HCI 2024 Prelims Paper 1 Solutions

Qn	Suggested Solutions
1 [4] **(a)**	Translation in the positive y direction by 1 unit $y=\mathrm{f}(x)\xrightarrow{\text{replace } y \text{ with } y-1} y=\mathrm{f}(x)+1$ $\left(0,\frac{1-p}{p}\right)\xrightarrow{\text{replace } y \text{ with } y-1}\left(0,\frac{1}{p}\right)$ **Note**: [only possible to state the y-intercept] $(1-p,0)\xrightarrow{\text{replace } y \text{ with } y-1}(1-p,1)$ doesn't cut the x-axis.
(b)	Translation in the positive x direction by p unit $y=\mathrm{f}(x)\xrightarrow{\text{replace } x \text{ with } x-p} y=\mathrm{f}(x-p)$ $(1-p,0)\xrightarrow{\text{replace } x \text{ with } x-p}(1,0)$ **Note**: [only possible to state the x-intercept] $\left(0,\frac{1-p}{p}\right)\xrightarrow{\text{replace } x \text{ with } x-p}\left(p,\frac{1-p}{p}\right)$ doesn't cut the y-axis.
(c)	**Step 1**: Translation in the positive x direction by p unit **Method 1**: From Part **(b)** **Step 2**: Scale parallel to the x axis by a factor of $\frac{1}{3}$ $(1-p,0)\xrightarrow{\text{replace } x \text{ with } x-p}(1,0)\xrightarrow{\text{replace } x \text{ with } 3x}\left(\frac{1}{3},0\right)$ **Note**: [only possible to state the x-intercept] $\left(0,\frac{1-p}{p}\right)\longrightarrow\left(\frac{p}{3},\frac{1-p}{p}\right)$ doesn't cut the y-axis.
	<u>Method 2</u>: **Step 1**: Scale parallel to the x axis by a factor of $\frac{1}{3}$ **Step 2**: Translation in the positive x direction by $\frac{p}{3}$ unit

$$y=\mathrm{f}(x)\xrightarrow[3x]{\text{replace } x \text{ with}}\mathrm{f}(3x)\xrightarrow[x-\frac{p}{3}]{\text{replace } x \text{ with}}\mathrm{f}(3x-p)$$

$$(1-p,0)\xrightarrow[3x]{\text{replace } x \text{ with}}\left(\frac{1-p}{3},0\right)\xrightarrow[x-\frac{p}{3}]{\text{replace } x \text{ with}}\left(\frac{1}{3},0\right)$$

Note:
[only possible to state the *x*-intercept]

$\left(0,\frac{1-p}{p}\right)\longrightarrow\left(\frac{p}{3},\frac{1-p}{p}\right)$ doesn't cut the *y*-axis.

(d)

$$y=\mathrm{f}(x)\xrightarrow{A}y=\mathrm{f}^{-1}(x)$$

A: Reflection about the line $y=x$

$$\left(0,\frac{1-p}{p}\right)\longrightarrow\left(\frac{1-p}{p},0\right)$$

$$(1-p,0)\longrightarrow(0,1-p)$$

2 [5] (a)

If $\mathrm{g}(x)>0$ for all $x\in\mathbb{R}$, then $\mathrm{f}(x)\geq\mathrm{g}(x)$.
Otherwise $\mathrm{f}(x)\geq\mathrm{g}(x)$ may not always be true.

(b)

$$\frac{2x^2-x-9}{x^2-x-6}\geq1$$

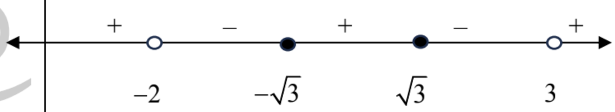
$$\frac{2x^2-x-9}{x^2-x-6}-1\geq0$$

$$\frac{\left(2x^2-x-9\right)-\left(x^2-x-6\right)}{x^2-x-6}\geq0$$

$$\frac{x^2-3}{x^2-x-6}\geq0$$

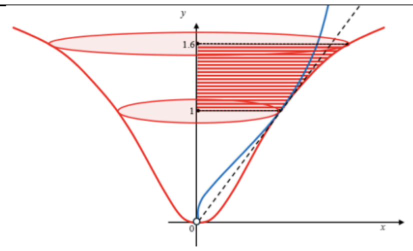
$$\frac{\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)}{(x-3)(x+2)}\geq0$$

Critical values are: $-2,-\sqrt{3},\ \sqrt{3},3$



$\{x\in\mathbb{R}:\ x<-2$ or $-\sqrt{3}\leq x\leq\sqrt{3}$ or $x>3\}$

3 [5] (a)



Volume generated

$$V_1 = \pi \int_1^{1.6} x^2 \, dy$$

$$= \pi \int_1^{1.6} \left[\frac{y}{\sqrt{2y - y^2}} \right]^2 dy$$

$$= \pi \int_1^{1.6} \left(\frac{y^2}{2y - y^2} \right) dy$$

$$= \pi \int_1^{1.6} \left(\frac{y}{2 - y} \right) dy$$

$$= \pi \int_1^{1.6} \left(-1 + \frac{2}{2 - y} \right) dy$$

$$= \pi \left[-y - 2\ln|2 - y| \right]_1^{1.6}$$

$$= \pi \left[-0.6 - 2\left(\ln\left| \frac{2 - 1.6}{2 - 1} \right| \right) \right]$$

$$= \pi \left[-0.6 - 2\ln\left(\frac{2}{5} \right) \right]$$

$$= 2\pi \left[\ln\left(\frac{5}{2} \right) - 0.3 \right] \text{ unit}^3$$

(b) New volume generated

$$V_2 = \pi \int_1^{1.6} x^2 \, dy$$

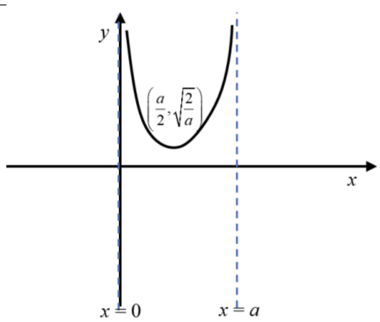
$$= \pi \int_1^{1.6} \left[\frac{by}{\sqrt{2y - y^2}} \right]^2 dy$$

$$= b^2 \pi \int_1^{1.6} \left[\frac{y}{\sqrt{2y - y^2}} \right]^2 dy$$

Required ratio: $1 : b^2$

4 **[7]** **(a)**	Let a be the first term of AP and d be the common difference. $\frac{a+22d}{a+14d}=\frac{a+14d}{a+10d}$ $(a+22d)(a+10d)=(a+14d)^2$ $a^2+32ad+220d^2=a^2+28ad+196d^2$ $4ad+24d^2=0$ $4d(a+6d)=0$ $d=0$ (reject) or $a=-6d$ Common ratio of GP = $\frac{a+14d}{a+10d}=\frac{-6d+14d}{-6d+10d}=\frac{8d}{4d}=2$
(b)	$v_n=S_n-S_{n-1}$ $=\frac{3^{n+2}-(-2)^{n+2}-5}{6}-\frac{3^{n+1}-(-2)^{n+1}-5}{6}$ $=\frac{1}{6}\left[3^{n+2}-(-2)^{n+2}-5-3^{n+1}+(-2)^{n+1}+5\right]$ $=\frac{1}{6}\left[9(3^n)-3(3^n)-(-2)^2(-2)^n+(-2)(-2)^n\right]$ $=\frac{1}{6}\left[6(3^n)-6(-2)^n\right]$ $=3^n-(-2)^n$

5 **[7]** **(a)**	$x = \sec\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \sec\theta\tan\theta$ When $x = \sqrt{2}$, $\dfrac{1}{\cos\theta} = \sqrt{2} \quad \Rightarrow \quad \dfrac{1}{\cos\theta} = \dfrac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \dfrac{\pi}{4}$ When $x = 2$, $\dfrac{1}{\cos\theta} = 2 \quad \Rightarrow \quad \dfrac{1}{\cos\theta} = \dfrac{1}{2} \quad \Rightarrow \quad \theta = \dfrac{\pi}{3}$ $\int_{\sqrt{2}}^{2} \dfrac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{1}{\sqrt{\sec^2\theta - 1}}\left(\sec\theta\tan\theta\right)\mathrm{d}\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{1}{\tan\theta}\left(\sec\theta\tan\theta\right)\mathrm{d}\theta$, Since $\sqrt{\tan^2\theta} = \lvert\tan\theta\rvert = \tan\theta$ where $0 < \theta < \dfrac{\pi}{2}$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\theta\,\mathrm{d}\theta$
	$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\theta\,\mathrm{d}\theta$ $= \left[\ln\lvert\sec\theta + \tan\theta\rvert\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= \ln\left[2+\sqrt{3}\right] - \ln\left[\sqrt{2}+1\right]$ $= \ln\left[\dfrac{2+\sqrt{3}}{\sqrt{2}+1}\right]$

6(a)	$f: x \mapsto \ln\left[(x+4)^2 - 9\right]$ $(x+4)^2 - 9 > 0$ $(x+4)^2 - 3^2 > 0$ $[x+4-3][x+4+3] > 0$ $(x+1)(x+7) > 0$ $x < -7 \quad \text{or} \quad x > -1$ Minimum $k = -1$
6(b)	$g\left(\frac{3}{2}\right) = f^{-1}(\alpha)$ $f\left[g\left(\frac{3}{2}\right)\right] = f\left[f^{-1}(\alpha)\right]$ $f\left[\frac{3-2\left(\frac{3}{2}\right)}{1+2\left(\frac{3}{2}\right)}\right] = f\left[f^{-1}(\alpha)\right]$ $f(0) = ff^{-1}(\alpha)$ $f(0) = \alpha$ $\alpha = \ln\left[(0+4)^2 - 9\right]$ $\alpha = \ln 7$
(c)	 **<u>Method 1</u>**: By observation, $x = \frac{a}{2}$, $y = \frac{1}{\sqrt[4]{\frac{a^2}{4}}} = \sqrt{\frac{2}{a}}$ **Coordinates of stationary (minimum) point** is $\left(\frac{a}{2}, \sqrt{\frac{2}{a}}\right)$ **Equations of 2 vertical asymptotes**: $x = 0$, $x = a$

Method 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x(a-x)\right]^{-\frac{1}{4}}$$

$$=-\frac{1}{4}\left[x(a-x)\right]^{-\frac{5}{4}}\left[a-2x\right]$$

$$=-\frac{a-2x}{4\sqrt[4]{\left[x(a-x)\right]^5}}$$

For stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow x=\frac{a}{2}$,

$$y=\frac{1}{\sqrt[4]{\frac{a^2}{4}}}=\sqrt{\frac{2}{a}}$$

Coordinates of stationary (minimum) point is $\left(\frac{a}{2},\sqrt{\frac{2}{a}}\right)$

Equations of 2 vertical asymptotes:

$x=0$, $x=a$

(d)

$\mathrm{g}:x\mapsto\frac{3-2x}{1+2x}$, for $x\geq\frac{1}{2}$,

$\mathrm{h}:x\mapsto\frac{1}{\sqrt[4]{x(a-x)}}$, for $0<x<a$,

$\mathrm{R_h}=\left[\sqrt{\frac{2}{a}},\infty\right)$, $\mathrm{D_g}=\left[\frac{1}{2},\infty\right)$

For gh to exist, $\mathrm{R_h}\subseteq\mathrm{D_g}$

$$\sqrt{\frac{2}{a}}\geq\frac{1}{2}$$

$$\frac{2}{a}\geq\frac{1}{4}$$

$$\frac{a}{2}\leq 4$$

$$a\leq 8$$

Since $a>0$, $0<a\leq 8$

(e)

$\left[\mathrm{h}(x)\right]^2 = 1$

$\mathrm{h}(x) = \dfrac{1}{\mathrm{h}(x)}$

<u>Method 1</u>:

Consider minimum point $\left(\dfrac{a}{2}, \sqrt{\dfrac{2}{a}}\right)$ of $y = \mathrm{g}(x)$ intersecting

Maximum of point of $\left(\dfrac{a}{2}, \sqrt{\dfrac{a}{2}}\right)$ of $y = \dfrac{1}{\mathrm{g}(x)}$ at exactly one point:

$\sqrt{\dfrac{2}{a}} = \sqrt{\dfrac{a}{2}}$

$\dfrac{2}{a} = \dfrac{a}{2}$

$a^2 = 4$

$a = \pm 2$

Since $a > 0$, $a = 2$

<u>Method 2</u>:

$\left[x(a-x)^{-\frac{1}{4}}\right]^2 = 1$

$\left[x(a-x)\right]^{\frac{1}{2}} = 1$

$x(a-x) = 1$

$x^2 - ax + 1 = 0$

For repeated roots,

$a^2 - 4 = 0$

$a = \pm 2$

Since $a > 0$, $a = 2$

7 [9] (a)	$\mathrm{f}(-x)=a(-x)^5+b(-x)^3+c(-x)$ $=-\left(ax^5+bx^3+cx\right)$ $=-\mathrm{f}(x)$
(b)	$\mathrm{f}(x)=ax^5+bx^3+cx=0$ Since all coefficients are real, by Conjugate Root Theorem, if $p+q\mathrm{i}$ is a root, then $p-q\mathrm{i}$ is also a root. Also from part **(a)**, $\mathrm{f}(-x)=-\mathrm{f}(x)$ We know that f is an odd function and If $\mathrm{f}(x)=0$, $\mathrm{f}(-x)=0$ and hence $-p-q\mathrm{i}$ and $-p+q\mathrm{i}$ are also non-real roots. Since $\mathrm{f}(-x)=-\mathrm{f}(x)$, So $-p-q\mathrm{i}$ and $-p+q\mathrm{i}$ are also the roots.

(c)

	$\int_{-3}^{3}\mathrm{f}(x)\,\mathrm{d}x$ $=\int_{-3}^{0}\mathrm{f}(x)\,\mathrm{d}x+\int_{0}^{3}\mathrm{f}(x)\,\mathrm{d}x$ Since f is an odd function and $\int_{0}^{3}\mathrm{f}(x)\,\mathrm{d}x=-5$ $\int_{-3}^{0}\mathrm{f}(x)\,\mathrm{d}x+\int_{0}^{3}\mathrm{f}(x)\,\mathrm{d}x$ $=5+(-5)$ $=0$

$\int_{-3}^{3}\mathrm{f}(|x|)\,\mathrm{d}x$

$=\int_{-3}^{0}\mathrm{f}(-x)\,\mathrm{d}x+\int_{0}^{3}\mathrm{f}(x)\,\mathrm{d}x$

$=\int_{3}^{0}-\mathrm{f}(x)\,\mathrm{d}x+\int_{0}^{3}\mathrm{f}(x)\,\mathrm{d}x$

$=\int_{0}^{3}\mathrm{f}(x)\,\mathrm{d}x+\int_{0}^{3}\mathrm{f}(x)\,\mathrm{d}x$

$=-5+(-5)$

$=-10$

(d)	$\mathrm{f}(x)=x^5+3x^3+cx$ $\mathrm{f}'(x)=5x^4+9x^2+c$ At stationary points, $5x^4+9x^2+c=0$ $x^2=\dfrac{-9\pm\sqrt{9^2-4(5)(c)}}{2(5)}$ Note : $x^2=\dfrac{-9-\sqrt{9^2-4(5)(c)}}{2(5)}$ (rejected) If they are 2 stat points, $x^2>0$ $-9+\sqrt{9^2-4(5)(c)}>0$ $\sqrt{9^2-4(5)(c)}>9$ $81-20(c)>81$ $c<0$
8[12] **(a)**	$\dfrac{\mathrm{d}x}{\mathrm{d}t}=2t$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{1}{t}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}\div\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{1}{2t^2}$ At the point with parameter *t*, Equation of tangent to C at $\left(t^2,\ln t\right)$ is $y-\ln t=\dfrac{1}{2t^2}\left(x-t^2\right)$ $y=\dfrac{1}{2t^2}\left(x-t^2\right)+\ln t$ $y=\dfrac{1}{2t^2}x-\dfrac{1}{2}+\ln t$
(b)	Equation of *L*, the tangent at *P*: $y=\dfrac{1}{2p^2}x-\dfrac{1}{2}+\ln p$ Given that *L* passes through $\left(1,\ \dfrac{p^2+1}{2p^2}\right)$, $\dfrac{p^2+1}{2p^2}=\dfrac{1}{2p^2}(1)-\dfrac{1}{2}+\ln p$ $\ln p=1$ $p=\mathrm{e}$

(c)	$\int \ln x \, \mathrm{d}x$ $u = \ln x \ , \ v' = \int 1 \, \mathrm{d}x$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}, v' = x$ $\therefore \int \ln x \, \mathrm{d}x$ $= x\ln x - \int x\left(\frac{1}{x}\right)\mathrm{d}x$ $= x\ln x - \int 1 \, \mathrm{d}x$ $= x\ln x - x + C$
(d)	Cartesian equation of curve C_2: Since $t > 0$, $t = \sqrt{x}, \quad y = \ln t$ $\Rightarrow \quad y = \ln\left(\sqrt{x}\right)$ $= \frac{1}{2}\ln x$ Area bounded = $\int_1^{\mathrm{e}^2} \left(y_1 - y_2\right)\mathrm{d}x$ $= \int_1^{\mathrm{e}^2}\left(\frac{1}{2\mathrm{e}^2}x + \frac{1}{2} - \frac{1}{2}\ln x\right)\mathrm{d}x$ $= \left[\frac{1}{4\mathrm{e}^2}x^2 + \frac{1}{2}x\right]_1^{\mathrm{e}^2} - \frac{1}{2}\int_1^{\mathrm{e}^2}\left(\ln x\right)\mathrm{d}x$ $= \left\{\frac{1}{4\mathrm{e}^2}\left(\mathrm{e}^2\right)^2 + \frac{1}{2}\mathrm{e}^2 - \frac{1}{4\mathrm{e}^2} - \frac{1}{2}\right\} \quad -\frac{1}{2}\left[x\ln x - x\right]_1^{\mathrm{e}^2}$ $= \left\{\frac{3}{4}\mathrm{e}^2 - \frac{1}{4\mathrm{e}^2} - \frac{1}{2}\right\} - \frac{1}{2}\left\{2\mathrm{e}^2\ln \mathrm{e} - 1^2\ln 1 - \mathrm{e}^2 + 1\right\}$ $= \left(\frac{3}{4}\mathrm{e}^2 - \frac{1}{4\mathrm{e}^2} - \frac{1}{2}\right) - \left(\mathrm{e}^2 - \frac{1}{2}\mathrm{e}^2 + \frac{1}{2}\right)$ $= \left(\frac{1}{4}\mathrm{e}^2 - \frac{1}{4\mathrm{e}^2} - 1\right)$ unit^2

9 [14] (a)

$\overrightarrow{OA}=\begin{pmatrix}-5\\-4\\1\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}-3\\6\\2\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}-3\\-4\\2\end{pmatrix}$

$\overrightarrow{AB}=\begin{pmatrix}-3\\6\\2\end{pmatrix}-\begin{pmatrix}-5\\-4\\1\end{pmatrix}=\begin{pmatrix}2\\10\\1\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}-3\\-4\\2\end{pmatrix}-\begin{pmatrix}-5\\-4\\1\end{pmatrix}=\begin{pmatrix}2\\0\\1\end{pmatrix}$

$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}2\\10\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}10\\0\\-20\end{pmatrix}$

Area of triangle ABC

$=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$

$=\frac{1}{2}\left|\begin{pmatrix}10\\0\\-20\end{pmatrix}\right|$

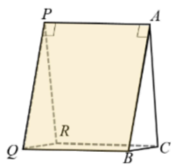
$=\frac{1}{2}\sqrt{100+0+400}$

$=\frac{1}{2}\sqrt{500}$ or $=5\sqrt{5}$ unit^2

(b)

$\underset{\sim}{n}_{ABC}=k\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)$

Plane ABC is parallel to Plane PQR



$\underset{\sim}{n}_{ABC}=\begin{pmatrix}1\\0\\-2\end{pmatrix}$

Vector equation of π_{ABC} in **scalar product** form:

$\underset{\sim}{r}\bullet\underset{\sim}{n}_{ABC}=\underset{\sim}{a}\bullet\underset{\sim}{n}_{ABC}$

$\underset{\sim}{r}\bullet\begin{pmatrix}1\\0\\-2\end{pmatrix}=\begin{pmatrix}-5\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}1\\0\\-2\end{pmatrix}=-7$

Vector equation of π_{ABC} in **cartesian** form:

$x-2z=-7$

(c) **Method 1**:

Let N be a point that lies on π_{PQR}

$$\pi_{PQR}: \quad \underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -2$$

By observation, $\overrightarrow{ON} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\overrightarrow{NA} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix}$$

Shortest distance from A to π_{PQR}

= Perpendicular height of the prism is

$$= \frac{\left| \overrightarrow{NA} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right|}{\sqrt{5}}$$

$$= \frac{\left| \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right|}{\sqrt{5}}$$

$$= \sqrt{5}$$

Volume of prism

$= \left(5\sqrt{5}\right)\left(\sqrt{5}\right)$

$= 25$ unit^3

Method 2:

$$\pi_{PQR}: \quad \underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -2,$$

$$\underset{\sim}{r} \bullet \frac{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}{\sqrt{5}} = \frac{-2}{\sqrt{5}}$$

Perpendicular height of the prism is

$$\frac{|-7-(-2)|}{\sqrt{5}} = \sqrt{5} \text{ units}$$

Volume of prism

$= \left(5\sqrt{5}\right)\left(\sqrt{5}\right)$

$= 25$ unit^3

(d) **Method 1**: Use intersection of l_{AP} and π_{PQR}

Let P be the foot of the perpendicular.

$$l_{AP}: \quad \underset{\sim}{r} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

$$\overrightarrow{OP} = \begin{pmatrix} -5+\lambda \\ -4 \\ 1-2\lambda \end{pmatrix}, \qquad \text{for some } \lambda \in \mathbb{R}$$

$$\begin{pmatrix} -5+\lambda \\ -4 \\ 1-2\lambda \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -2$$

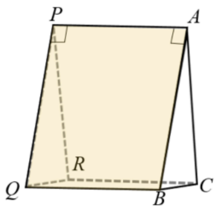
$$-5+\lambda-2(1-2\lambda) = -2$$

$$-7+5\lambda = -2$$

$$\lambda = 1$$

$$\overrightarrow{OP} = \begin{pmatrix} -5+1 \\ -4 \\ 1-2 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix}$$

$P(-4,-4,-1)$



(d) **Method 2:** Use $\overrightarrow{AP} // \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Let P be the foot of the perpendicular.

$$l_{AP}: \quad \underset{\sim}{r} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

$$\overrightarrow{OP} = \begin{pmatrix} -5+\lambda \\ -4 \\ 1-2\lambda \end{pmatrix}, \qquad \text{for some } \lambda \in \mathbb{R}$$

$$\overrightarrow{AP} = \begin{pmatrix} -\lambda \\ 0 \\ -2\lambda \end{pmatrix}$$

From part **(c)**, $\left|\overrightarrow{AP}\right| = \sqrt{5}$

$$\left|\begin{pmatrix} -\lambda \\ 0 \\ -2\lambda \end{pmatrix}\right| = \sqrt{5}$$

$$|\lambda| = 1$$

$$\lambda = \pm 1$$

When $\lambda = 1, \overrightarrow{OP} = \begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -2$

$\therefore P(-4,-4,-1)$ lies on π_{PQR}

When $\lambda = -1, \overrightarrow{OP} = \begin{pmatrix} -6 \\ -4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 0$

$\therefore P(-6,-4,3)$ does not lie on π_{PQR}

$\therefore P(-4,-4,-1)$

(d)

<u>Method 3</u>: Using projection vector, $\overrightarrow{NA}$ projected onto normal vector

Let N be a point that lies on the plane $\underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -2$

$$\overrightarrow{ON} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\overrightarrow{NA} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix}$$

$$\overrightarrow{PA} = \frac{\begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}{\sqrt{5}} \frac{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}{\sqrt{5}}$$

$$\overrightarrow{PA} = -\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$\overrightarrow{OA} - \overrightarrow{OP} = -\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$\overrightarrow{OP} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix}$$

$P(-4, -4, -1)$

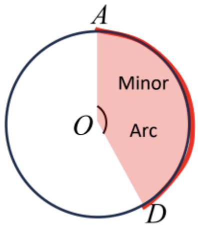
(e)

$$\overrightarrow{OA}=\begin{pmatrix}-5\\-4\\1\end{pmatrix},\ \overrightarrow{OD}=\begin{pmatrix}1\\5\\-4\end{pmatrix}$$

Let the angle between $\overrightarrow{OA}$ and $\overrightarrow{OD}=\theta$

$$\begin{pmatrix}-5\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}1\\5\\-4\end{pmatrix}=\left(\sqrt{42}\right)^2\cos\theta$$

$$\cos\theta=\frac{-5-20-4}{42}=-\frac{29}{42}$$



Since, $\theta=\cos^{-1}\left(-\frac{29}{42}\right)$

$\theta=133.6678153°$

(Minor) Arc length

$=r\theta$, where θ is in radians.

$=\sqrt{42}\left[\cos^{-1}\left(-\frac{29}{42}\right)\right]$

$=15.1192$

$=15.119$ (3 d.p.)

OR

(minor) Arc length

$=\frac{\theta}{360}[2\pi r]$, where θ is in degrees.

$=15.119$ (3 d.p.)

10 [14] (a)

$y=24-2x$

(b)

$z=24-3x$

$\frac{dx}{dt}\ \alpha\ (yz)$ or $\frac{dx}{dt}=k_1(yz)$ where k_1 is a positive constant

$$\frac{dx}{dt}=k_1(24-2x)(24-3x)$$
$$=6k_1(12-x)(8-x)$$
$$=k(x-12)(x-8)$$

$\therefore\ \frac{dx}{dt}=k(x-12)(x-8)$, where k is a positive constant

(c) **<u>Method 1</u>**:

$$\frac{dx}{dt}=k(x-12)(x-8)$$

$$\frac{dt}{dx}=\frac{1}{k(x-12)(x-8)}$$

$$t=\frac{1}{k}\int\frac{1}{(x-8)(x-12)}\,dx$$

$$t=\frac{1}{k}\left[\int\frac{1}{4\left(x-12\right)}\,dx-\int\frac{1}{4\left(x-8\right)}\,dx\right]$$

$$t=\frac{1}{4k}\ln\left|\frac{x-12}{x-8}\right|+C$$

$$4kt-4C=\ln\left|\frac{x-12}{x-8}\right|$$

$$\frac{x-12}{x-8}=\pm e^{4kt}e^{-4C}$$

$$=Ae^{4kt}$$

where $A=\pm e^{-4C}$ is an arbitrary constant

When $t=0$, $x=0$:

$$\frac{0-12}{0-8}=Ae^{4k(0)}$$

$$A=\frac{3}{2}$$

$$\frac{x-12}{x-8}=\frac{3}{2}e^{4kt}$$

$$2x-24=(3x-24)e^{4kt}$$

$$x=\left[\frac{24(1-e^{4kt})}{2-3e^{4kt}}\right]$$

(c) **Method 2**:

$$\frac{dx}{dt} = k(x-12)(x-8)$$

$$\frac{1}{(x-12)(x-8)}\frac{dx}{dt} = k$$

$$\int \frac{1}{(x-8)(x-12)}\,dx = kt + C$$

$$\int \frac{1}{(x-10)^2 - 2^2}\,dx = kt + C$$

$$\frac{1}{2(2)}\ln\left|\frac{x-10-2}{x-10+2}\right| = kt + C$$

$$\ln\left|\frac{x-12}{x-8}\right| = 4kt + 4C$$

$$\frac{x-12}{x-8} = \pm e^{4kt}e^{4C}$$

$$= Ae^{4kt}$$

where $A = \pm e^{4C}$ is an arbitrary constant

When $t = 0$, $x = 0$:

$$\frac{0-12}{0-8} = Ae^{4k(0)}$$

$$A = \tfrac{3}{2}$$

$$\frac{x-12}{x-8} = \frac{3}{2}e^{4kt}$$

$$2x - 24 = (3x-24)e^{4kt}$$

$$x = \left[\frac{24(1-e^{4kt})}{2-3e^{4kt}}\right]$$

d

$x \to 8$ as $t \to \infty$

Theoretical Mass = 8g

e

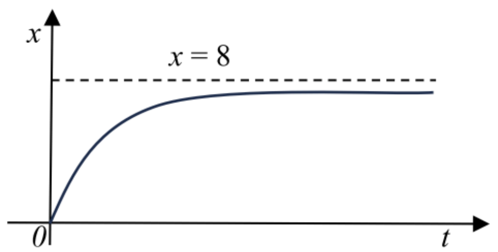
When $t = 5$, $x = 4$:

$$\frac{4-12}{4-8} = \frac{3}{2}e^{4k(5)}$$

$$\frac{-8}{-4} = \frac{3}{2}e^{4k(5)}$$

$$k = \frac{1}{20}\ln\left(\frac{4}{3}\right)$$

f



11 **[12]** **(a)**	By Pythagoras Theorem, $l^2 = h^2 + \left(\frac{x}{2}\right)^2$
(b)	$A =$ Area of Square + Area of 4 Triangles $A = x^2 + 4\left[\frac{1}{2}(x)\sqrt{h^2 + \frac{x^2}{4}}\right]$ $\therefore A = x^2 + 2x\sqrt{h^2 + \frac{x^2}{4}}$ (shown)
(c)	From part **(b)**, $A - x^2 = 2x\sqrt{h^2 + \frac{x^2}{4}}$ $\frac{A - x^2}{2x} = \sqrt{h^2 + \frac{x^2}{4}}$ $\left(\frac{A - x^2}{2x}\right)^2 = h^2 + \frac{x^2}{4}$ $h^2 = \left(\frac{A - x^2}{2x}\right)^2 - \frac{x^2}{4}$ Volume of a right pyramid $= \frac{1}{3} \times$ base area $\times$ height $V = \frac{1}{3}x^2 h$ $V^2 = \frac{1}{9}x^4\left[\left(\frac{A - x^2}{2x}\right)^2 - \frac{x^2}{4}\right]$ $= \frac{x^2\left(A - x^2\right)^2 - x^6}{36}$ $= \frac{x^2\left(A^2 - 2Ax^2 + x^4\right) - x^6}{36}$
	$V^2 = \frac{A^2x^2 - 2Ax^4}{36}$ $V^2 = \frac{Ax^2\left(A - 2x^2\right)}{36}$

(d)	$V^2 = \dfrac{Ax^2\left(A-2x^2\right)}{36}$ **Method 1**: $2V\dfrac{\mathrm{d}V}{\mathrm{d}x} = \dfrac{A\left(2Ax-8x^3\right)}{36}$ For stationary values, $\dfrac{\mathrm{d}V}{\mathrm{d}x}=0$ $2Ax-8x^3=0$ $2x\left(A-4x^2\right)=0$ $x\neq 0$, $x\neq -\sqrt{\dfrac{A}{4}}$, $\therefore$ $x=\dfrac{1}{2}\sqrt{A}$ **Method 2**: $V=\dfrac{\sqrt{A}x\sqrt{\left(A-2x^2\right)}}{6}$ $\dfrac{\mathrm{d}V}{\mathrm{d}x}=\dfrac{1}{6}\sqrt{A}\sqrt{A-2x^2}+\dfrac{1}{6}\sqrt{A}x\left[\dfrac{1}{2}\left(A-2x^2\right)^{-\frac{1}{2}}(-4x)\right]$ $\dfrac{\mathrm{d}V}{\mathrm{d}x}=\dfrac{\sqrt{A}\left(A-2x^2\right)-2\sqrt{A}x^2}{6\sqrt{A-2x^2}}$ $=\dfrac{\sqrt{A}\left(A-4x^2\right)}{6\sqrt{A-2x^2}}$ For stationary values, $\dfrac{\mathrm{d}V}{\mathrm{d}x}=0$ $A-4x^2=0$ $x\neq 0$, $x\neq -\sqrt{\dfrac{A}{4}}$, $\therefore$ $x=\dfrac{1}{2}\sqrt{A}$ Maximum $V^2=\dfrac{A^2\left(\dfrac{A}{4}\right)-2A\left(\dfrac{A^2}{16}\right)}{36}$ Maximum $V=\sqrt{\dfrac{A^3}{288}}=\dfrac{\sqrt{A^3}}{12\sqrt{2}}=\dfrac{\sqrt{2}\sqrt{A^3}}{24}$

(e)

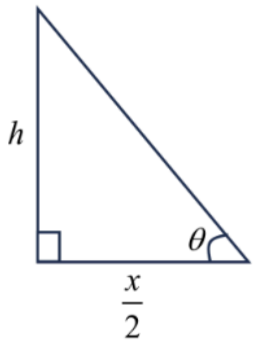
$$\frac{1}{3}(x)^2 h = \frac{\sqrt{A^3}}{12\sqrt{2}}$$

$$\frac{h}{x} = \frac{3\sqrt{A^3}}{12\sqrt{2}} \div \left(\frac{\sqrt{A^3}}{2^3}\right)$$

$$\frac{h}{x} = \frac{2^3}{4\sqrt{2}}$$

$$\therefore \frac{h}{x} = \sqrt{2}$$

Let the angle the lateral face make with the horizontal be θ.



$$\tan\theta = \frac{h}{\frac{x}{2}}$$

$$\tan\theta = \frac{2h}{x}$$

$$\tan\theta = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(2\sqrt{2}\right)$$

$$\theta = 70.529^\circ$$

$\theta = 71^\circ$ (nearest degree)