) Using the more appropriate model in part **(iii)**, find the equation of the regression line giving the values of a and b. Interpret in context, the meaning of a. [2]

(v) Re-write your equation so that it can be used when the population size, y, is given in millions. [1]

(vi) Find the product moment correlation coefficient for the chosen model in part **(iii)**. Give two reasons why it would be reasonable to use the equation to estimate the value of y when $x = 6$. [3]

9 Mr Hsu takes the train to office for work every weekday and is supposed to arrive at office by 7.30 a.m. On average, he reaches the train station at 6.00 a.m. His arrival time at the train station is normally distributed with a standard deviation of 10 minutes. Every morning, there are only two trains, each departing 5 minutes apart, with the first train departing at 6:10 a.m. sharp. The time taken for the train journey follows a normal distribution with mean 60 minutes and standard deviation 4 minutes.

After alighting from the train, the time taken to walk from the train station to his office follows a normal distribution with mean 10 minutes and standard deviation 3 minutes. Mr Hsu will be late for work if he misses the second train or arrives at office after 7.30 a.m. Assume that all travelling and waiting times are independent of each other.

(a) On a randomly chosen day, given that Mr Hsu takes the first train, find the probability that he is late for work. [2]

(b) On a randomly chosen day, show that the probability that Mr Hsu is late for work is 0.101. [3]

(c) On a randomly chosen day, Mr Hsu has a briefing to attend before work. Given that he takes the first train, find the earliest starting time of the briefing (correct to the nearest minutes) for which the probability that he is late for the briefing is not more than 0.1. [2]

According to Mr Hsu's office policy, employees who are late for work will face a pay deduction of $D\%$, where D is calculated as 5 times the number of days the employee is late in a month. Given that there are 20 workdays in a month,

(d) find the probability that Mr Hsu receives between 60% and 80% inclusive, of his salary in a randomly chosen month. [3]

10 For quality control, the production manager of a company wishes to take a random sample of a certain type of chocolate bar from his factory. He wants to check that the mean mass of the bars is 52 grams, as stated on the packets.

(a) State what it means for a sample to be random in this con[illegible] [1]

The masses, x grams, of a random sample of 80 chocolate bars are summarised as follows.

$$n = 80 \qquad \sum(x-52) = -37 \qquad \sum(x-52)^2 = 310.7$$

(b) Calculate the unbiased estimates of the population mean and variance. [2]

(c) What do you understand by the term 'unbiased estimate' ? [1]

(d) Carry out a suitable hypothesis test, explaining the choice between a 1-tail test and a 2-tail test. You should state your hypotheses and define any symbols you use.
Referring to the p-value for your test, explain what it indicates. [6]

State 2 assumptions

~~**(e)** Explain whether there is a need for the manager to know anything about the population distribution of the masses of the chocolate bars.~~ [2]

11 Wildlife biologists are studying the bird populations in a large nature reserve. In a study of one species of birds during breeding season, it was recorded that on average, 3 out of 5 chicks will survive to leave their nests.

(a) State, in context, two assumptions required for the number of chicks, which survive to leave their nest to be well-modelled by a binomial distribution. [2]

Assume now that all nests initially have 4 chicks and that the number of chicks which survive to leave their nest has a binomial distribution.

(b) For a randomly selected nest, find the probability that exactly half the number of chicks will survive to leave their nest. [1]

Individual nests are grouped together to form breeding zones. Each breeding zone comprises 15 such nests. A biologist considers a breeding zone successful if there are more than 10 nests where at least 2 chicks will survive to leave their nest.

(c) Find the probability that exactly two out of three randomly selected breeding zones are successful. [3]

Another study looks at the abundance of several bird species in a specific area in the nature reserve. For this study, the number of birds N, for a given species in the area is observed to follow a probability distribution with parameter α, where $0<\alpha<1$, given by $\mathrm{P}(N=r)=\dfrac{A}{\ln(1-\alpha)}\left(\dfrac{\alpha^r}{r}\right)$, where $r \in \mathbb{Z}^+$, and A is a constant.

You may use the following results without proof.

For $0<x<1$,

- $\displaystyle\sum_{k=1}^{\infty}\left(\frac{x^k}{k}\right)=-\ln(1-x)$
- $\displaystyle\sum_{k=1}^{\infty}\left(kx^k\right)=\frac{x}{(1-x)^2}$.

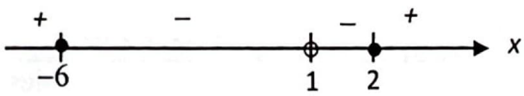
(d) Find the value of A. [2]

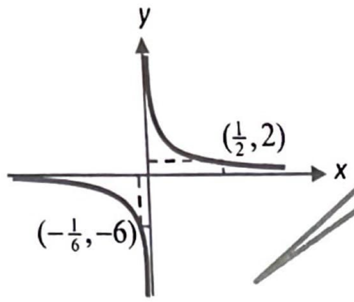
11 [Continued]

(e) If $\alpha = 0.3$, find $\mathrm{P}(4 \le N \le 30)$. [1]

(f) Find $\mathrm{E}(N)$ and $\mathrm{Var}(N)$ in terms of α, leaving your answers as a single fraction. [4]

2024 Year 6 H2 Math Prelim Exam P1 solution and comments

Qn	Suggested Solution	Comments
1(a)	$\dfrac{2x^2+2x-11}{x^2-2x+1} \geq 1$ $\dfrac{2x^2+2x-11}{(x-1)^2} - 1 \geq 0$ $\dfrac{x^2+4x-12}{(x-1)^2} \geq 0$ $\dfrac{(x+6)(x-2)}{(x-1)^2} \geq 0$  $\therefore x \leq -6$ or $x \geq 2$ **Alternative** $\dfrac{2x^2+2x-11}{x^2-2x+1} \geq 1$ $\dfrac{2x^2+2x-11}{(x-1)^2} \geq 1$ $2x^2+2x-11 \geq (x-1)^2 \quad (\because x \neq 1 \Rightarrow (x-1)^2 > 0 \ \forall x \in \mathbb{R})$ Note that this reasoning is required to justify the inequality sign being unchanged $x^2+4x-12 \geq 0$ $(x+6)(x-2) \geq 0$ $\therefore x \leq -6$ or $x \geq 2$	
(b)	$\dfrac{2+2x-11x^2}{x^2-2x+1} \geq 1$ For $x \neq 0$, Replace x with $\dfrac{1}{x}$, $\dfrac{\frac{2}{x^2}+\frac{2}{x}-11}{1-\frac{2}{x}+\frac{1}{x^2}} \geq 1$ (multiply by $\dfrac{x^2}{x^2}$ and -1 on both side) $\dfrac{11x^2-2x-2}{x^2-2x+1} \leq -1$ $\therefore$ For $x \neq 0$, $\dfrac{1}{x} \leq -6$ or $\dfrac{1}{x} \geq 2$	Students are reminded to make use of previous result in (a) when seeing the word "Hence" in question instead of using the same method to solve the question

	 So we have $-\frac{1}{6}\le x<0$ or $0<x\le\frac{1}{2}$ In addition, note that $x=0$ also satisfies the given inequality in (b) thus $-\frac{1}{6}\le x\le\frac{1}{2}$	The solution $-\frac{1}{6}\le x<0$ or $0<x\le\frac{1}{2}$ is valid only for $x\neq 0$, Hence students need to sub $x=0$ into inequality $\frac{11(0)^2-2(0)-2}{(0)^2-2(0)+1}\le -1$ $-2\le -1$ (True)
		Total marks: 5

Qn	Suggested Solution	Comments
2(a)	$z=\dfrac{\left(\cos\left(-\frac{\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right)^2}{\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}}=\dfrac{\left(e^{\mathrm{i}\left(-\frac{\pi}{12}\right)}\right)^2}{e^{\mathrm{i}\left(\frac{\pi}{6}\right)}}$ $=e^{\mathrm{i}\left(-\frac{\pi}{6}-\frac{\pi}{6}\right)}=e^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$ $\cos\left(\frac{\pi}{12}\right)-\mathrm{i}\sin\left(\frac{\pi}{12}\right)$ $=\cos\left(-\frac{\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{\pi}{12}\right)$ Note $re^{i\theta}=r\left(\cos\theta+i\sin\theta\right)$ hence both modulus of complex number in numerator and denominator is 1 (not $\sqrt{2}$)	Useful properties For $0<\theta<\frac{\pi}{2}$, $\cos\theta=\cos(-\theta)$ $\sin\theta=-\sin(-\theta)$ (can use graph to deduce) For power $/\times/\div$ operation of complex numbers, it is always more helpful to express in exponential form first
(b)	**<u>Method 1 (half-power)</u>** $(1+z)^3=\left(1+e^{\mathrm{i}\left(-\frac{\pi}{3}\right)}\right)^3$ $=\left[e^{\mathrm{i}\left(-\frac{\pi}{6}\right)}\left(e^{\mathrm{i}\left(\frac{\pi}{6}\right)}+e^{\mathrm{i}\left(-\frac{\pi}{6}\right)}\right)\right]^3$ $=\left(e^{\mathrm{i}\left(-\frac{\pi}{6}\right)}\right)^3\left(2\cos\left(\frac{\pi}{6}\right)\right)^3$ $=e^{\mathrm{i}\left(-\frac{\pi}{2}\right)}\left(2\times\frac{\sqrt{3}}{2}\right)^3$ $=-3\sqrt{3}\,\mathrm{i}$, where $p=-3\sqrt{3}$	Refer to Ch13B, p16 Ex6 of notes on **"Taking Out Half Power"** technique. Again students are encouraged to express complex numbers in exponential form first before applying power operation Students are reminded to revise on converting exponential to trigo to cartesian form of complex number (vice-versa)

Method 2 (1+z in exponential form)

$$(1+z)^3 = (1+\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right))^3$$
$$=\left(\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)^3$$
$$=\left(\sqrt{3}e^{-\frac{\pi}{6}i}\right)^3$$
$$=\sqrt{3}^3 e^{-\frac{\pi}{2}i}$$
$$=-3\sqrt{3}\,i$$

Method 3 (apply binomial expansion)

$$(1+z)^3 = 1+3z+3z^2+z^3$$
$$=1+3e^{i\left(-\frac{\pi}{3}\right)}+3\left(e^{i\left(-\frac{\pi}{3}\right)}\right)^2+\left(e^{i\left(-\frac{\pi}{3}\right)}\right)^3$$
$$=1+3\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$$
$$+3\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)+e^{i(-\pi)}$$
$$=1+3\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)+3\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)+(-1)$$
$$=-3\sqrt{3}\,i$$

$$(1+z)^3+(1+z^*)^3$$
$$=(1+z)^3+\left[(1+z)^*\right]^3$$
$$=(1+z)^3+\left[(1+z)^3\right]^*$$
$$=-3\sqrt{3}\,i+\left(3\sqrt{3}\,i\right)$$
$$=0$$

Learn to observe

$1+z^*$ is conjugate of $1+z$

$(1+z)^3$ is conjugate of $\left[(1+z)^*\right]^3$

Hence $(1+z)^3$ and $(1+z^*)^3$ are conjugates of each other

Refer to Ch13A, p8 of notes - **Properties of Complex Conjugates.**

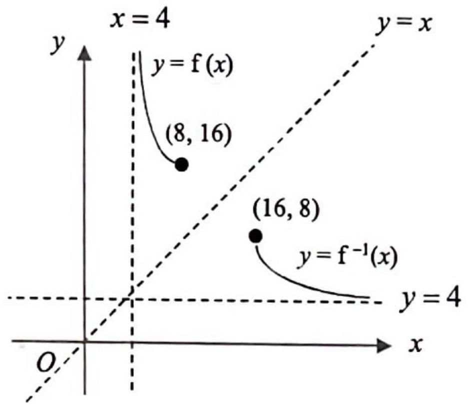
Total marks: 5

Qn	Suggested Solution	Comments
3	$z=x\dfrac{dy}{dx}$ $\dfrac{dz}{dx}=x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}$ ... (1) Given $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=\dfrac{\ln x}{x}$ ... (2) Substitute (1) in LHS (2) : $\dfrac{dz}{dx}=\dfrac{\ln x}{x}$ $z=\int\dfrac{\ln x}{x}\,dx=\dfrac{1}{2}(\ln x)^2+C$ $x\dfrac{dy}{dx}=\dfrac{1}{2}(\ln x)^2+C$ $\dfrac{dy}{dx}=\dfrac{1}{2}\dfrac{(\ln x)^2}{x}+\dfrac{C}{x}$ At $x=1,\dfrac{dy}{dx}=1,$ $1=\dfrac{1}{2}\dfrac{(\ln 1)^2}{1}+\dfrac{C}{1}$ $\Rightarrow C=1$ $\dfrac{dy}{dx}=\dfrac{1}{2}\dfrac{(\ln x)^2}{x}+\dfrac{1}{x}$ (shown) To find eqn of tangent at $\left(e,\dfrac{7}{6}\right)$, $\dfrac{dy}{dx}=\dfrac{1}{2}\dfrac{(\ln e)^2}{e}+\dfrac{1}{e}=\dfrac{3}{2e}$ $y-\dfrac{7}{6}=\dfrac{3}{2e}(x-e)$ $y=\dfrac{3}{2e}x-\dfrac{1}{3}$	Note that x, y & z are variables, $z=x\dfrac{dy}{dx}$ is to be differentiated wrt x & substituted into $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=\dfrac{\ln x}{x}$ to simplify to $\dfrac{dz}{dx}=f(x)$. It's useful to view $z=\int\dfrac{\ln x}{x}\,dx=\int\dfrac{1}{x}\times(\ln x)\,dx$ $=\int f'(x)\times(f(x))^1\,dx=\dfrac{(\ln x)^2}{2}+C$ To show $\dfrac{dy}{dx}=\dfrac{1}{2}\dfrac{(\ln x)^2}{x}+\dfrac{1}{x}$, the initial conditions $\dfrac{dy}{dx}=1$ when $x=1$ must be used & workings need to be explicit. After finding the gradient at $\left(e,\frac{7}{6}\right)$, the exact equation of the tangent to the curve $y=f(x)$ at the same point can be found.
		Total marks: 7

MF 26 formula once you have completed the square. Check that coefficient of x must be 1.

Qn	Suggested Solution	Comments
4(a)	$\int \frac{14+3x}{\sqrt{9-8x-x^2}}\,dx$ $=\int \frac{2}{\sqrt{9-8x-x^2}}-\frac{3}{2}\left(\frac{(-8-2x)}{\sqrt{9-8x-x^2}}\right)dx$ $=2\int \frac{1}{\sqrt{25-(x+4)^2}}\,dx-\frac{3}{2}\int \frac{(-8-2x)}{\sqrt{9-8x-x^2}}\,dx$ $=2\sin^{-1}\left(\frac{x+4}{5}\right)-\frac{3}{2}\left(\frac{\sqrt{9-8x-x^2}}{-\frac{1}{2}+1}\right)+c$ $=2\sin^{-1}\left(\frac{x+4}{5}\right)-3\sqrt{9-8x-x^2}+c$ Use 2(a) formula. On first impressions, this integration looks like 2(a). Thus, try to rewrite $14+3x=a(-2x-8)+b$	Concepts: 1. Check MF 26 for formulas. (remember that coefficient of x must be 1). 2. Recall the 3 formulas for integration and check whether it can be used, namely: a. $\int f'(x)[f(x)]^n\,dx$ b. $\int \frac{f'(x)}{f(x)}\,dx$ c. $\int f'(x)e^{f(x)}\,dx$ 3. Use by parts as last resort. Note: substitution will be given if qn wants you to do integration by substitution. You will likely have to do some algebraic manipulation in order to use the above procedure.
(b)	$\int_0^2 3x^2e^{kx}\,dx$ $=\left[\frac{3x^2e^{kx}}{k}\right]_0^2-\int_0^2\frac{6xe^{kx}}{k}\,dx$ $=\frac{12}{k}e^{2k}-\frac{1}{k}\left(\left[\frac{6xe^{kx}}{k}\right]_0^2-\int_0^2\frac{6e^{kx}}{k}\,dx\right)$ $=\frac{12}{k}e^{2k}-\frac{12}{k^2}e^{2k}+\frac{6}{k^2}\left[\frac{e^{kx}}{k}\right]_0^2$ $=\frac{12}{k}e^{2k}-\frac{12}{k^2}e^{2k}+\frac{6}{k^2}\left[\frac{e^{2k}}{k}-\frac{1}{k}\right]$ $=\frac{6e^{2k}\left(2k^2-2k+1\right)-6}{k^3}$ **Method 1** $\frac{6e^{2k}\left(2k^2-2k+1\right)-6}{k^3}=-\frac{6}{k^3}$ $6e^{2k}\left(2k^2-2k+1\right)=0$ Since $e^{2k}>0$, $2k^2-2k+1=0$ but there are no solutions for k as Discriminant $=2^2-4(2)(1)<0$. If you were to follow the procedure, you will see that you have no choice but to use by parts. Choose x^2 as u (LIATE) Do by parts one more time to remove the algebra. Choose x as u (LIATE). Note: Do not reverse the order for u and dv/dx when doing by parts twice. You will get back the same integral as before.	

	Method 2 Since $3x^2e^{kx} \geq 0$ for $0 \leq x \leq 2$, $\int_0^2 3x^2e^{kx}\,dx$ denotes the area bounded by the x-axis, the curve with equation $y = 3x^2e^{kx}, x = 0$ and $x = 2$ which is positive. Thus it can never be equal to $-\dfrac{6}{k^3}$ which is always negative for all real positive number k i.e. there are no solutions for k.		
			Total marks: 8

Qn	Suggested Solution	Comments
5(a)	Let $y = \dfrac{x^2}{x-4}$. $xy - 4y = x^2 \Rightarrow x^2 - xy + 4y = 0$ Using quad formula, $x = \dfrac{y \pm \sqrt{y^2 - 16y}}{2}$ (or by completing the square) Pick a point from $4 < x \leq 8$ say $x = 5, y = \dfrac{5^2}{5-4} = 25$. Check $x = \dfrac{25 \pm \sqrt{25^2 - 16(25)}}{2}$ $= \dfrac{25 \pm 15}{2} = 5 \text{ or } 20$ $\therefore x = \dfrac{y - \sqrt{y^2 - 16y}}{2}$ Since $4 < x \leq 8$, take a point (5,25) to check $\Rightarrow x = \dfrac{y - \sqrt{y^2 - 16y}}{2}$ $f^{-1}(x) = \dfrac{x - \sqrt{x^2 - 16x}}{2}$ $D_{f^{-1}} = R_f = [16, \infty)$  	• When it's difficult to make x the subject, using the quadratic formula/complete the square are the preferred methods. • When sketching graphs of $y = f(x)$ and $y = f^{-1}(x)$, note the following : ✓ both axes of equal scale ✓ symmetrical about $y = x$ ✓ approach asymptotes ✓ label end points (8,16) & (16,8) clearly

(b)

Method 1

Area of the region R = area of region Q

$$= \int_5^8 f(x)\,dx = \int_5^8 \frac{x^2}{x-4}\,dx$$

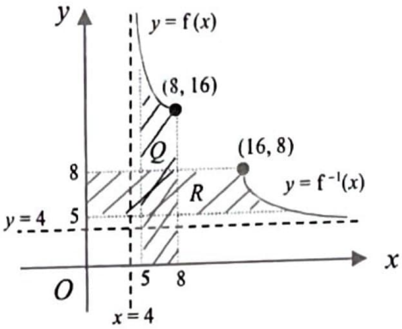
$$= \int_5^8 x+4+\frac{16}{x-4}\,dx$$

$$= \left[\frac{x^2}{2}+4x+16\ln|x-4|\right]_5^8$$

$$= \left[\left(\frac{64}{2}+32+16\ln 4\right)-\left(\frac{25}{2}+20\right)\right]$$

$$= 16\ln 4 + 31.5$$

$$= 32\ln 2 + 31.5 \text{ unit}^2$$



- The symmetrical property of $y=f(x)$ and $y=f^{-1}(x)$ about the line $y=x$, can be used to find the area of R. Because of the symmetry, area R = area Q.

Method 2

Area region $R = \int_5^8 x\,dy = \int_5^8 f(y)\,dy$

$$= \int_5^8 \frac{y^2}{y-4}\,dy$$

$$= \left[\frac{y^2}{2}+4y+16\ln|y-4|\right]_5^8$$

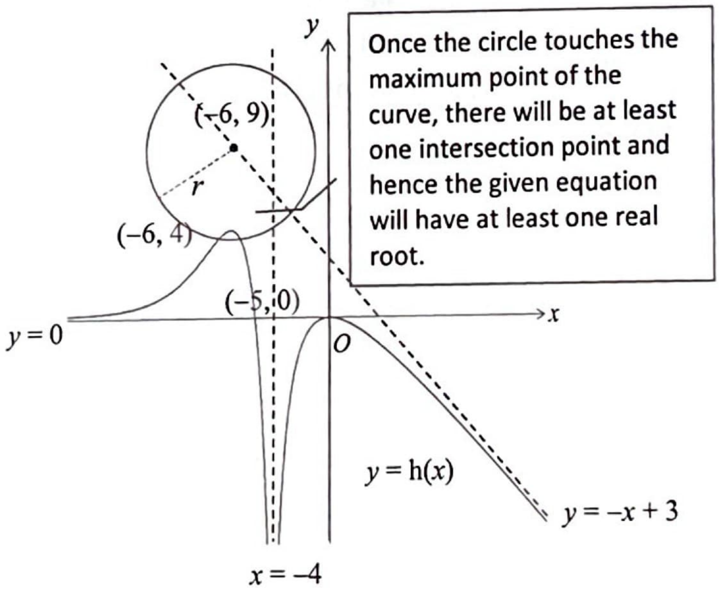
$$= \left[\left(\frac{64}{2}+32+16\ln 4\right)-\left(\frac{25}{2}+20\right)\right]$$

$$= 16\ln 4 + 31.5 = 32\ln 2 + 31.5 \text{ unit}^2$$

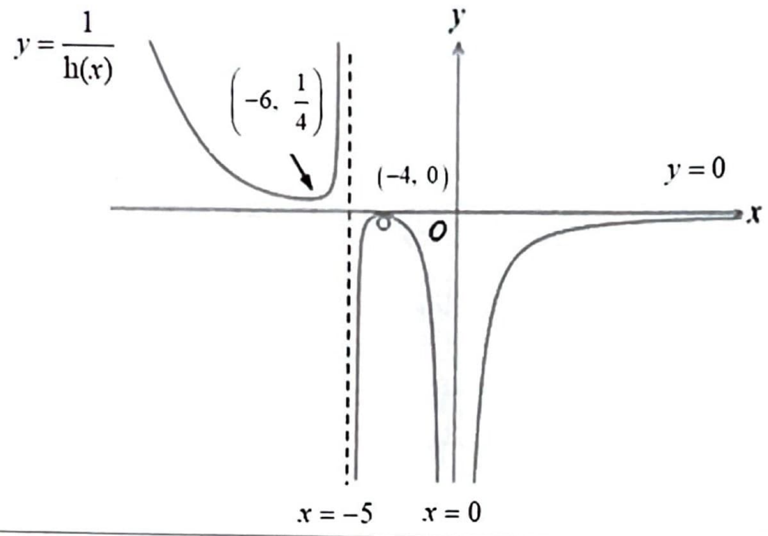
- Alternatively, if we do it directly, $x=f(y)=\frac{y^2}{y-4}$.

So $\int_5^8 x\,dy = \ldots = \int_5^8 \frac{y^2}{y-4}\,dy$ which turns out to be identical to Method 1.

Total marks: 7

Qn	Suggested Solution	Comments
6a(i)		Recall: Equation of a circle of the form $(x-h)^2+(y-k)^2=r^2$ has centre (h,k) and radius r.
	Observe that the centre of the circle lies vertically above the max pt $(-6,4)$ of the curve and on the oblique asymptote $y=-x+3$. Required range : $r \geq 5$	In this case, the centre of the circle is $(-6,9)$

(a)(ii)



Recall the key feature between graph and its reciprocal:
Max pt → min pt
VA → x-intercept
x-intercept → VA
Increasing → decreasing
Decreasing → increasing
When $h(x)$ is positive, the reciprocal will still be positive.

Mark out the above features first and shape of reciprocal will be out.

(b)

Method 1

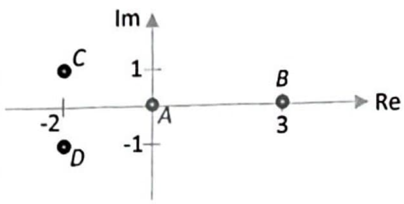
Algebraic manipulation	Transformation
$y = 10 - \lvert x+1 \rvert$	
replace y with $y+10$	Translate the curve ($y = 10 - \lvert x+1 \rvert$) 10 units in the negative y-direction.
$y = -\lvert x+1 \rvert$	
replace y with $-y$	Reflect the curve ($y = -\lvert x+1 \rvert$) about the x-axis.
$y = \lvert x+1 \rvert$	
replace x with $x-2$	Translate the curve ($y = \lvert x+1 \rvert$) 2 units in the positive x-direction.
$y = \lvert x-1 \rvert$	

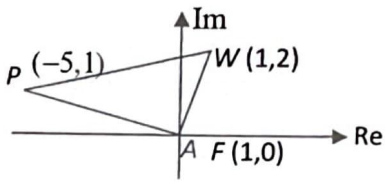
Method 2

Algebraic manipulation	Transformation
$y = 10 - \lvert x+1 \rvert$	
replace y with $-y$	Reflect the curve ($y = 10 - \lvert x+1 \rvert$) about the x-axis.
$y = -10 + \lvert x+1 \rvert$	
replace y with $y-10$	Translate the curve ($y = -10 + \lvert x+1 \rvert$) 10 units in the positive y-direction
$y = \lvert x+1 \rvert$	

For transformation question, it is important to be clear about the corresponding algebraic manipulation.

		replace x with $x-2$ **OR** replace x with $-x$ $y=\lvert x-1\rvert$	Translate the curve ($y=\lvert x+1\rvert$) 2 units in the positive x-direction. Reflect the curve ($y=\lvert x+1\rvert$) about the y-axis.
			Total marks: 9

Qn	Suggested Solution	Comments
7(a)	By conjugate root theorem, if $-2+\text{i}$ is a root, then $-2-\text{i}$ is also a root. $\text{f}(z)$ $=z(z-3)(z-(-2+\text{i}))(z-(-2-\text{i}))$ $=(z^2-3z)((z+2)-\text{i})((z+2)+\text{i})$ $=(z^2-3z)((z+2)^2-\text{i}^2)$ $=(z^2-3z)(z^2+4z+5)$	State the use of conjugate root theorem clearly with clear justification for its use. Take note of the question requirement to leave your answer as the product of two quadratic factors with all real coefficients.
(b)	Let D represent the complex number $-2-\text{i}$. 	Labels for points representing complex numbers should be in capital letters. When labelling points representing complex numbers in cartesian form on Argand diagrams, the following are acceptable: 1. $A(0,0)$, $B(3,0)$, $C(-2,1)$, $D(-2,-1)$ (or equivalently marking the values of the Re and Im parts on the axes as shown in the diagram on the left) 2. $A(0)$, $B(3)$, $C(-2+\text{i})$, $D(-2-\text{i})$ Note that you should not be finding the polar form of the complex numbers $re^{\text{i}\theta}$ for this part.
(c)	Therefore $w=\frac{c}{\text{i}}=-ic$. The point W is obtained by rotating the point representing c about the origin by 90° degrees clockwise. Therefore the angle CAW is 90°. This implies that $\overrightarrow{AC}\bullet\overrightarrow{AW}=0$. **<u>Method 1</u>**	For this question, you should use the vector representations of the complex numbers when performing the associated operations like dot and cross products. For example, ☑ $\overrightarrow{AC}\cdot\overrightarrow{AW}=\begin{pmatrix}-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}=0$ ☑ $\overrightarrow{AC}\cdot\overrightarrow{AW}=\begin{pmatrix}-2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix}=0$

	$\overrightarrow{AP}=\begin{pmatrix}-5\\1\\0\end{pmatrix}$, $\overrightarrow{AW}=\begin{pmatrix}1\\2\\0\end{pmatrix}$ Area of triangle APW $=\frac{1}{2}\left\lvert\overrightarrow{AP}\times\overrightarrow{AW}\right\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}-5\\1\\0\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}\right\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}0\\0\\-11\end{pmatrix}\right\rvert=5.5\text{ units}^2$ **Method 2** Area of triangle APW $=\frac{1}{2}\begin{vmatrix}0 & -5 & 1 & 0\\0 & 1 & 2 & 0\end{vmatrix}$ $=\frac{1}{2}\lvert -10-1\rvert$ $=5.5\text{ units}^2$	☒ $\overrightarrow{AC}\cdot\overrightarrow{AW}=(-2+\mathrm{i})\cdot(1+2\mathrm{i})=-4+3\mathrm{i}$ Next, note that the cross product is only defined for 3D vectors so you can think of the vector representations as lying on the xy − plane i.e. $z=0$. Once the 3D vectors are defined correctly, you may proceed to find the area of the triangle. It is inefficient to use the formula $\frac{1}{2}\left\lvert\overrightarrow{AP}\times\overrightarrow{AW}\right\rvert=\frac{1}{2}\left\lvert\overrightarrow{AP}\right\rvert\left\lvert\overrightarrow{AW}\right\rvert\sin\angle PAW$.  Also note that $\angle PAW$ is obtuse. You can see this once you draw an Argand diagram marking out these three points A, P and W. Do not add points P and W on your diagram in part (b). You may confuse the marker.

Qn	Suggested Solution	Comments
8(a)	$\overrightarrow{AB}=\begin{pmatrix}-1\\-1\\3\end{pmatrix}-\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}-2\\-3\\-1\end{pmatrix}$ $l_1: r=\begin{pmatrix}1\\2\\4\end{pmatrix}+\beta\begin{pmatrix}2\\3\\1\end{pmatrix},\ \beta\in\mathbb{R}$ l_2: $\mathrm{r}=\begin{pmatrix}3\\-5\\2\end{pmatrix}+t\begin{pmatrix}6\\-7\\2\end{pmatrix},\ t\in\mathbb{R}$ Since $\begin{pmatrix}2\\3\\1\end{pmatrix}\neq k\begin{pmatrix}6\\-7\\2\end{pmatrix}$ for any $k\in\mathbb{R}$, l_1 and l_2 are not parallel. Let $\begin{pmatrix}1\\2\\4\end{pmatrix}+\beta\begin{pmatrix}2\\3\\1\end{pmatrix}=\begin{pmatrix}3\\-5\\2\end{pmatrix}+t\begin{pmatrix}6\\-7\\2\end{pmatrix}$. $\Rightarrow\begin{cases}2\beta-6t=2\\3\beta+7t=-7\\\beta-2t=-2\end{cases}$ Solving using the GC, there is **no solution found** for β and t. Hence l_1 and l_2 do not intersect.	• Non-parallel lines may still intersect each other. Therefore, it is not sufficient to only prove that both lines are non-parallel. You would also need to show that there is no points of intersection between both lines. • You need to show your working to justify that the lines are non-parallel and non-intersecting. Simply stating "non-parallel" and "non-intersecting" is not sufficient. • When you solve $\overrightarrow{OA}=\begin{pmatrix}3\\-5\\2\end{pmatrix}+t\begin{pmatrix}6\\-7\\2\end{pmatrix}$, you are trying to determine whether point A lies on line l_2. When you do not get a consistent value of t, it only means that point A

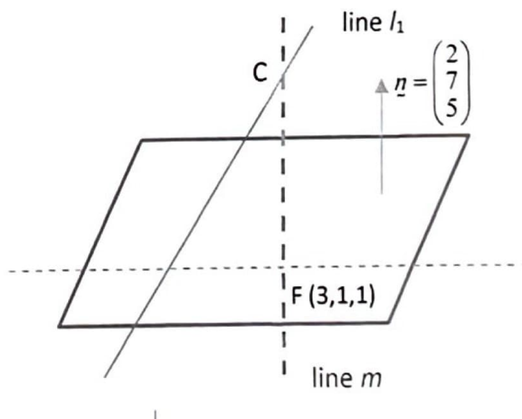
<table>
<tr><td></td><td>Since l_1 and l_2 are non-parallel and non-intersecting lines, they are skew lines (shown).</td><td>does not lies on line l_2. It does not mean that the lines are skew.

• The word "constant" means differently from "consistent". When you want to mean no common values of t is found, you may say that the values are "not consistent".</td></tr>
<tr><td>(b)</td><td>Midpoint of AB = $=\left(\frac{1-1}{2},\frac{2-1}{2},\frac{4+3}{2}\right)=\left(0,\frac{1}{2},\frac{7}{2}\right)$

$\begin{pmatrix}3\\-5\\2\end{pmatrix}-\begin{pmatrix}0\\0.5\\3.5\end{pmatrix}=\begin{pmatrix}3\\-5.5\\-1.5\end{pmatrix}$
Let the normal of the required plane be n.
$\begin{pmatrix}3\\-5.5\\-1.5\end{pmatrix}\times\begin{pmatrix}6\\-7\\2\end{pmatrix}=\begin{pmatrix}-21.5\\-15\\12\end{pmatrix}$
$\mathrm{n}=\begin{pmatrix}-43\\-30\\24\end{pmatrix}$
Hence, an equation of the required plane:
$\mathrm{r}\bullet\begin{pmatrix}-43\\-30\\24\end{pmatrix}=\begin{pmatrix}0\\0.5\\3.5\end{pmatrix}\bullet\begin{pmatrix}-43\\-30\\24\end{pmatrix}=69$
$\mathrm{r}\bullet\begin{pmatrix}-43\\-30\\24\end{pmatrix}=69$</td><td>• Use cross product of two direction vectors of the plane to find the normal vector of the plane.

• Leave your answer in scalar product form, as stated in the question.</td></tr>
<tr><td>(c)</td><td><u>Method 1</u>
Let line m be a line that is perpendicular to π_1 and passes through (3, 1, 1).
$m: r=\begin{pmatrix}3\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}2\\7\\5\end{pmatrix},\ \alpha\in\mathbb{R}$
The 2 lines will intersect at C.

$\begin{pmatrix}1+2\beta\\2+3\beta\\4+\beta\end{pmatrix}=\begin{pmatrix}3+2\alpha\\1+7\alpha\\1+5\alpha\end{pmatrix}$

By GC, $\alpha=1,\ \beta=2$.
Point C has coordinates (5, 8, 6)</td><td>• To help you visualise, this is how the diagram would look like.

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Method 2

Since point C lies on l_1,

$\overrightarrow{OC}=\begin{pmatrix}1+2\lambda\\2+3\lambda\\4+\lambda\end{pmatrix}$ for particular value of λ

Since $\overrightarrow{CF}$ is perpendicular to π_1, then $\overrightarrow{CF}$ is parallel to $\underset{\sim}{n}_{\pi_1}$.

$\overrightarrow{CF}=k\begin{pmatrix}2\\7\\5\end{pmatrix}$

$\begin{pmatrix}2-2\lambda\\-1-3\lambda\\-3-\lambda\end{pmatrix}=k\begin{pmatrix}2\\7\\5\end{pmatrix}$

Solving using GC, $k=-1, \lambda=2$.

Therefore, $\overrightarrow{OC}=\begin{pmatrix}1+2(2)\\2+3(2)\\4+2\end{pmatrix}=\begin{pmatrix}5\\8\\6\end{pmatrix}$

The coordinates of C is (5, 8, 6).

</td><td>

- $\begin{pmatrix}2\\7\\5\end{pmatrix}$ is the normal vector of the plane, meaning it is not parallel to π_1. This means that $\overrightarrow{CF}\cdot\begin{pmatrix}2\\7\\5\end{pmatrix}=0$ is not correct.
- Leave your answer in coordinates form, as stated in the question.

</td></tr>
<tr><td>

(d)

</td><td>

Given l_1 does not intersect π_2,

$\Rightarrow l_1 \,//\, \pi_2 \Rightarrow l_1 \perp \mathbf{n_2}$

$\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\\lambda\end{pmatrix}=0$

$-6+\lambda=0$

$\lambda=6$

$\cos\theta=\dfrac{\left|\begin{pmatrix}2\\7\\5\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\6\end{pmatrix}\right|}{\sqrt{78}\sqrt{61}}$

$\cos\theta=\dfrac{8}{\sqrt{78}\sqrt{61}}$

$\theta=83.3^\circ$ *or* 1.45rad

</td><td>

- Acute angle is required, so remember to include the modulus sign in your formula.
- Indicate your units clearly.

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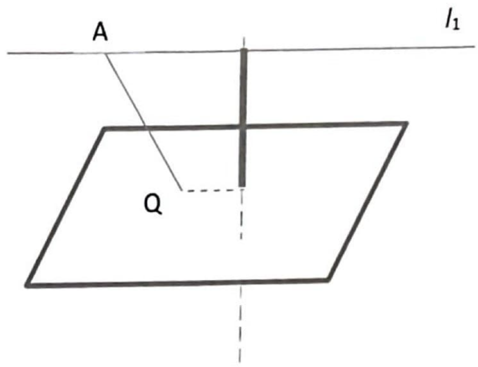
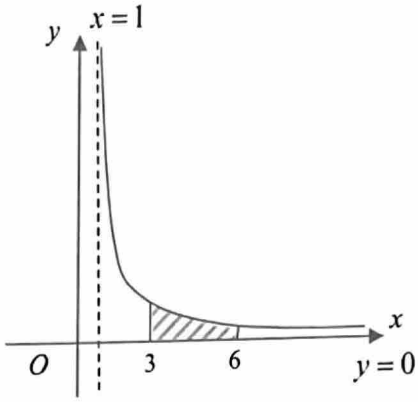
(e)	Let $Q\left(\frac{1}{3}\mu,0,0\right)$ be a point on π_2. $\left\|\overrightarrow{AQ}\cdot\hat{n}\right\|=2$ $\frac{\left\|\begin{pmatrix}1-\frac{1}{3}\mu\\2\\4\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\6\end{pmatrix}\right\|}{\left\|\begin{pmatrix}3\\-4\\6\end{pmatrix}\right\|}=2$ $\frac{\|19-\mu\|}{\sqrt{61}}=2$ $19-\mu=2\sqrt{61}$ or $19-\mu=-2\sqrt{61}$ $\mu=19-2\sqrt{61}$ or $19+2\sqrt{61}$	• This is the length of projection of $\overrightarrow{AQ}$ onto the normal vector of π_2. Here's the diagram to help your visualisation.  • Note that point A can be on the opposite side of the plane as well, hence the modulus sign in the formula.
		Total marks: 14

Qn	Suggested Solution	Comments
9(a)	$V=\pi x^2 h$ $h=\frac{V}{\pi x^2}$	
(b)	$C=2\pi xh+k\left(\pi x^2\right)(2)$ $=2\pi x\left(\frac{V}{\pi x^2}\right)+2\pi x^2 k$ $=\frac{2V}{x}+2\pi kx^2$ $\frac{\mathrm{d}C}{\mathrm{d}x}=-\frac{2V}{x^2}+4\pi kx$ For minimum, $\frac{\mathrm{d}C}{\mathrm{d}x}=0$ $-\frac{2V}{x^2}+4\pi kx=0$ $x^3=\frac{V}{2\pi k}$ $\frac{\mathrm{d}^2C}{\mathrm{d}x^2}=\frac{4V}{x^3}+4\pi k>0$ ($\because V,x,k>0$) C is a minimum.	Write the quantity we want to minimise/maximise accurately, otherwise it will affect the rest of your solution – many marks may be lost. Note that both h and x are variables – from **(a)** we can see that for a fixed constant V, as x varies, h also varies. Thus, we cannot perform differentiation until the expression we want to differentiate is solely in terms of just one variable (either x or h for this question but it is easier to express C in terms of x for this case). We also cannot substitute certain values of x or h here before differentiation with the aim of "removing" the other variable – this is equivalent to us treating the variable as some constant value.

	$\frac{x}{h}=\frac{x}{\left(\frac{V}{\pi x^2}\right)}$ $=\frac{\pi x^3}{V}$ $=\frac{\pi}{V}\left(\frac{V}{2\pi k}\right)$ $=\frac{1}{2k}$ (shown)	The first and second derivative test should be shown clearly and accurately done, with values stated or explanations provided to justify why it is positive, etc.
(c)(i)	$V=\pi x^2 h \Rightarrow h=\frac{V}{\pi x^2}$ $\frac{\mathrm{d}h}{\mathrm{d}x}=-\frac{2V}{\pi x^3}$ Using chain rule $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$ or differentiating w.r.t t directly, we obtain $\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{2V}{\pi x^3}\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{2h}{x}\frac{\mathrm{d}x}{\mathrm{d}t}$ (shown)	The chain rule should be clearly shown. The question stated to use **(a)** but many students used **(b)** instead and ended up with an incorrect derivative because k varies as t varies (no longer a constant), hence, h, x, k are all not constants.
(c)(ii)	$\frac{x}{h}=\frac{1}{2k}$ $2kx=h$ Differentiating w.r.t t directly: $2\left(x\frac{\mathrm{d}k}{\mathrm{d}t}+k\frac{\mathrm{d}x}{\mathrm{d}t}\right)=\frac{\mathrm{d}h}{\mathrm{d}t}$ At $x=1$, $k=2$, $h=4$ and using part **(c)(i)**, $2\left(\frac{\mathrm{d}k}{\mathrm{d}t}+2\frac{\mathrm{d}x}{\mathrm{d}t}\right)=-\frac{2h}{x}\frac{\mathrm{d}x}{\mathrm{d}t}$ $2\left(0.1+2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\right)=-8\frac{\mathrm{d}x}{\mathrm{d}t}$ $12\frac{\mathrm{d}x}{\mathrm{d}t}=-0.2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{60}$ The rate of change of x is $-\frac{1}{60}$ units per month. **Alternative** Using $x^3=\frac{V}{2\pi k}$ and differentiating w.r.t t directly: $3x^2\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{V}{2\pi}\left(\frac{-1}{k^2}\right)\frac{\mathrm{d}k}{\mathrm{d}t}$ At $x=1$, $k=2$, $h=4$ and $V=4\pi$, $3(1)^2\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{4\pi}{2\pi}\left(\frac{-1}{(2)^2}\right)(0.1)$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{60}$ The rate of change of x is $-\frac{1}{60}$ units per month.	If using $\frac{x}{h}=\frac{1}{2k}$, it will be more easily done (or rather, less chances of error) if students differentiated implicitly w.r.t t directly. Some students started getting confused because there are 3 variables h, x, k here. Students tend to be able to handle the alternative method better because it only involves 2 variables since V is a fixed constant. Some students started substituting in values of either $x=1$, $k=2$ or $h=4$ to "remove" variables from the start before differentiating – as mentioned in (b), this is equivalent to us taking the variables as some fixed constant. We should never substitute in values for any variable at the start before differentiating. The values should only be substituted in after the differentiation is done.

			Total marks: 12

Qn	Suggested Solution	Comments
10(a)	$x=2t^2-t,\quad y=\dfrac{4}{t^3-t}$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=4t-1$  $x=3\ ,\ 3=2t^2-t\Rightarrow t=\dfrac{3}{2}$ $x=6,\ 6=2t^2-t\Rightarrow t=2$ Area under the curve $\int_3^6 y\,\mathrm{d}x$ $=\int_{\frac{3}{2}}^{2}\dfrac{4}{t^3-t}(4t-1)\,\mathrm{d}t$ $=3.4864$ $=3.486\text{ units}^2$	Common errors include: - using wrong formula - forgetting to change limit values (to limits w.r.t. t) - not replacing $\mathrm{d}x$ in terms of $\mathrm{d}t$ or doing so incorrectly Remember that the GC can be used to evaluate the integral and to leave your answer to 3d.p. as specified in the question. Convert to Cartesian form in order to find the area – method not advised for this question as not all parametric equations can be convert to Cartesian form
(b)(i)	The equation of curve C is $y=18-(x-3)^2$	
(b)(ii)	$y=(x-3)^2$ $x=3\pm\sqrt{y}$ Required volume below the line $y=9$, $=\pi\int_d^9\left(\left[3+\sqrt{y}\right]^2-\left[3-\sqrt{y}\right]^2\right)\mathrm{d}y$ $=\pi\int_d^9 12\sqrt{y}\,\mathrm{d}y$ $=\pi 12\left(\dfrac{2}{3}\right)\left[y^{3/2}\right]_d^9$ $=8\pi\left[27-(d)^{3/2}\right]$ Required volume above the line $y=9$, $=\pi\int_9^{9+d}\left(\left[3+\sqrt{18-y}\right]^2-\left[3-\sqrt{18-y}\right]^2\right)\mathrm{d}y$	Quite a handful of students used the wrong formula – take note that this is a very costly error and you might end up losing almost all (if not all) the marks for this section. Common formula errors include using: - no π or using 2π - making use of areas to get volumes of solids of revolutions - $\pi\int y^2\mathrm{d}x$

<table>
<tr><td></td><td>

$=\pi\int_9^{9+d} 12\sqrt{18-y}\ dy$

$=\pi 12\left(-\frac{2}{3}\right)\left[(18-y)^{3/2}\right]_9^{9+d}$

$=-8\pi\left[(9-d)^{3/2}-(9)^{3/2}\right]$

$=8\pi\left[27-(9-d)^{3/2}\right]$

Or alternatively, for the required volume above the line $y=9$, due to symmetry, replace d with $9-d$,
volume is $8\pi\left[27-(9-d)^{3/2}\right]$

Total volume is
$8\pi\left[27-(d)^{3/2}\right]+8\pi\left[27-(9-d)^{3/2}\right]=8\pi\left[54-(d)^{3/2}-(9-d)^{3/2}\right]$

</td><td>

- $\pi\int x\ dy$
- $\pi\int (x_1-x_2)^2\ dy$

*Note that the correct formula used should have been of the form $\pi\int x_1^2-x_2^2\ dy$ and $x_1^2-x_2^2\neq(x_1-x_2)^2$

No integration mark was awarded if errors led to a simpler integral to solve.

Most failed to notice that there will be a "hollow" volume that needs to be subtracted and some took them as cylinders/cones which was incorrect.

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<tr><td>(iii)</td><td>

Using symmetry, to get the max volume, d must be 4.5
Volume of ornament
$=8\pi\left[54-(4.5)^{3/2}-(9-4.5)^{3/2}\right]$
$=877.34=877$ units^3

</td><td>

This part was often done by differentiation, which is perfectly fine, but would have depended on your answer in (ii) which could have been incorrect. Do notice that it could have been "observed" from the given diagram using symmetry.

Some students misread that d should be given as an integer – d can be any positive real value where $0<d<9$, while V was to be left to the nearest integer.

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<tr><td></td><td></td><td>Total marks: 12</td></tr>
</table>

Qn	Suggested Solution	Comments
11(a)	Let u_n be the amount of caffeine in day n. $u_1 = 200$ $\therefore u_2 = 0.2(200)+100$ $= 140$	• Question is asking for **amount of caffeine remaining,** not the amount decreased

(b)

$$u_1 = 200$$
$$u_2 = 0.2(200)+100$$
$$u_3 = 0.2[0.2(200)+100]+100$$
$$= 0.2^2(200)+0.2(100)+100$$

Do not evaluate u_2 and u_3 as we want to see the pattern to derive u_n

$$\ldots$$
$$u_n = 0.2^{n-1}(200)+0.2^{n-2}(100)+0.2^{n-3}(100)+\ldots+0.2(100)+100$$
$$= 0.2^{n-1}(200)+100\left(1+0.2+0.2^2+\ldots+0.2^{n-2}\right)$$
$$= 0.2^{n-1}(200)+100\left[\frac{1-(0.2)^{n-1}}{1-0.2}\right]$$

There are n-1 terms for the sum of GP, hence $(0.2)^{n-1}$ for sum of GP formula

$$= 0.2^{n-1}(200)+125\left[1-(0.2)^{n-1}\right]$$
$$= 0.2^{n-1}(75)+125$$

(c)

$$(0.2)^{n-1}(75)+125 < 125.1$$
$$(0.2)^{n-1} < \frac{0.1}{75}$$

Should be $<$, not $\leq$

$$n-1 > \frac{\ln\frac{0.1}{75}}{\ln 0.2}$$
$$n > 5.11328$$

Least $n = 6$

It will be on the 6th day.

__Alternative__

$(0.2)^{n-1}(75)+125 < 125.10$

n	$(0.2)^{n-1}(75)+125$
5	$125.12 > 125.1$
6	$125.02 < 125.1$
7	$125.0048 < 125.1$

From the GC, least $n = 6$

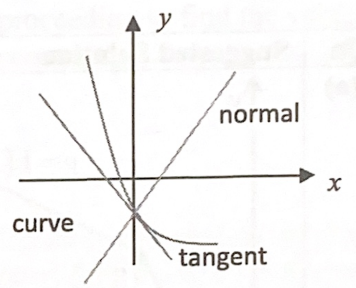
It will be on the 6th day.

(d)	$u_n = 200\left(1-\frac{q}{100}\right)^{n-1} + 100\left(1+\left(1-\frac{q}{100}\right)+\left(1-\frac{q}{100}\right)^2+\ldots+\left(1-\frac{q}{100}\right)^{n-2}\right)$ $= 200\left(1-\frac{q}{100}\right)^{n-1} + 100\left[\frac{1-\left(1-\frac{q}{100}\right)^{n-1}}{1-\left(1-\frac{q}{100}\right)}\right]$ $= 200\left(1-\frac{q}{100}\right)^{n-1} + 100\left[\frac{1-\left(1-\frac{q}{100}\right)^{n-1}}{\frac{q}{100}}\right]$ $= 200\left(1-\frac{q}{100}\right)^{n-1} + \frac{10000}{q}\left[1-\left(1-\frac{q}{100}\right)^{n-1}\right]$ $= \frac{10000}{q} + \left(200-\frac{10000}{q}\right)\left(1-\frac{q}{100}\right)^{n-1}$ As $n \to \infty, u_n \to \frac{10000}{q}$ $(\because 0 < \left(1-\frac{q}{100}\right) < 1)$ For $25 < q < 50$, we have $200-\frac{10000}{q} < 0$ and $1-\frac{q}{100} > 0$ so u_n increases to $\frac{10000}{q} < 400$ (upper bound for $\frac{10000}{q}$ occurs when $q = 25$) as n increases. Since the maximum amount of caffeine in Travis's body is less than 400mg when $25 < q < 50$, Travis is not in danger of consuming too much caffeine	Not $\frac{q}{100}$ but $1-\frac{q}{100}$ for those bracketed terms $200\left[\left(1-\frac{q}{100}\right)^{n-1}\right]+125\left[1-\left(1-\frac{q}{100}\right)^{n-1}\right]$ is incorrect as 125 is obtained from $\frac{100}{1-0.2}$ instead of $\frac{100}{1-\left(1-\frac{q}{100}\right)}$ Determine the sign of $200-\frac{10000}{q}$ and $1-\frac{q}{100}$ to see the behaviour for u_n. You may also graph u_n for $25 < q < 50$.
		Total marks: 12

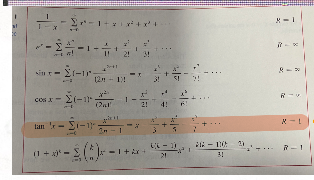
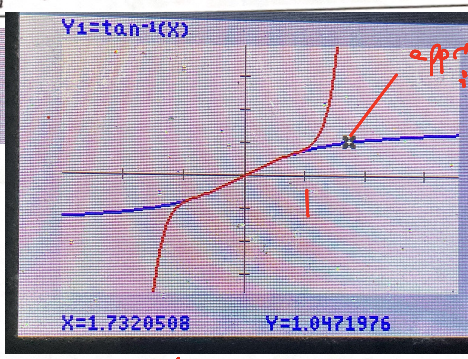
2024 Year 6 H2 Math Prelim Exam P2 solution and comments

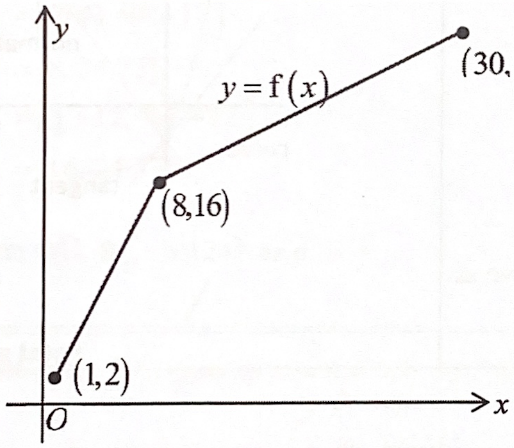
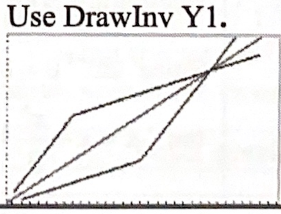
Section A: Pure Mathematics [40 marks]

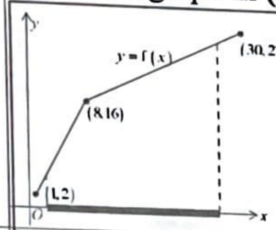
Qn	Suggested Solution	Comments
1(a)	$\sum_{r=1}^{n}\left(\mathrm{f}(r+1)-\mathrm{f}(r)\right)$ $= \left[\cancel{\mathrm{f}(2)}-\mathrm{f}(1)\right]$ $+\left[\cancel{\mathrm{f}(3)}-\cancel{\mathrm{f}(2)}\right]$ $\vdots$ $+\left[\cancel{\mathrm{f}(n)}-\cancel{\mathrm{f}(n-1)}\right]$ $+\left[\mathrm{f}(n+1)-\cancel{\mathrm{f}(n)}\right]$ $=\mathrm{f}(n+1)-\mathrm{f}(1)$ $=(n+1)^3-1$	As stated in the question, **MOD** is to be applied. Clear cancellation of terms must be evident to arrive at the final answer (in terms of n).
(b)	$\mathrm{f}(r+1)-\mathrm{f}(r)=(r+1)^3-r^3=3r^2+3r+1$ $\sum_{r=1}^{n}\left(3r^2+3r+1\right)=(n+1)^3-1$ $\sum_{r=1}^{n}\left(3r^2\right)+\sum_{r=1}^{n}(3r+1)=(n+1)^3-1$ $\sum_{r=1}^{n}\left(3r^2\right)+\frac{n}{2}(3n+5)=(n+1)^3-1$ $\sum_{r=1}^{n}\left(3r^2\right)=(n+1)^3-1-\frac{n}{2}(3n+5)$ $\sum_{r=1}^{n}\left(3r^2\right)=n^3+3n^2+3n-\frac{n}{2}(3n+5)$ $\sum_{r=1}^{n}\left(3r^2\right)=\frac{n}{2}\left(2n^2+6n+6-3n-5\right)$ $=\frac{n}{2}\left(2n^2+3n+1\right)$ $=\frac{n}{2}(n+1)(2n+1)$ $\sum_{r=1}^{n}\left(r^2\right)=\frac{n}{6}(n+1)(2n+1)$ (shown)	As required, $\mathrm{f}(r+1)-\mathrm{f}(r)$ has to be evaluated before applying the result in **part (a)**. $\sum_{r=1}^{n}\left(3r^2+3r+1\right)$ needs to be split so that the AP formula can be applied to evaluate $\sum_{r=1}^{n}(3r+1)$ to find $\sum_{r=1}^{n}r^2$. As the end result $\sum_{r=1}^{n}r^2=\frac{n}{6}(n+1)(2n+1)$ is given, clear workings must be shown; for e.g., the factorisation of the cubic expression.
		Total marks: 6

Qn	Suggested Solution	Comments
2	$y^3+8=3xy$ $3y^2\dfrac{dy}{dx}=3\left(x\dfrac{dy}{dx}+y\right)$ $(y^2-x)\dfrac{dy}{dx}=y$ $\left(2y\dfrac{dy}{dx}-1\right)\dfrac{dy}{dx}+(y^2-x)\dfrac{d^2y}{dx^2}=\dfrac{dy}{dx}$ $2y\left(\dfrac{dy}{dx}\right)^2-2\dfrac{dy}{dx}=(x-y^2)\dfrac{d^2y}{dx^2}$ ---- (1) $2\dfrac{dy}{dx}\left(\dfrac{dy}{dx}\right)^2+4y\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right)-2\dfrac{d^2y}{dx^2}$ $=\left(1-2y\dfrac{dy}{dx}\right)\dfrac{d^2y}{dx^2}+(x-y^2)\dfrac{d^3y}{dx^3}$ ---- (2) When $x=0$, $y=-2$ and $\dfrac{dy}{dx}=-\dfrac{1}{2}$ From (1), $-4\left(\dfrac{1}{4}\right)+1=-4\dfrac{d^2y}{dx^2}\Rightarrow\dfrac{d^2y}{dx^2}=0$ From (2), $2\left(-\dfrac{1}{2}\right)^3=-4\dfrac{d^3y}{dx^3}\Rightarrow\dfrac{d^3y}{dx^3}=\dfrac{1}{16}$ Hence $y=-2-\dfrac{1}{2}x+\dfrac{1}{96}x^3$ Hence, the equation of the normal to the curve at $x=0$ is $y=-2+2x$	• Differentiate using implicit differentiation. It is impossible to make y the subject for this equation. • Remember to apply chain rule when differentiating y^2 with respect to x. • Refer to the Maclaurin's Series expansion in the MF26 booklet, so that you may apply the formula correctly. • From the diagram below, you can see that the y-intercept of the tangent and normal remains the same, and gradient of normal $=-\dfrac{1}{\text{gradient of tangent}}$ 
		Total marks: 7

Qn	Suggested Solution	Comments
3(a)(i)	$\dfrac{d}{dx}(\tan^{-1}x)=(1+x^2)^{-1}$	This can be found in MF26.
(a)(ii)	$(1+x^2)^{-1}=1-x^2+x^4-\ldots,$ Integrating both sides, $\tan^{-1}x=C+x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots$ Since $\tan^{-1}0=0$, $C=0$. $\tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots$	Standard series refer to the series expansion of $(1+x)^n, e^x, \sin x, \cos x$ and $\ln(1+x)$ etc. It is inefficient to use the method of repeated differentiation here. When performing integration, remember the arbitrary constant and sub in value to solve for it.

	Coefficient of $x^{2n-1}=\dfrac{(-1)^{n+1}}{2n-1}$ or $\dfrac{(-1)^{n-1}}{2n-1}$	To deduce the coefficient of x^{2n-1}, you may refer to part (b), i.e. $z_n=\mathrm{e}^{\mathrm{i}\frac{(-1)^{n+1}a^{2n+1}}{2n-1}}$
(b)	$z_n=\mathrm{e}^{\mathrm{i}\frac{(-1)^{n+1}a^{2n+1}}{2n-1}}$ $z_1=\mathrm{e}^{\mathrm{i}a^3}$ $z_2=\mathrm{e}^{-\mathrm{i}\frac{a^5}{3}}$ $z_3=\mathrm{e}^{\mathrm{i}\frac{a^7}{5}}$ $\arg(z_1z_2z_3)=a^3-\dfrac{a^5}{3}+\dfrac{a^7}{5}=a^2\left(a-\dfrac{a^3}{3}+\dfrac{a^5}{5}\right)$ where $k=a^2, b=3, c=5$	Y1=tan⁻¹(X) Y2=X − X³/3 + X⁵/5 − X⁷/7 + X⁹/9  $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots$ $R=1$ $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ $R=\infty$ $\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$ $R=\infty$ $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$ $R=\infty$ $\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$ $R=1$ $(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots$ $R=1$ R = radius of convergence  approximation is not good for $x>1$ we should set $a<1$
(c)	$\lim_{n\to\infty}\arg(z_1z_2\ldots z_n)=\cancel{3\tan^{-1}(\sqrt{3})=\pi}$ $=\frac{1}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{1}{3}\cdot\frac{\pi}{6}=\frac{\pi}{18}$	Note that $\arg(z_1z_2\ldots z_n)=a^2\left(a-\dfrac{a^3}{3}+\dfrac{a^5}{5}+\ldots+\dfrac{(-1)^{n+1}a^{2n-1}}{2n-1}\right)$
		Total marks: 7

Qn	Suggested Solution	Comments
4(a)		You can use the GC to sketch the piecewise function. MATH NUM CMPLX PROB FRAC 5↑ˣ√ 6:fMin(7:fMax(8:nDeriv(9:fnInt(0:summation Σ(A:logBASE(B:piecewise(C:Numeric Solver… Plot1 Plot2 Plot3 Y1 = { 2X; 1≤X<8 ; .5X+12; 8≤X≤30 To access the inequality signs, use 2nd + MATH TEST LOGIC CONDITIONS 1:= 2:≠ 3:> 4:≥ 5:< 6:≤
(b)	Since the graphs of f and f^{-1} intersect only at the line $y=x$, to solve $f(x)=f^{-1}(x)$ is to solve $f(x)=x$. $f(x)=x$ $2x=x$ or $\dfrac{x}{2}+12=x$ $x=0$ $\qquad x=24$ Check using GC	Plot1 Plot2 Plot3 Y1 = { 2X; 1≤X<8 ; .5X+12; 8≤X≤30 Y2=X Use DrawInv Y1. 

	$x \in D_f \cap D_{f^{-1}}$ $\Rightarrow x \in D_f \cap R_f$ $\Rightarrow x \in [1,30] \cap [2,27]$ $\Rightarrow x \in [2,27]$ $\therefore x = 24$ **Alternative** Use GC to find the intersection between the graph of f and the line $y = x$. $\therefore x = 24.$	Note that the equation $f(x) = f^{-1}(x)$ is only defined for $x \in D_f \cap D_{f^{-1}}$. As f is a piecewise function, f^{-1} and ff will also be piecewise functions. Solving $f(x) = f^{-1}(x)$ directly or $ff(x) = x$ is a lot more complicated (both are not recommended).
(c)	Since $R_f = [2,27] \subseteq [1,30] = D_f$, f^2 exists. From the graph in **(a)**, $R_{f^2} = \left[2(2), \frac{27}{2}+12\right] = [4, 25.5]$.  Use the idea of restricted domain i.e. $R_f \xrightarrow{f} R_{f^2}$.	Be clear on the differences between the tests to show the existence of f^{-1} and composite function gf. f^{-1} exists (i.e. f is one-one): Use horizontal line test or show that f is increasing. gf exists: $R_f \subseteq D_g$.
(d)	$R_{f^3} = \left[2(4), \frac{25.5}{2}+12\right]$ $= [8, 24.75]$ $R_{f^4} = \left[\frac{8}{2}+12, \frac{24.75}{2}+12\right]$ $= [16, 24.375]$ From GC, $R_{f^n} \to \{24\}$ as $n \to \infty$. In the main GC window, recursively perform 0.5Ans+12.	Using the same method as above, it is not hard to find $R_{f^3} = [8, 24.75]$. Next, to find R_{f^n} for $n \geq 4$ we continue to use the same idea of $R_{f^{n-1}} \xrightarrow{f} R_{f^n}$. Observe that we can now focus on the 2nd rule of the piecewise function i.e. $\frac{x}{2}+12$ since inputs are now greater than or equal to 8. There are other methods to deduce the range of f^n as $n \to \infty$ but the method shown here is the easiest.
		Total marks: 8

GC screen (left):

8	
	8
0.5Ans+12	
	16
0.5Ans+12	
	20
0.5Ans+12	
	22
⋮	
0.5Ans+12	
	23.99993896
0.5Ans+12	
	23.99996948
0.5Ans+12	
	23.99998474
0.5Ans+12	
	23.99999237

GC screen (right):

24.75	
	24.75
0.5Ans+12	
	24.375
0.5Ans+12	
	24.1875
0.5Ans+12	
	24.09375
⋮	
0.5Ans+12	
	24.00000143
0.5Ans+12	
	24.00000072
0.5Ans+12	
	24.00000036
0.5Ans+12	
	24.00000018

$f^3(x) = f f^2(x)$

$R_{f^2} \xrightarrow{f} R_{f^3}$

GC seq mode

$u_n = \frac{1}{2}u_{n-1} + 12$

$u_1 = 8, 16, 24.375$

$R_{f^n} \to \{24\}$

$u_n = \frac{1}{2}u_{n-1} + 12$

$u_2 = \frac{1}{2}u_1 + 12$

$u_3 = \frac{1}{2}u_2 + 12 = \frac{1}{2}\left[\frac{1}{2}u_1 + 12\right] + 12 = \frac{1}{2^2}u_1 + \frac{1}{2}(12) + 12$

$u_4 = \frac{1}{2}u_3 + 12 = \frac{1}{2}\left[\frac{1}{2^2}u_1 + \frac{1}{2}(12) + (12)\right] + 12 = \frac{1}{2^3}u_1 + \frac{1}{2^2}\left[12 + \frac{1}{2}(12) + 12\right.$

$n \to \infty$, $u_n \to \frac{12}{1-\frac{1}{2}} = 24$ (regardless of u_1 as $\frac{1}{2^{n-1}} \to 0$ as $n \to \infty$)

GP $a = 12$, $r = \frac{1}{2}$

	Suggested Solution	Comments

5(a)

$$X : Y : Z \qquad X : Y : Z$$
$$1 : 2 : 3 \Rightarrow \tfrac{1}{3} : \tfrac{2}{3} : 1$$

$$\frac{dz}{dt} = p\left(10 - \tfrac{1}{3}z\right)\left(15 - \tfrac{2}{3}z\right)$$

Amount of X in the c

$X = 10 - \frac{1}{3}z$

$$\frac{dz}{dt} = \frac{p}{9}(30 - z)(45 - 2z)$$

Amount of Y in the c

$$= k(30 - z)(45 - 2z) \text{ (shown)}$$

$Y = 15 - \frac{2}{3}z$

Comments:
- Even though the ratio of mass of $X{:}Y{:}Z = 1{:}2{:}3$, it doesn't mean $X + 2Y = 3Z$.
- As this is a "show" qn, clear working needs to be presented to be awarded full credit.

(b) **Method 1 (partial fractions)**

$$\frac{dz}{dt} = k(30 - z)(45 - 2z)$$

$$\int \frac{1}{(30-z)(45-2z)}\, dz = \int k \; dt$$

$$\int \frac{-\frac{1}{15}}{(30-z)} + \frac{\frac{1}{7.5}}{(45-2z)}\, dz = \int k \; dt$$

$$\frac{1}{7.5}\frac{\ln|45-2z|}{-2} - \frac{1}{15}\frac{\ln|30-z|}{-1} = kt + C$$

$$\frac{1}{15}\ln\left|\frac{30-z}{45-2z}\right| = kt + C$$

$$\left|\frac{30-z}{45-2z}\right| = e^{15kt + 15C}$$

$$\frac{30-z}{45-2z} = \pm e^{15C} \cdot e^{15kt} = Ae^{Bt} \quad (\text{where } A = \pm e^{15C}, B = 15k)$$

Comments:
- Modulus needs to be applied after integration
- Remove modulus first before proceeding to find the value of the constants.

When $t = 0, z = 0$: $\dfrac{30-0}{45-0} = Ae^0 \Rightarrow A = \dfrac{2}{3}$

$$\therefore \frac{30-z}{45-2z} = \frac{2}{3}e^{Bt}$$

When $t = 5, z = 10$: $\dfrac{30-10}{45-20} = \dfrac{2}{3}e^{5B}$

$$e^{5B} = 1.2 \Rightarrow e^B = (1.2)^{0.2}$$

$$\therefore \frac{30-z}{45-2z} = \frac{2}{3}(1.2)^{0.2t}$$

$$90 - 3z = (90 - 4z)(1.2)^{0.2t}$$

$$4z(1.2)^{0.2t} - 3z = 90(1.2)^{0.2t} - 90$$

$$z\left[4(1.2)^{0.2t} - 3\right] = 90\left[(1.2)^{0.2t} - 1\right]$$

$$z = \frac{90\left[(1.2)^{0.2t} - 1\right]}{\left[4(1.2)^{0.2t} - 3\right]} = \frac{90\left[1 - (1.2)^{-0.2t}\right]}{\left[4 - 3(1.2)^{-0.2t}\right]}$$

Comments:
- It's recommended to use the initial condition: $t = 0, z = 0$ first to find A, as there is only 1 unknown. If you use $t = 5, z = 10$ first, you will end up with 2 unknowns: A & k.

Alternatively, from $e^B = (1.2)^{0.2}$

$$\Rightarrow B = 15k = \ln(1.2)^{0.2} \text{ or } \frac{1}{5}\ln\left(\frac{6}{5}\right)$$

$$\therefore k = \frac{1}{75}\ln\left(\frac{6}{5}\right) \text{ or } 0.0365 \text{ (3 sf)}$$

- Make z the subject, as required by the question

Other alternative answers include :

$$z = \frac{90\left[e^{0.0365t} - 1\right]}{\left[4e^{0.0365t} - 3\right]} \text{ or } z = \frac{30\left[e^{\frac{1}{5}\ln\left(\frac{6}{5}\right)t} - 1\right]}{\left[\frac{4}{3}e^{\frac{1}{5}\ln\left(\frac{6}{5}\right)t} - 1\right]}$$

Method 2 (complete the square)

$$\frac{dz}{dt} = k(30-z)(45-2z)$$

$$\int \frac{1}{(30-z)(45-2z)}\,dx = \int k\ dt$$

$$\int \frac{1}{2z^2-105z+1350}\,dx = \int k\ dt$$

$$\frac{1}{2}\int \frac{1}{z^2-52.5z+675}\,dx = \int k\ dt$$

$$\frac{1}{2}\int \frac{1}{(z-26.25)^2-(3.75)^2}\,dx = \int k\ dt$$

$$\frac{1}{2}\times\frac{1}{2(3.75)}\ln\left|\frac{(z-26.25)-3.75}{(z-26.25)+3.75}\right| = kt + C$$

$$\frac{1}{15}\ln\left|\frac{z-30}{z-22.5}\right| = kt + C$$

$$\left|\frac{z-30}{z-22.5}\right| = e^{15kt+15C}$$

$$\frac{z-30}{z-22.5} = \pm e^{15C}\bullet e^{15kt} = Ae^{Bt}\ \ (\text{where } A=\pm e^{15C}, B=15k)$$

When $t=0, z=0$: $\dfrac{0-30}{0-22.5} = Ae^0 \Rightarrow A = \dfrac{4}{3}$

$$\therefore \frac{z-30}{z-22.5} = \frac{4}{3}e^{Bt}$$

When $t=5, z=10$: $\dfrac{10-30}{10-22.5} = \dfrac{4}{3}e^{5B}$

$$e^{5B} = 1.2 \Rightarrow e^B = (1.2)^{0.2}$$

$$\therefore \frac{z-30}{z-22.5} = \frac{4}{3}(1.2)^{0.2t}$$

$$3z-90 = (4z-90)(1.2)^{0.2t}$$

$$4z(1.2)^{0.2t} - 3z = 90(1.2)^{0.2t} - 90$$

$$z\left[4(1.2)^{0.2t}-3\right] = 90\left[(1.2)^{0.2t}-1\right]$$

$$z = \frac{90\left[(1.2)^{0.2t}-1\right]}{\left[4(1.2)^{0.2t}-3\right]} = \frac{90\left[1-(1.2)^{-0.2t}\right]}{\left[4-3(1.2)^{-0.2t}\right]}$$

(c)

As $t\to\infty$, $(1.2)^{-0.2t}\to 0$, $z \to \dfrac{90(1-0)}{(4-0)} = 22.5$

Max possible mass of Z = 22.5 g

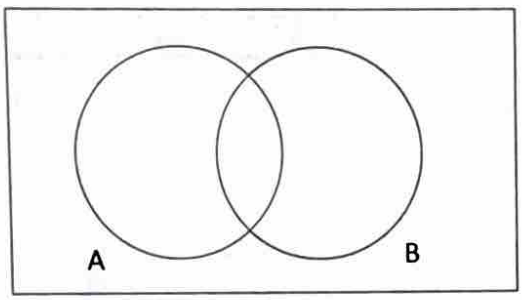
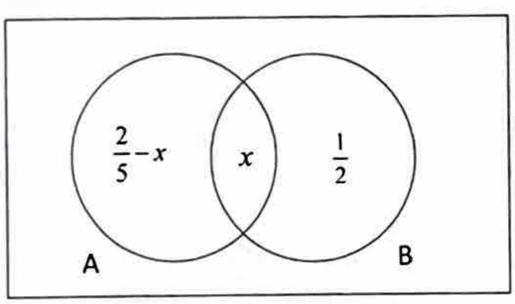
Remaining mass of $X = 10 - 22.5\left(\frac{1}{3}\right) = 2.5$ g

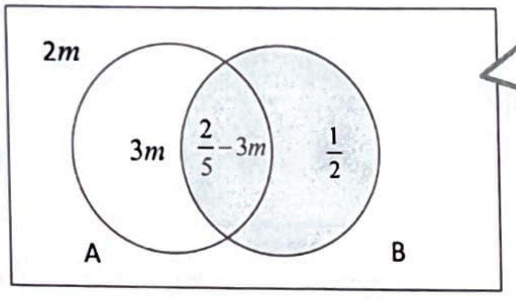
Remaining mass of $Y = 15 - 22.5\left(\frac{2}{3}\right) = 0$ g

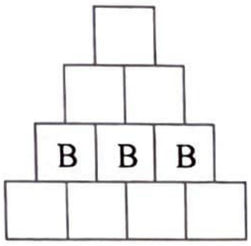
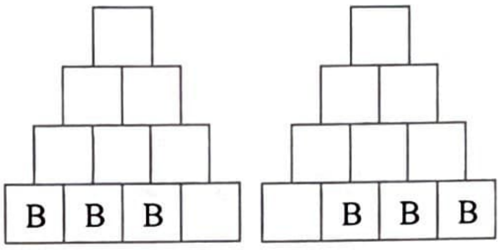
- Students can deduce the answer from part (b) ***OR*** reason that mass of Y = 15g is the limiting factor. Hence max mass of X = 7.5g which gives 15+7.5 = 22.5g of Z.

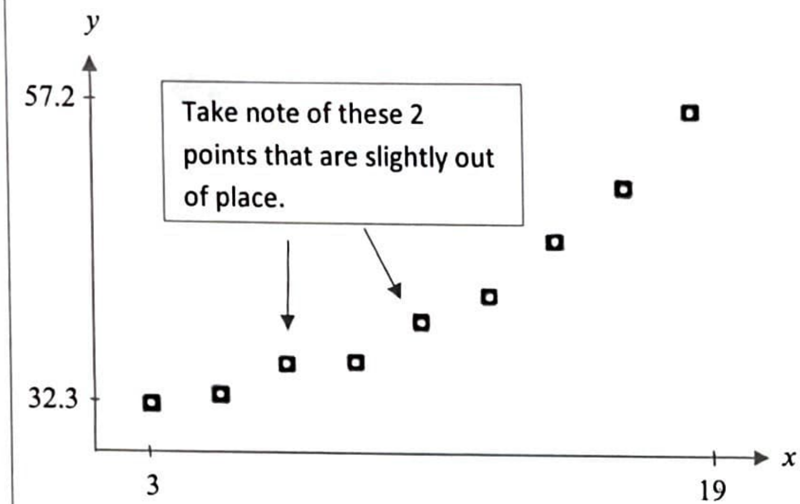
Total = 12 marks

Section B: Probability and Statistics [60 marks]

Qn	Suggested Solution	Comments
6(a)	$\mathrm{P}\left(B \mid A'\right)=\dfrac{\mathrm{P}(B\cap A')}{\mathrm{P}(A')}=\dfrac{5}{6}$ $\mathrm{P}(A'\cap B)=\left(\dfrac{5}{6}\right)\left(1-\dfrac{2}{5}\right)=0.5$	
(b)	 $\mathrm{P}(A'\cap B')$ $=1-\mathrm{P}(A\cup B)$ $=1-\left(\mathrm{P}(A)+\mathrm{P}(A'\cap B)\right)$ $=0.1$	Concepts: Always facilitate yourself with a Venn diagram for this type of question.
(c)	<u>Method 1</u>  Let $\mathrm{P}(A\cap B)$ be x $\mathrm{P}\left(A \mid B'\right)=\dfrac{\mathrm{P}(A\cap B')}{\mathrm{P}(B')}=\dfrac{\mathrm{P}(A\cap B')}{1-\mathrm{P}(B)}$ $\dfrac{3}{5}=\dfrac{\frac{2}{5}-x}{1-(x+0.5)}=\dfrac{0.4-x}{0.5-x}$ $1.5-3x=2-5x$ $2x=0.5 \Rightarrow x=0.25$ $\therefore\ \mathrm{P}(A\cap B)=0.25$	Use the Venn diagram and put in the information from the question to devise a strategy

	<u>Method 2 (not advisable if you are weak in probability)</u>  We just need to find the value of m to get to the answer. Total probability = 1 $2m+3m+\left(\frac{2}{5}-3m\right)+\frac{1}{2}=1$ $m=\frac{1}{20}$ $\mathrm{P}(A\cap B)=\frac{2}{5}-3m=\frac{2}{5}-\frac{3}{20}=\frac{1}{4}$ <u>Method 3</u> $\mathrm{P}(A'\mid B')=1-\mathrm{P}(A\mid B')=\frac{2}{5}$ $\frac{\mathrm{P}(A'\cap B')}{\mathrm{P}(B')}=\frac{2}{5}$ $\mathrm{P}(B')=\frac{\mathrm{P}(A'\cap B')}{\frac{2}{5}}=\frac{0.1}{0.4}=\frac{1}{4}$ $\mathrm{P}(B)=\frac{3}{4}$ $\mathrm{P}(A'\cap B)+\mathrm{P}(A\cap B)=\frac{3}{4}$ $\mathrm{P}(A\cap B)=\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$	Since $\mathrm{P}(A\mid B')=\frac{3}{5}$, if we were to reduce the sample space to B' in the venn diagram, the ratio in terms of $A\cap B'$ and B' is 3 : 5, so the ratio of $A\cap B'$ and $A'\cap B'$ will be 3 : 2. Find P(B') from conditional probability and find P(B). Use (i) answer to find $\mathrm{P}(A\cap B)$.
	Since $\mathrm{P}(A\mid B')\neq\mathrm{P}(A)$, events A and B are not independent. **<u>Alternative</u>** $\mathrm{P}(A)\cdot\mathrm{P}(B)=(0.4)(0.25+0.5)=0.3$ Since $\mathrm{P}(A)\cdot\mathrm{P}(B)\neq\mathrm{P}(A\cap B)$, events A and B are not independent.	
		Total marks: 7

Qn	Suggested Solution	Comments
7(a)	No. of ways $=\dfrac{10!}{2!3!5!}=2520$	• Ways to give 2R,3B,5G cupcakes to 10 children is like ways to arrange 2R,3B,5G in a row.
(b)	**Case 1**  **Case 2**  Case 1 : Ways to arrange all the cupcakes $=\dfrac{7!}{2!5!}=21$ Case 2 : Ways to arrange all the cupcakes $=\dfrac{7!}{2!5!}\times 2=42$ Total ways $=21+42=63$	• Cases 1 & 2 use the similar idea that after placing 3Bs in either row 3 or 4, the rest of the cupcakes 2R5G can be arranged in $\dfrac{7!}{2!5!}$ ways.
(c)	Ways $=\dfrac{10!}{2!3!}-1$ $=302399$	• There is only 1 way to arrange the letters in alphabetical order : ABCDEEELLT
(d)	DELECTABLE = 3Es, 2Ls, A.B,C,D,T **Case 1 : all different letters** Ways $={}^7C_3\times 3!=210$ (or ${}^7P_3$ or $7\times6\times5$) **Case 2 : two letters same** EE_ or LL_ Ways $={}^2C_1\times{}^6C_1\times\dfrac{3!}{2!}=36$ ${}^2C_1$: either EE_ or LL_ ${}^6C_1$: 6 other letters to choose for last slot of EE_ or LL_ $\dfrac{3!}{2!}$: ways to arrange EE_ or LL_ **Case 3 : three letters same** EEE Way = 1 Total $=210+36+1=247$	• Even though ${}^3C_3=1$, it's not a correct concept to use here as technically there is only 1 way to get the EEE codeword. There's nothing to choose. • Due to the above misconception, some students wrongly applied ${}^3C_2$ to choose 2Es out of 3Es in case 2. Obviously, there is still only 1 way to get 2Es.

Qn	Suggested Solution	Comments
8(a)	**Any one possible answer** The scatter diagram helps to 1. confirm the relationship/trend/pattern between the two variables. 2. identify outliers or suspicious observations.	Focus on what advantages a scatter diagram has over the value of r.
(b)(i)		Range of the data points must be indicated. The scatter diagram must show that as x increases, y is increases at an increasing rate.
(ii)	From G.C : possible $(x_{10}, y_{10}) = (\bar{x}, \bar{y}) = (11, 41.4)$	
(iii)	From the scatter diagram, it's observed that as x increases, y increases at an increasing rate. Model (A) : as x increases, y increases at an increasing rate Model (B) : as x increases, y increases at a decreasing rate Hence (A) $y = a + bx^2$ is the more appropriate model.	
(iv)	Equation of regression line for Model A: $y = 31.276 + 0.068783x^2 = 31.3 + 0.0688x^2$ "a" represents the **estimated/predicted** population of the city in the Year 2000.	Key concept: A regression line is a best fit line thus the information from this line is an estimation. You need to be aware the difference between data points and points on the regression line.
(v)	New eqn : $1000y = 31.3 + 0.0688x^2$	
(vi)	Prod moment correlation coeff $r = 0.996$ (3 sf) <u>Reasons</u> 1) **r or $\|r\|$ is close to 1** 2) **interpolation** since $x = 6$ is **within the data range** $(3 \le x \le 19)$	
		Total = 10 marks

Qn	Suggested Solution	Comments
9(a)	Let X and W be the time taken (in mins) for a randomly chosen train journey and walk from train station to office respectively. $X \sim \mathrm{N}(60, 4^2)$ $\qquad$ $W \sim \mathrm{N}(10, 3^2)$ $X + W \sim \mathrm{N}(60 + 10, 4^2 + 3^2)$ $X + W \sim \mathrm{N}(70, 5^2)$ Req probability $= \mathrm{P}(X + W > 80)$ $= 0.0228$ (3 s.f.)	We should always define our random variables clearly. This is a conditional probability question. Keyword (**Given that**) is in the question. We used the reduced sample idea here. Since we know that Mr Hsu takes the first train (ie, 6.10am) [this has happened], for him to be late (arrive after 7.30am), we just need to calculate the probability that the total travel time must take more than 80 minutes.
(b)	Let A be the number of minutes after 6.00 a.m that Mr Hsu takes to reach train station platform $A \sim \mathrm{N}(0, 10^2)$ $\mathrm{P}(\text{late for work}) = \mathrm{P}(A > 15) + \mathrm{P}(A < 10)\mathrm{P}(X + W > 80)$ $+\mathrm{P}(10 < A < 15)\mathrm{P}(X + W > 75)$ $= 0.10052$ $= 0.101$ (3 s.f) (shown)	Definition of random variable (r.v.) used should be properly defined. It is easier to use the reference time as 6am. In order that Mr Hse is late, there are 3 cases. Case 1: he misses both trains, ie, arrives after 6.15pm Case 2: takes the first train and late Case 3: takes the second train and late
(c)	$\mathrm{P}(X + W > t) \le 0.1$ $\mathrm{P}(X + W > 76.4078) = 0.1$  $\Rightarrow t \ge 76.4$ (3 s.f.) The earliest starting time for briefing is 7.27 a.m.	Similar to part (a), this is conditional probability and we are using the reduced sample idea. Given that he takes the first train, for him to be late for the briefing, total travelling time must exceeds t mins. For area to be smaller, 'move right'. Smallest integer t value here is 77mins. Hence, 77 mins away from taking the first train at 6.10am will be 7.27am
(d)	Let L be the number of days, out of 20, that Mr Hsu is late for work. $L \sim \mathrm{B}(20, 0.101)$ Let S be the percentage of salary Mr Hsu receives in the month	For Mr Hsu to receive 60%-80% of salary, he will have a pay reduction of 20%-40%. This in turn imply that he will be late for $\frac{20}{5} = 4$ to $\frac{40}{5} = 8$ days of being late.

	$P(60 \le S \le 80)$ $= P(60 \le 100 - 5L \le 80)$ $= P(20 \le 5L \le 40)$ $= P(4 \le L \le 8)$ $= P(L \le 8) - P(L \le 3)$ $= 0.137$ (3 s.f.)	The values of L that we want are from 4 to 8. We use the binomcdf (≤) idea here for faster computation.
		Total marks: 10

Qn	Suggested Solution	Comments
10(a)	Each chocolate bar in the population has an **equal chance** of being chosen and the chocolate bars are **chosen independently**.	**Both** points are to be explained.
(b)	$\bar{x} = \frac{-37}{80} + 52 = 51.5375$ $s^2 = \frac{1}{79}\left[310.7 - \frac{(-37)^2}{80}\right] = 3.7163 \approx 3.72$ (3 s.f.)	
(c)	An estimate is unbiased when the **expected value of the estimator** used to obtain the estimate **is equal to the value of the population parameter**.	The **definition** needs to be understood & remembered.
(d)	Let μ be the population mean mass in grams. $H_0 : \mu = 52$ $H_1 : \mu \neq 52$ A 2-tail test is used as the manager just wanted to know if the mean mass is 52 grams or not. Under H_0, $\overline{X} \sim N\left(52, \frac{3.7163}{80}\right)$ approximately by Central Limit Theorem since $n = 80$ is large. From GC, p-value = 0.0319 (3 s.f.) It indicates that **if the level of significance is 3.19% or more**, the **null hypothesis** (population mean mass is 52 grams) will be **rejected**. Otherwise, the null hypothesis will not be rejected. __Alternative 1__ The null hypothesis is rejected at the 5% significance level, but not at the 1% significance level. There is therefore some, but not very strong, evidence to reject the null hypothesis that the mean mass of the bars is 52 grams, as stated on the packets. __Alternative 2__ The p-value indicates some evidence (though not very strong evidence) that the population mean mass is not 52 grams; i.e., we reject the null hypothesis if $\alpha = 5\%$ while we do not reject the null hypothesis if $\alpha = 3\%$.	μ must be defined with the correct hypotheses. The choice of a 2-tail test needs to be explained. $\overline{X}$ follows a normal distribution by CLT, & the variance of $\overline{X}$ must be divided by 80 (sample size). The p-value is **NOT** the level of significance. In this case, the p-value must be explained in the **context** of the question & used in statistical decision-making. Note that the conclusion will not be valid if the p-value is incorrect.

(c)	There is **no need** for the manager to know anything about the population distribution of the masses of the chocolate bars since $\overline{X} \sim \mathrm{N}(\mu, \frac{s^2}{n})$ **approximately** by Central Limit Theorem as $n =$ 80 is large and z-test can be used.	Note that $\overline{X}$ follows a normal distribution by CLT, **not *X*.**
		Total marks: 12

Qn	**Suggested soln**	**Comments**
11(a)	1. Whether one chick survives to leave its nest is independent of whether another chick can do so. 2. The probability of a chick being able to leave its nest is constant at 0.6.	• We need the independence of the event (a chick surviving to leave its nest), **NOT** the probability. • The conditions that there will be a finite number of chicks in a nest and that a chick can either survive to leave its nest or not are obvious in this context. No need to make them assumptions. • Answers must be worded in the context of the question. Avoid simply using words like "trials" and "outcomes" without qualifying what they are in context.
(b)	Let W be the number of chicks, in a nest of 4, that will survive to independently leave its nest. $W \sim \mathrm{B}(4, 0.6)$ $\mathrm{P}(W=2)=0.3456$	• Pls define r.v. clearly. • Since the answer is exact at 0.3456, there is no need to round off to 3.s.f.
(c)	$\mathrm{P}(W \geq 2)=1-\mathrm{P}(W \leq 1)$ $=1-0.1792$ $=0.8208$ Let Y be the number of nests of 4 chicks, out of 15 nests, where at least 2 chicks survive to independently leave its nest. $Y \sim \mathrm{B}(15, 0.8208)$ $\mathrm{P}(\text{successful})=\mathrm{P}(Y>10)$ $=1-\mathrm{P}(Y \leq 10)$ $=1-0.11497$ $=0.88503$	• Pls define r.v. clearly. • Intermediate working to be left to at least 2 more degrees of accuracy compared to final answer.

	$\text{Required probability} = \binom{3}{2} \cdot \mathrm{P}(Y_1 > 10) \cdot \mathrm{P}(Y_2 > 10) \cdot \mathrm{P}(Y_3 \le 10)$ $= \binom{3}{2} \left[\mathrm{P}(Y > 10)\right]^2 \cdot \mathrm{P}(Y_1 \le 10)$ $= \binom{3}{2} (0.88503)^2 (1 - 0.88503)$ $= 0.27016$ $= 0.270$ (3 s.f.) **Alternative** Let U be the number of breeding zones, out of 3, which are considered successful. $U \sim \mathrm{B}(3, 0.88503)$ $\mathrm{P}(U = 2) = 0.270$	
(d)	$\sum_{r=1}^{\infty} \mathrm{P}(N = r) = 1$ $\sum_{r=1}^{\infty} \frac{A}{\ln(1-\alpha)} \left(\frac{\alpha^r}{r}\right) = 1$ $\frac{A}{\ln(1-\alpha)} \sum_{r=1}^{\infty} \left(\frac{\alpha^r}{r}\right) = 1$ $\frac{A}{\ln(1-\alpha)} \left[-\ln(1-\alpha)\right] = 1$ $A = -1$	• Since $r \in \mathbb{Z}^+$, this means N can take values from 1 through infinity. • Key concept: total sum of probabilities = 1 • Observe that $\frac{A}{\ln(1-\alpha)}$ is independent of r. Hence, it be brought out of the summation sign.
(e)	From GC, $\mathrm{P}(4 \le N \le 30) = -\sum_{r=4}^{30} \frac{1}{\ln(0.7)} \left(\frac{0.3^r}{r}\right) = 0.00750$	• Use GC to evaluate the sum.
(f)	$\mathrm{E}(N) = \sum_{r=1}^{\infty} r\mathrm{P}(N = r)$ $= \sum_{r=1}^{\infty} r \frac{-1}{\ln(1-\alpha)} \left(\frac{\alpha^r}{r}\right)$ $= \frac{-1}{\ln(1-\alpha)} \underbrace{\sum_{r=1}^{\infty} (\alpha^r)}_{\text{GP sum to infinity}}$ $= \frac{-1}{\ln(1-\alpha)} \left[\alpha + \alpha^2 + \alpha^3 + \ldots\right]$ $= \frac{-1}{\ln(1-\alpha)} \cdot \frac{\alpha}{1-\alpha}$ $= \frac{-\alpha}{(1-\alpha)\ln(1-\alpha)}$	• Key concepts: $\mathrm{E}(N) = \sum_{\text{All } r} r\mathrm{P}(N = r)$ $\mathrm{E}(N^2) = \sum_{\text{All } r} r^2\mathrm{P}(N = r)$ $\mathrm{Var}(N) = \mathrm{E}(N^2) - \left[\mathrm{E}(N)\right]^2$ • Observe that $\frac{-1}{\ln(1-\alpha)}$ is independent of r. Hence, it be brought out of the summation sign.

	$$\mathrm{E}(N^2) = \sum_{r=1}^{\infty} r^2 \mathrm{P}(N=r)$$ $$= \sum_{r=1}^{\infty} r^2 \frac{-1}{\ln(1-\alpha)}\left(\frac{\alpha^r}{r}\right)$$ $$= \frac{-1}{\ln(1-\alpha)} \sum_{r=1}^{\infty}(r\alpha^r)$$ $$= \frac{-1}{\ln(1-\alpha)} \cdot \frac{\alpha}{(1-\alpha)^2}$$ $$= \frac{-\alpha}{(1-\alpha)^2 \ln(1-\alpha)}$$ $$\mathrm{Var}(N) = \mathrm{E}(N^2) - [\mathrm{E}(N)]^2$$ $$= \frac{-\alpha}{(1-\alpha)^2 \ln(1-\alpha)} - \left[\frac{-\alpha}{(1-\alpha)\ln(1-\alpha)}\right]^2$$ $$= \frac{-\alpha}{(1-\alpha)^2 \ln(1-\alpha)} - \frac{\alpha^2}{(1-\alpha)^2[\ln(1-\alpha)]^2}$$ $$= \frac{-\alpha}{(1-\alpha)^2 \ln(1-\alpha)}\left[1 + \frac{\alpha}{\ln(1-\alpha)}\right]$$ $$= -\frac{\alpha}{(1-\alpha)^2 \ln(1-\alpha)}\left[\frac{\ln(1-\alpha)+\alpha}{\ln(1-\alpha)}\right]$$ $$= -\frac{\alpha[\ln(1-\alpha)+\alpha]}{(1-\alpha)^2[\ln(1-\alpha)]^2}$$	
		Total marks: 13