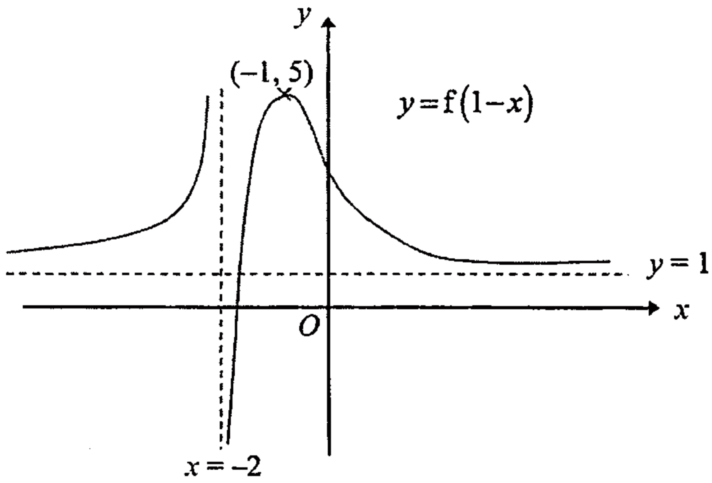
DHS H2 Mathematics Promo 2023

1 The parallelogram $OABC$ has points F and G that divide AB and BC in the ratio of 2:1 and 4:1 respectively. The lines OF and AG intersect at the point E. By letting $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$, find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. Hence find the ratio $OE : EF$. [5]

2 Given that $x < 0$, solve the inequality $\frac{2}{x} < |x| - a$, where a is constant such that $a > 3$. [4]

3 The diagram shows the curve $y = \mathrm{f}(1-x)$. The curve has a maximum point at $(-1, 5)$ and two asymptotes $x = -2$ and $y = 1$.



On separate diagrams, sketch the graphs of

(a) $y = \mathrm{f}(x)$, [2]

(b) $y = \mathrm{g}'(x)$, where $\mathrm{g}(x) = \mathrm{f}(1-x)$. [2]

4 It is given that $\mathrm{e}^y = 2 + \mathrm{e}^x$.

(a) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{\mathrm{d}y}{\mathrm{d}x}$. Hence find the Maclaurin series for y up to and including the term in x^2, giving the coefficients in exact form. [4]

(b) By using appropriate standard series expansions from the List of Formulae (MF26), verify the correctness of the series expansion found in **part (a)**. [3]

5 With reference to the origin O, the points A and B have coordinates $(1, 2, 0)$ and $(-5, 1, 0)$.

(a) Find the angle AOB. [3]

The vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are projected along the direction of $\begin{pmatrix}1\\2\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\\2\end{pmatrix}$ to form the vectors $\overrightarrow{O'A'}$ and $\overrightarrow{O'B'}$ respectively.

(b) Find the vectors $\overrightarrow{O'A'}$ and $\overrightarrow{O'B'}$. [3]

(c) Find the exact area of the triangle $O'A'B'$. [2]

6 A curve C has parametric equations

$$x = 3t^3,\quad y = \frac{3}{1+t^2},\qquad t \in \ .$$

(a) Find $\frac{dy}{dx}$ in terms of t. Explain why C has no stationary points. [2]

(b) Sketch C, indicating clearly the equation of the asymptote if any. [2]

(c) Show that the equation of the tangent L to C at the point $\left(-3, \frac{3}{2}\right)$ is $6y = x + 12$. [1]

(d) For $t < 0$, find the exact value of the area of the region bounded by C, L and the y-axis. [4]

7 (a) It is given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

(i) Find $\sum_{r=1}^{n}\left(2^{r+1} + 3r - r^2\right)$ in the form $A(2^n - 1) + f(n)$, where A is a constant and $f(n)$ is in fully factorised form. [3]

(ii) Using your answer in part a(i), find $\sum_{r=4}^{N}\left(2^r + 3r - (r-1)^2\right)$, leaving your answer in the form $B\left(2^N\right) + C\left[N(N-1)(5-N)\right] + DN + E$ where B, C, D and E are constants to be determined. [4]

(b) A sequence is such that $v_1 = p$, where p is a constant, and

$$v_{n+1} = 3v_n - 2, \quad \text{for } n \geq 1.$$

Describe how the sequence behaves when

(i) $p = 5$, [1]

(ii) $p = 1$. [1]

8 (a) Find $\int \sin x \sin 4x \, dx$. [2]

(b) Using the substitution $x = \sec\theta$, where $0 < \theta < \frac{\pi}{2}$, find the exact value of $\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x} \, dx$. [4]

(c) Find $\int \frac{x}{\sqrt{1-m^2x^2}} \, dx$, where m is a constant. Hence find $\int \left(\sin^{-1} mx\right) \frac{x}{\sqrt{1-m^2x^2}} \, dx$. [4]

9 (a) The functions f and g are defined by

$$f : x \mapsto \frac{1}{x+a}, \quad x \in \ , \quad x \neq -a,$$

$$g : x \mapsto x^2 + 2, \quad x \in \ ,$$

where a is a positive constant.

(i) Show that f^{-1} exists and define f^{-1} in a similar form. [3]

(ii) Show that the composite function gf exists and find its exact range. [3]

(b) The function h is defined by

$$h : x \mapsto \begin{cases} \tan^{-1} x & \text{for } -1 < x \leq 1, \\ -\frac{\pi}{4}x + \frac{\pi}{2} & \text{for } 1 < x \leq 3. \end{cases}$$

It is further given that $h(x+4) = h(x)$ for all real values of x.

(i) Sketch the graph of $y = h(x)$ for $-2 \leq x \leq 6$. [3]

(ii) Write down $\int_{-2}^{6} h(x) \, dx$. [1]

10 The curves C_1 and C_2 have equations $y = \sqrt{15x+4}$ and $y = x^2 - 2$ respectively. The region R is bounded by the y-axis, C_1 and C_2.

(a) Show algebraically that the point with coordinates (3,7) is the only point of intersection between C_1 and C_2. [3]

(b) Without the use of a graphing calculator, find the exact area of R. [4]

(c) Find the volume of revolution when R is rotated 2π radians about the y-axis. [3]

11 An ecologist is investigating the population of rabbits in a forest ecosystem. The population growth rate of the rabbits is influenced by both the natural growth rate and the effects of predation by foxes. It is modelled by the differential equation

$$\frac{dP}{dt} = 2P - 0.02P^2,$$

where P, in thousands, is the population of rabbits over t number of weeks. It is estimated that the initial number of rabbits in the forest is P_0 thousands.

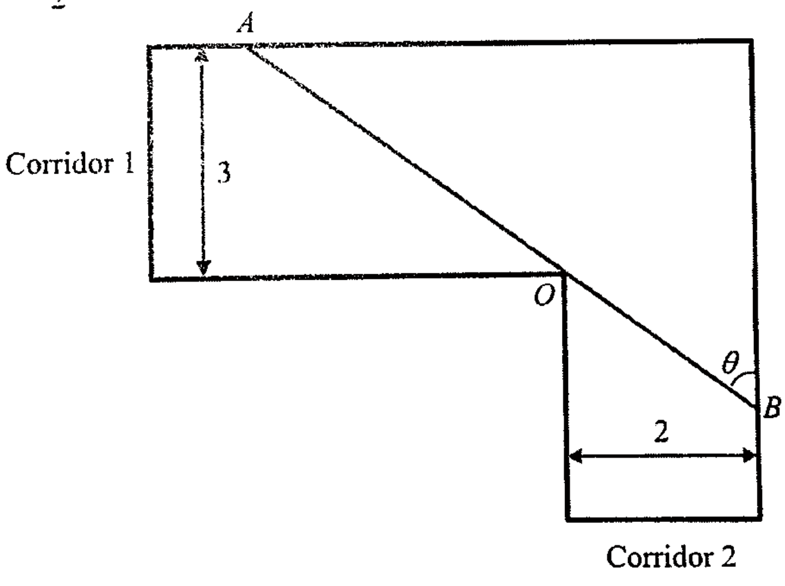
(a) Determine the range of values of P for which the population of the rabbits is increasing. [2]

(b) Show that $P = \dfrac{100P_0e^{2t}}{100 - P_0 + P_0e^{2t}}$. [6]

(c) What can be said about the population of rabbits eventually? Justify your answer. [2]

(d) Sketch, on the same diagram, the graphs of the population of rabbits over time when $P_0 = 80$ and $P_0 = 120$ respectively. [2]

12 In a particular building, there is an L-shaped corridor (see diagram below). Corridor 1 has a fixed width of 3 m and Corridor 2, which is perpendicular to the first corridor, has a fixed width of 2 m. The points A and B are on one side of the wall along Corridor 1 and Corridor 2 respectively, such that line AB touches the corner O and makes an angle of θ radians with one of the walls as shown where $0 \le \theta \le \frac{\pi}{2}$.

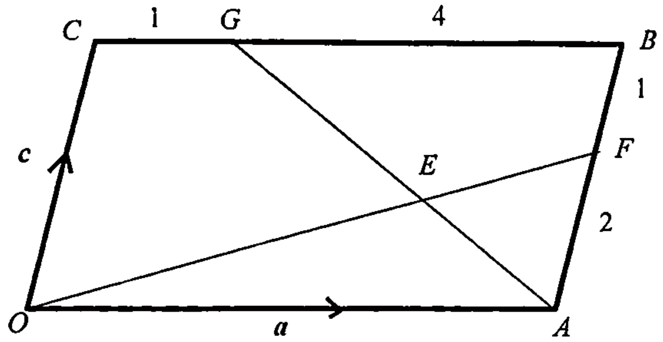


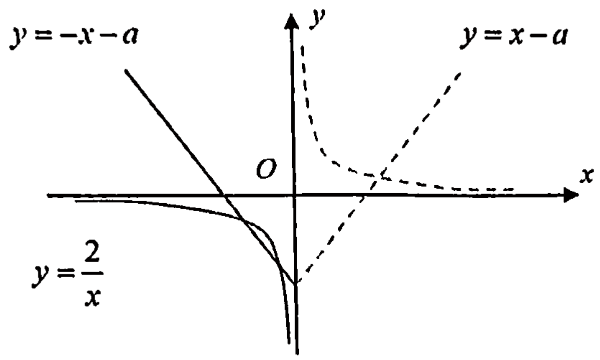
(a) Show that $AB = \dfrac{\alpha}{\cos\theta} + \dfrac{\beta}{\sin\theta}$, where α and β are constants to be determined. [1]

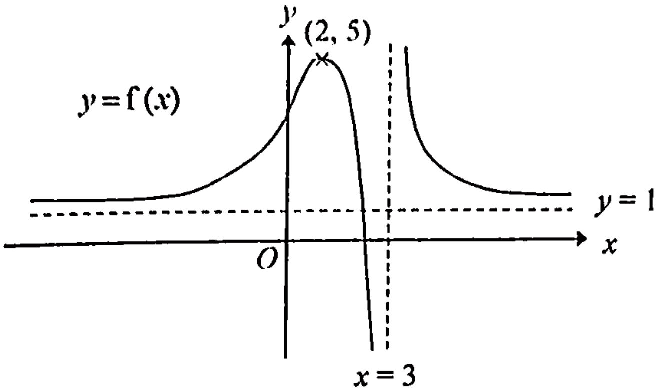
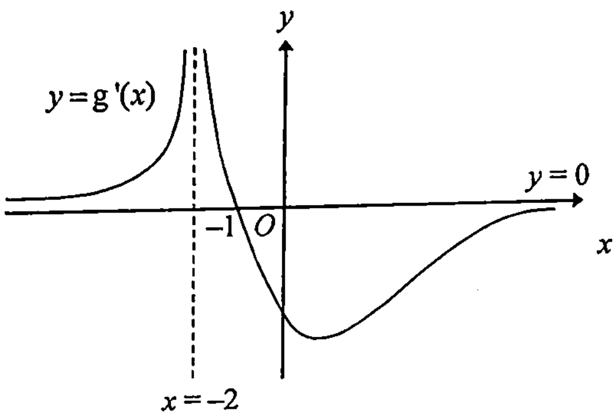
(b) If θ is decreasing at 0.1 radians per second, find the rate at which AB is increasing at the instant when $OB = 4$ m. [4]

(c) A pole of length p m is carried through this L-shaped corridor parallel to the ground. By using differentiation, show that the maximum possible value of p is $\left[(2)^k + (3)^k\right]^{\frac{1}{k}}$, where k is a constant to be determined exactly. [7]

2023 Year 5 H2 Math Promotional Examination Soln

Qn	Suggested Solution
1	 $\overrightarrow{OF} = \mathbf{a} + \frac{2}{3}\mathbf{c}$, $\overrightarrow{AG} = \mathbf{c} - \frac{4}{5}\mathbf{a}$ $\overrightarrow{OE} = \lambda\overrightarrow{OF} = \lambda\left(\mathbf{a} + \frac{2}{3}\mathbf{c}\right)$ $\overrightarrow{OE} = \mathbf{a} + \mu\overrightarrow{AG} = \mathbf{a} + \mu\left(\mathbf{c} - \frac{4}{5}\mathbf{a}\right)$ By comparing the coefficients, $\lambda = 1 - \frac{4}{5}\mu$---(1) $\frac{2}{3}\lambda = \mu$---(2) Subst (2) into (1), $\lambda = 1 - \frac{4}{5}\left(\frac{2}{3}\lambda\right)$ $\frac{23}{15}\lambda = 1$ $\lambda = \frac{15}{23}$ $\overrightarrow{OE} = \frac{15}{23}\left(\mathbf{a} + \frac{2}{3}\mathbf{c}\right) = \frac{15}{23}\mathbf{a} + \frac{10}{23}\mathbf{c}$ $\therefore OE : EF = 15 : 8$

Qn	Suggested Solution
2	 For $x<0$, solve $\frac{2}{x}=-x-a$ to determine the x-coordinate of the point of intersection. $x^2+ax+2=0$ $\left(x+\frac{a}{2}\right)^2+2-\frac{a^2}{4}=0$ $x=-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-2}=-\frac{a}{2}\pm\sqrt{\frac{a^2-8}{4}}$ Since $a^2>9$, then $a^2-8>0$. From the graph, the soln is $-\frac{a}{2}+\sqrt{\frac{a^2-8}{4}}<x<0$ or $x<-\frac{a}{2}-\sqrt{\frac{a^2-8}{4}}$.

Qn	Suggested Solution
3(a)	
(b)	

Qn	Suggested Solution
4(a)	$e^y = 2 + e^x$ Differentiate with respect to x $e^y \frac{dy}{dx} = e^x$ Differentiate with respect to x $e^y \left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) + e^y \frac{d^2y}{dx^2} = e^x$ $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{e^x}{e^y}$ $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$ (shown) When $x = 0$, $y = \ln 3$, $\frac{dy}{dx} = \frac{1}{3}$, $\frac{d^2y}{dx^2} = \frac{2}{9}$

	$y = \ln 3 + x\left(\frac{1}{3}\right) + \frac{x^2}{2!}\left(\frac{2}{9}\right) + \ldots$ $= \ln 3 + \frac{x}{3} + \frac{x^2}{9} + \ldots$
(b)	$y = \ln\left(2 + e^x\right)$ $= \ln\left(2 + 1 + x + \frac{x^2}{2!} + \ldots\right)$ $= \ln 3\left[1 + \frac{1}{3}\left(x + \frac{x^2}{2!} + \ldots\right)\right]$ $= \ln 3 + \ln\left[1 + \frac{1}{3}\left(x + \frac{x^2}{2!} + \ldots\right)\right]$ $y = \ln\left(2 + e^x\right) = \ln 3 + \ln\left[1 + X\right]$ $\quad \left[\text{let } X = \frac{1}{3}\left(x + \frac{x^2}{2!} + \ldots\right)\right]$ $= \ln 3 + \left[X - \frac{1}{2}X^2 + \ldots\right]$ (apply std series of $\ln(1+X)$) $= \ln 3 + \left[\frac{1}{3}\left(x + \frac{x^2}{2!} + \ldots\right) - \frac{1}{2}\left(\frac{1}{3}\left(x + \frac{x^2}{2!} + \ldots\right)\right)^2 + \ldots\right]$ $= \ln 3 + \left[\frac{x}{3} + \frac{x^2}{6} - \frac{x^2}{18} + \ldots\right]$ $= \ln 3 + \frac{x}{3} + \frac{x^2}{9} + \ldots$ (verified)

Qn	Suggested Solution
5(a)	$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}$ Let $\theta = \square AOB$, $\overrightarrow{OA} \square \overrightarrow{OB} = \left\|\overrightarrow{OA}\right\| \left\|\overrightarrow{OB}\right\| \cos\theta$ $\cos\theta = \dfrac{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \square \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{1^2 + 2^2}\sqrt{5^2 + 1^1}} = \dfrac{-5 + 2}{\sqrt{5}\sqrt{26}}$ $\cos\theta = -\dfrac{3}{\sqrt{5}\sqrt{26}}$ $\theta = 105.3°$

(b)

$$\overrightarrow{O'A'} = \left(\overrightarrow{OA} \square \frac{\begin{pmatrix}1\\2\\2\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\2\end{pmatrix}\right|} \right) \frac{\begin{pmatrix}1\\2\\2\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\2\end{pmatrix}\right|} = \left(\frac{\begin{pmatrix}1\\2\\0\end{pmatrix} \square \begin{pmatrix}1\\2\\2\end{pmatrix}}{\left(\sqrt{1^2+2^2+2^2}\right)^2} \right) \begin{pmatrix}1\\2\\2\end{pmatrix} = \frac{5}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\overrightarrow{O'B'} = \left(\overrightarrow{OB} \square \frac{\begin{pmatrix}2\\1\\2\end{pmatrix}}{\left|\begin{pmatrix}2\\1\\2\end{pmatrix}\right|} \right) \frac{\begin{pmatrix}2\\1\\2\end{pmatrix}}{\left|\begin{pmatrix}2\\1\\2\end{pmatrix}\right|} = \left(\frac{\begin{pmatrix}-5\\1\\0\end{pmatrix} \square \begin{pmatrix}2\\1\\2\end{pmatrix}}{\left(\sqrt{2^2+1^2+2^2}\right)^2} \right) \begin{pmatrix}2\\1\\2\end{pmatrix} = -\begin{pmatrix}2\\1\\2\end{pmatrix}$$

(c)

Area of triangle $O'A'B'$

$$= \frac{1}{2}\left|\overrightarrow{O'A'} \times \overrightarrow{O'B'}\right|$$

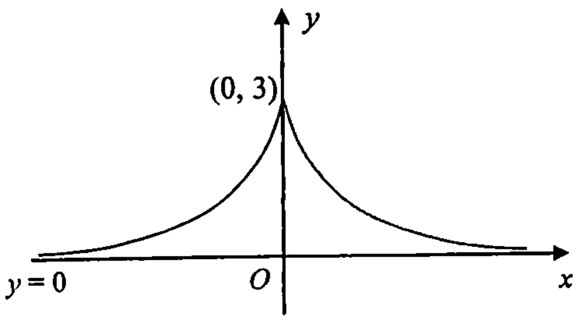
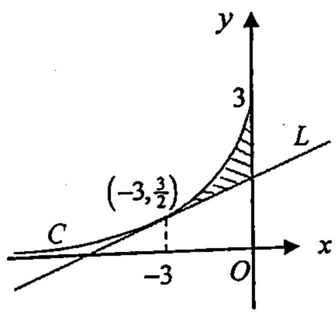
$$= \frac{5}{18}\left|\begin{pmatrix}1\\2\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\2\end{pmatrix}\right| \quad \because |\mathbf{a} \times (-\mathbf{b})| = |\mathbf{a} \times \mathbf{b}|$$

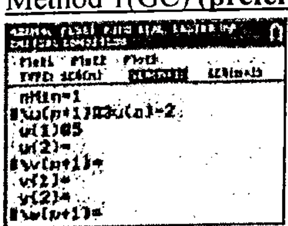
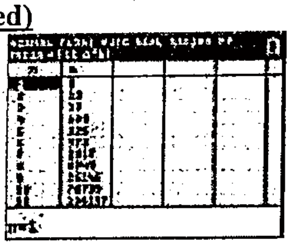
$=$

$$= \frac{5}{18}\left|\begin{pmatrix}2\\2\\-3\end{pmatrix}\right|$$

$$= \frac{5}{18}\sqrt{17} \text{ units}^2$$

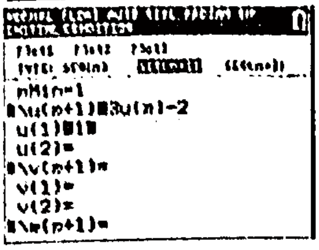
Qn	Suggested Solutions
6(a)	$\frac{dy}{dt} = -\frac{3}{(1+t^2)^2}(2t) = \frac{-6t}{(1+t^2)^2}$ $\frac{dx}{dt} = 9t^2$ $\frac{dy}{dx} = \frac{-6t}{(1+t^2)^2} \times \frac{1}{9t^2} = \frac{-2}{3t(1+t^2)^2}$ There are no real solutions for t when $\frac{dy}{dx} = 0$. Curve C has no stationary points.

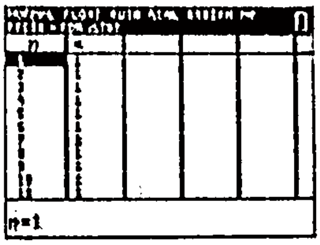
	Alternatively, $\frac{-2}{3t(1+t^2)^2}=0 \Rightarrow -2=0$ (Inconsistent) . Thus C has no stationary points.
(b)	
(c)	At $t=-1$, $\frac{dy}{dx}=\frac{-2}{3(-1)(1+(-1)^2)^2}=\frac{1}{6}$ Equation of tangent is $y-\frac{3}{2}=\frac{1}{6}(x+3)$ $y=\frac{1}{6}x+2$ $6y=x+12$ (shown)
(d)	Area required $=\int_{-3}^{0} y\ dx-\frac{1}{2}\left(\frac{3}{2}+2\right)(3)$ $=\int_{-1}^{0} y\left(\frac{dx}{dt}\right)dt-\frac{21}{4}$ $=\int_{-1}^{0}\frac{3}{1+t^2}\ (9t^2)\,dt-\frac{21}{4}$ $=27\int_{-1}^{0}\frac{t^2}{1+t^2}\ dt-\frac{21}{4}$ $=27\int_{-1}^{0}1-\frac{1}{1+t^2}\ dt-\frac{21}{4}$ $=27\left[t-\tan^{-1}t\right]_{-1}^{0}-\frac{21}{4}$ $=27\left(1-\frac{\pi}{4}\right)-\frac{21}{4}$ $=\frac{87}{4}-\frac{27}{4}\pi$ unit^2 

Qn	Suggested Solution
7a(i)	$\sum_{r=1}^{n}\left(2^{r+1}+3r-r^2\right)$ $=2\sum_{r=1}^{n}2^r+3\sum_{r=1}^{n}r-\sum_{r=1}^{n}r^2$ $=2\left[\frac{2\left(2^n-1\right)}{2-1}\right]+3\left[\frac{n}{2}(n+1)\right]$ $-\frac{1}{6}(n)(n+1)(2n+1)$ $=4\left(2^n-1\right)+\frac{n(n+1)}{6}(9-(2n+1))$ $=4\left(2^n-1\right)+\frac{1}{3}n(n+1)(4-n)$
a(ii)	Replace r with $r+1$ $\sum_{r=4}^{N}\left(2^r+3r-(r-1)^2\right)$ $=\sum_{r+1=4}^{r+1=N}\left(2^{r+1}+3(r+1)-(r+1-1)^2\right)$ $=\sum_{r=3}^{N-1}\left(2^{r+1}+3r-r^2\right)+\sum_{r=3}^{N-1}3$ $=\sum_{r=1}^{N-1}\left(2^{r+1}+3r-r^2\right)-\sum_{r=1}^{2}\left(2^{r+1}+3r-r^2\right)+3(N-1-3+1)$ $=4\left(2^{N-1}-1\right)+\frac{1}{3}(N-1)(N-1+1)(4-(N-1))-(16)+(3N-9)$ $=2\left(2^N\right)+\frac{1}{3}N(N-1)(5-N)+3N-29$ where $B=2, C=\frac{1}{3}, D=3, E=-29$
7b (i, ii)	$\underline{p=5}$ <u>Method 1(GC) (preferred)</u>   <u>Method 2 (algebraic)</u> $v_2=3v_1-2=3(5)-2=13$ $v_3=3v_2-2=3(13)-2=37$ $v_4=3v_3-2=3(37)-2=109$ … & so on $\therefore$ The sequence <u>**increases & diverges**</u>.

$v = 1$

Method 1(GC) (preferred)





Method 2 (algebraic)

$v_2 = 3v_1 - 2 = 3(1) - 2 = 1$

$v_3 = 3v_2 - 2 = 3(1) - 2 = 1$

$v_4 = 3v_3 - 2 = 3(1) - 2 = 1$

… & so on

$\therefore$ It is a **constant** sequence which **converges to 1**.

Qn	Suggested Solution
8(a)	$\int \sin x \sin 4x \, dx = -\frac{1}{2}\int \cos 5x - \cos 3x \, dx$ $= -\frac{1}{2}\left(\frac{\sin 5x}{5} - \frac{\sin 3x}{3}\right) + c$
(b)	$\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x} \, dx$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} (\sec\theta\tan\theta) \, d\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2\theta \, d\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec^2\theta - 1) \, d\theta$ $= [\tan\theta - \theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right)$ $= \sqrt{3} - 1 - \frac{\pi}{12}$ $dx = (\sec\theta\tan\theta) \, d\theta$ When $x = 2$, $\sec\theta = 2$ $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ When $x = \sqrt{2}$, $\sec\theta = \sqrt{2}$ $\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

(c)

$$\int \frac{x}{\sqrt{1-m^2x^2}}\,\mathrm{d}x = -\frac{1}{2m^2}\int -2m^2x\left(1-m^2x^2\right)^{-\frac{1}{2}}\,\mathrm{d}x$$

$$= -\frac{1}{2m^2}\frac{\left(1-m^2x^2\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$= -\frac{1}{m^2}\sqrt{1-m^2x^2}+c$$

$$\int\left(\sin^{-1}mx\right)\frac{x}{\sqrt{1-m^2x^2}}\,\mathrm{d}x$$

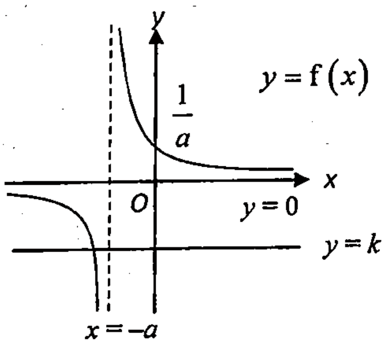
$$=\left(-\frac{\sin^{-1}mx}{m^2}\sqrt{1-m^2x^2}\right)-\int -\frac{1}{m}\,\mathrm{d}x$$

$$=\left(-\frac{\sin^{-1}mx}{m^2}\sqrt{1-m^2x^2}\right)+\frac{x}{m}+c$$

$u=\sin^{-1}mx$	$\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{x}{\sqrt{1-m^2x^2}}$
$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{m}{\sqrt{1-m^2x^2}}$	$v=-\frac{1}{m^2}\sqrt{1-m^2x^2}$

Qn	Suggested Solution

9(a)(i)



Every horizontal line $y=k$, $k\in\mathbb{R}$ cuts the graph of f at most once hence f is one-one and f^{-1} exist.

Let $y=\frac{1}{x+a}$

$\Rightarrow xy+ay=1$

$\Rightarrow xy=1-ay$

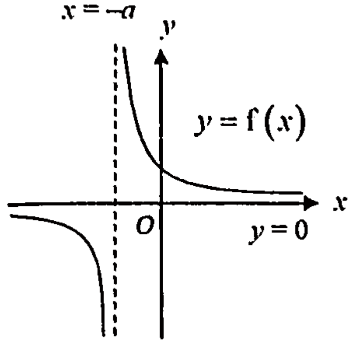
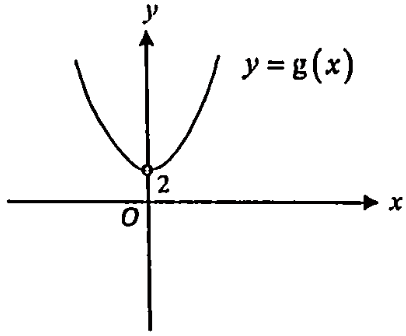
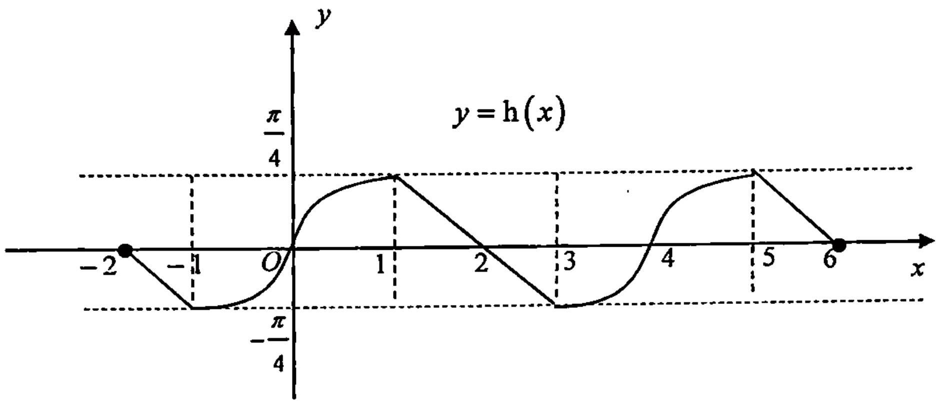
$\Rightarrow x=\frac{1-ay}{y}=\frac{1}{y}-a$

Therefore $\mathrm{f}^{-1}:x\mapsto\frac{1}{x}-a,\quad x\in\mathbb{R},\quad x\neq 0.$

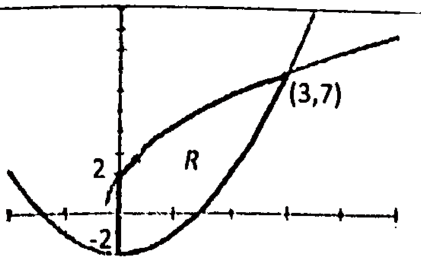
(a)(ii)

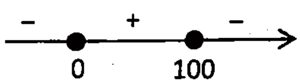
Since $\mathrm{R_f}=\mathbb{R}\setminus\{0\}$ and $\mathrm{D_g}=\mathbb{R}$

$\mathrm{R_f}\subseteq\mathrm{D_g}$, thus gf exists.

	  Using the graphs of f and g $D_f \to R_f \to R_{gf}$ $\mathbb{R}\setminus\{-a\} \to \mathbb{R}\setminus\{0\} \to (2,\infty)$ $R_{gf}=(2,\infty)$
b(i)	
b(ii)	$\int_{-2}^{6}\mathrm{h}(x)\ dx=0$

Qn	Suggested Solution
10(a)	$\sqrt{15x+4}=x^2-2$ $15x+4=(x^2-2)^2$ $15x+4=x^4-4x^2+4$ $x\left(x^3-4x-15\right)=0$ $x^3-4x-15=0 \quad \because x\neq 0$ (does not satisfy original eqn) $(x-3)(x^2+3x+5)=0$ Since $x^2+3x+5=\left(x+\frac{3}{2}\right)^2+\frac{11}{4}>0$ for all x, $\therefore x=3, y=7$ $\therefore (3,7)$ is the only point of intersection between the 2 curves.

(b)	 Area of the region R $=\int_0^3 \sqrt{15x+4}-(x^2-2)\,dx$ $=\left[\dfrac{(15x+4)^{\frac{3}{2}}}{\frac{3}{2}(15)}-\dfrac{x^3}{3}+2x\right]_0^3$ $=\left[\dfrac{2}{45}(45+4)^{\frac{3}{2}}-\dfrac{27}{3}+6-\dfrac{2}{45}(4)^{\frac{3}{2}}\right]$ $=\left[\dfrac{686}{45}-3-\dfrac{16}{45}\right]$ $=\dfrac{551-16}{45}$ $=\dfrac{535}{45}$ $=\dfrac{107}{9}$ unit^2
(c)	Volume of revolution $=\pi\left[\int_{-2}^{7} y+2\,dy-\dfrac{1}{225}\int_2^7\left(y^2-4\right)^2 dy\right]$ $=91.7$ units^3

Qn	Suggested Solutions
11(a)	For increasing population, $\dfrac{dP}{dt}>0$ $2P-0.02P^2>0$ $P(1-0.01P)>0$ $0<P<100$ 

(b)

$$\frac{dP}{dt} = 2P - 0.02P^2$$

$$\frac{dP}{dt} = 0.02\left(100P - P^2\right)$$

$$\int \frac{1}{100P - P^2}\, dP = \int 0.02\, dt$$

$$\int \frac{1}{P(100-P)}\, dP = 0.02\, dt$$

$$\frac{1}{100}\int \frac{1}{P} + \frac{1}{100-P}\, dP = \int 0.02\, dt$$

$$\frac{1}{100}\left(\ln P - \ln|100-P|\right) = 0.02t + C$$

$$\ln\left|\frac{P}{100-P}\right| = 2t + 100C$$

$$\frac{P}{100-P} = \pm e^{2t+100C}$$

$$\frac{P}{100-P} = \pm e^{100C} e^{2t}$$

$$\frac{P}{100-P} = Ae^{2t}, \text{ where } A = \pm e^{100C}$$

$$P = Ae^{2t}(100-P)$$

$$P = 100Ae^{2t} - Ae^{2t}P$$

$$P + Ae^{2t}P = 100Ae^{2t}$$

$$P = \frac{100Ae^{2t}}{1+Ae^{2t}}$$

When $t = 0$, $P = P_0$,

$$P_0 = \frac{100A}{1+A}$$

$$P_0(1+A) = 100A$$

$$A = \frac{P_0}{100-P_0}$$

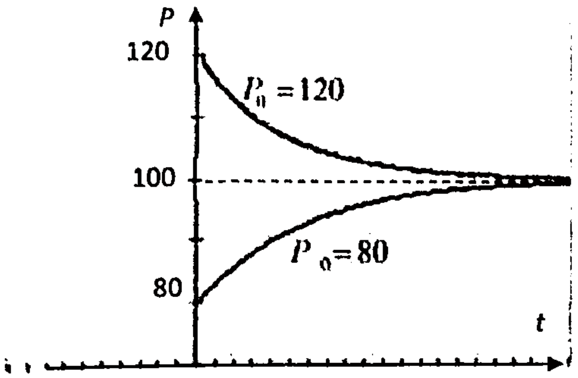
$$\therefore P = \frac{100\left(\frac{P_0}{100-P_0}\right)e^{2t}}{1+\left(\frac{P_0}{100-P_0}\right)e^{2t}} = \frac{100P_0e^{2t}}{100-P_0+P_0e^{2t}} \text{ (shown)}$$

(c)

As $t \to \infty$, $e^{2t} \to \infty$, $P = \frac{100P_0e^{2t}}{100-P_0+P_0e^{2t}} = \frac{100P_0}{\frac{100-P_0}{e^{2t}}+P_0} \to 100$

$\therefore$ population of rabbits approach 100,000

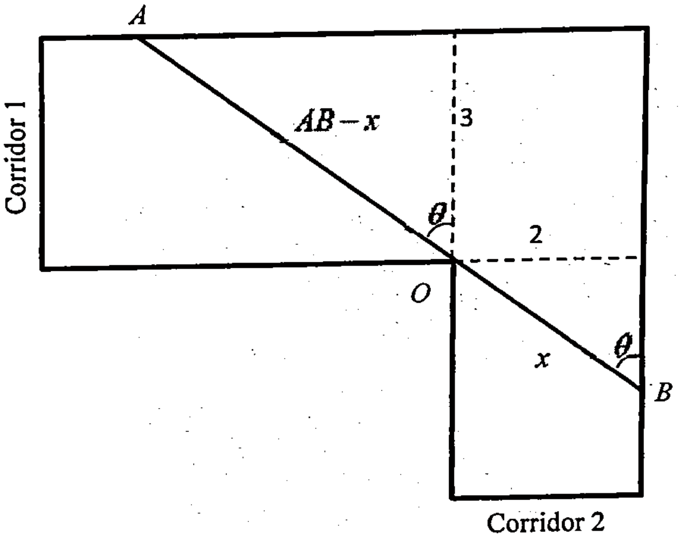
(d)



Qn	Suggested Solution

12(a)

Let OB be x m.



From diagram,

$\cos\theta = \dfrac{3}{AB - x}$ and $\sin\theta = \dfrac{2}{x}$.

So we have $AB = (AB - x) + x = \dfrac{3}{\cos\theta} + \dfrac{2}{\sin\theta}$, where $\alpha = 3, \beta = 2$.

(b)

Differentiating the result in part (a) with respect to t,

$$\frac{\mathrm{d}(AB)}{\mathrm{d}t} = (3\sec\theta\tan\theta - 2\operatorname{cosec}\theta\cot\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) \cdots (*)$$

When $OB = \dfrac{2}{\sin\theta} = 4$, $\theta = \frac{1}{6}\pi$.

Subst $\theta = \frac{1}{6}\pi$ and $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -0.1$ into (*):

Using GC, $\dfrac{\mathrm{d}(AB)}{\mathrm{d}t} = 0.493$ m/s (to 3 s.f.)

$\therefore$ Rate of increase in AB when $OB = 4$ is 0.493 m/s

(c)	$p \le AB$ for all possible lengths p and angles θ, i.e. max p = min AB. $AB = \frac{3}{\cos\theta} + \frac{2}{\sin\theta}$. $\frac{d(AB)}{d\theta} = \frac{3\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta} = \frac{3\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta}$ At min AB, $\frac{d(AB)}{d\theta} = 0$ $\Rightarrow 3\sin^3\theta = 2\cos^3\theta$ $\tan^3\theta = \frac{2}{3} \Rightarrow \tan\theta = \left(\frac{2}{3}\right)^{\frac{1}{3}}$ $\therefore \theta = \tan^{-1}\left(\frac{2}{3}\right)^{\frac{1}{3}}$.

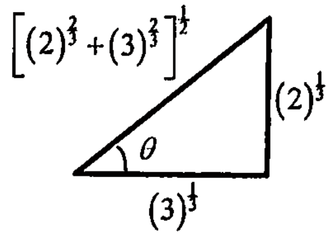
θ	0.717	$\tan^{-1}\left(\frac{2}{3}\right)^{\frac{1}{3}}$	0.719
$\frac{dAB}{d\theta}$	−0.0216	0	0.0205
Tangent	□	-	□

$\therefore AB$ is a minimum when $\theta = \tan^{-1}\left(\frac{2}{3}\right)^{\frac{1}{3}}$.

Alternative

$$\frac{d(AB)}{d\theta} = \frac{3\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta} = \frac{3\cos^3\theta\left(\tan^3\theta - \frac{2}{3}\right)}{\cos^2\theta\sin^2\theta}$$

θ	$\left(\tan^{-1}\left(\frac{2}{3}\right)^{\frac{1}{3}}\right)^-$	$\tan^{-1}\left(\frac{2}{3}\right)^{\frac{1}{3}}$	$\left(\tan^{-1}\left(\frac{2}{3}\right)^{\frac{1}{3}}\right)^+$
$\frac{dAB}{d\theta}$	negative	0	positive



Thus we have

$$AB=\frac{3\left[(2)^{\frac{2}{3}}+(3)^{\frac{2}{3}}\right]^{\frac{1}{2}}}{(3)^{\frac{1}{3}}}+\frac{2\left[(2)^{\frac{2}{3}}+(3)^{\frac{2}{3}}\right]^{\frac{1}{2}}}{(2)^{\frac{1}{3}}}$$

$$=\left[(2)^{\frac{2}{3}}+(3)^{\frac{2}{3}}\right]^{\frac{1}{2}}\left[(3)^{\frac{2}{3}}+(2)^{\frac{2}{3}}\right]$$

$$=\left[(2)^{\frac{2}{3}}+(3)^{\frac{2}{3}}\right]^{\frac{3}{2}}$$

$\max p=\left[(2)^{\frac{2}{3}}+(3)^{\frac{2}{3}}\right]^{\frac{3}{2}}$, where $k=\frac{2}{3}$.